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A Soergel bimodule approach to the character theory of real groups

Anna Romanov (UNSW)

Joint with Scott Larson (UGA)

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Dubrovnik Representation Theory XVII

PLAN:

① Motivation: What is the Lusztig-Vogan module?

② The LV module through the lens of categorification

③ An algebraic incarnation

④ Main results and discussion

① What is the LV module?

A PLOT SYNOPSIS

Some familiar players:

G connected complex reductive algebraic group $\quad \theta : G \to G$ holomorphic involution

$$\begin{tikzcd} & K \subset G^{\theta} \text{ finite index subgroup} \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\ \text{real reductive group } G_{\mathbb{R}} & & H \circlearrowright M_{LV} \text{ Lusztig-Vogan module} \end{tikzcd}$$

admissible representation theory ⟷ structure theory

irreducible character formulas ⟷ KLV polynomials (change of basis matrix)

Today's Goal: Describe two categorifications of M_{LV} - one geometric (living within the K-equivariant derived category of constructible sheaves on G/B), and one algebraic (a category of bimodules over polynomial rings)

① What is the LV module?

In more detail:

Fix G, θ, $K \subset G^{\theta}$. Set $\mathfrak{g} = \text{Lie } G$.

Running Example: $(G_{\mathbb{R}} = SL_2(\mathbb{R}))$

$G = SL_2(\mathbb{C})$, $\theta : g \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^{\times} \right\} \simeq \mathbb{C}^{\times}$

Recall:

An irreducible $(\mathfrak{g}, K)$-module V is <u>admissible</u> if in the decomposition

$$V = \bigoplus_{\sigma \in \widehat{K}} V_{\sigma}$$

each K-type V_{σ} appears with finite multiplicity.

(**Aside:** one motivation for this def'n is that

$$\widehat{G_{\mathbb{R}}}^{\text{unitary}} \subset \widehat{G_{\mathbb{R}}}^{\text{admissible}}$$
)

① What is the LV module?

Classification of irreducible admissible $(\mathfrak{g},K)$-modules:

Fix $\underset{\text{max'l torus}}{T} \subset \underset{\text{Borel}}{B} \subset G$. $K \curvearrowright G/B$ w/ finitely many orbits

$$\left\{ \begin{matrix} \text{irreducible admissible} \\ (\mathfrak{g},K)\text{-modules} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{irreducible } K\text{-equivariant} \\ \text{local systems on } K\text{-orbits} \end{matrix} \right\}$$

with a fixed regular integral infinitesimal character

Can realize this parameterization explicitly using Beilinson-Bernstein localization:

Harish-Chandra modules

$$\begin{tikzcd} \mathcal{M}_{f.g.}(\mathcal{U}_\theta, K) \arrow[r, bend left, "\Delta_\lambda"] \arrow[r, phantom, "\sim"] & \mathcal{M}_{coh}(\mathcal{D}_\lambda, K) \arrow[l, bend left, "\Gamma"] \end{tikzcd}$$

K-equivariant $\mathcal{D}_\lambda$-modules on G/B

fixed regular infinitesimal character $\to \theta$

$$\begin{tikzcd} M(\mathcal{O},\mathcal{L}) \arrow[d] & i_{\mathcal{O}!}(\mathcal{L}) \arrow[l, mapsto] \arrow[d] \\ L(\mathcal{O},\mathcal{L}) & i_{\mathcal{O}!*}(\mathcal{L}) \arrow[l, mapsto] \end{tikzcd}$$

irreducible admissible $(\mathfrak{g},K)$-module $\to L(\mathcal{O},\mathcal{L})$

$i_{\mathcal{O}} : \mathcal{O} \hookrightarrow G/B$ inclusion of K-orbit

$\mathcal{L}$ = irreducible K-eqvt connection on $\mathcal{O}$

① What is the LV module?

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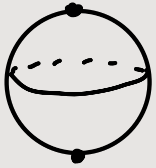
with a fixed regular integral infinitesimal character

Example: $SL_2(\mathbb{R})$

$K = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right\} \simeq \mathbb{C}^\times \curvearrowright$



$G/B \simeq \mathbb{C}\mathbb{P}^1$

3 orbits: $\overset{0}{\bullet}$ $\quad \underset{\infty}{\bullet}$ $\quad \mathbb{C}^\times$

closed $\qquad$ open

4 local systems: k_0, k_∞, $\underbrace{k_{\mathbb{C}^\times}, \overset{\text{"Möbius band"}}{\mathcal{L}}}$

← Why are there 2 K-eqvt local systems?

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \cdot [x:y] = [a^2 x : y]$$

$$\Rightarrow \mathrm{Stab}_K [x:y] = \pm \mathrm{id}$$

① What is the LV module?

What about character theory?

- Have finite-length abelian category $\mathcal{A}_\chi :=$ admissible $(\mathfrak{g}, K)$-modules w/ inf'l character χ with classification of simple objects given by

$$\{\text{standards}\} \twoheadrightarrow \{\text{simples}\}$$

- In Grothendieck group $[\mathcal{A}_\chi]$,

$$[M(\mathcal{O},\mathcal{L})] = \sum m^{\tau,\tau'}_{\mathcal{O},\mathcal{O}'} [L(\mathcal{O}',\mathcal{L}')]$$

↑

Question: What are these multiplicities?

Answer: Given by the Luszting-Vogan module of the associated Hecke algebra H

↓

My conventions:

unital $\mathbb{Z}[v,v^{-1}]$-algebra generated by $\{\delta_s \mid s\in S\}$ subject to relations

set of simple reflections in W

(i) $\delta_s^2 = 1 + (v^{-1}-v)\delta_s$ for all $s\in S$

(ii) $\underbrace{\delta_s\delta_t\delta_s\cdots}_{\text{order of } st} = \underbrace{\delta_t\delta_s\delta_t\cdots}_{\text{order of } st}$ for all $s\neq t\in S$

two natural bases:

$\{\delta_w\}$ standard basis

$\{b_w\}$ KL basis (self-dual)

① What is the LV module?

The Upshot: the Lusztig-Vogan module tells us about character theory of real groups.

How is M_{LV} constructed? Idea: imitate $\boxed{\text{Kazhdan-Lusztig theory}}$

• Set $D = \{(\mathcal{O}, \mathcal{L}) \mid \mathcal{O} \in K\backslash G/B,\ \mathcal{L} \in \mathrm{Loc}_K(\mathcal{O}) \text{ irred.}\}$	⟷	W = Weyl group
• length function on D: $\ell(\mathcal{O}, \mathcal{L}) = \dim \mathcal{O}$	⟷	$\ell(w) = \dim X_w$
• Set $M_{LV} :=$ free $\mathbb{Z}[v^{\pm 1}]$-module w/ basis D	⟷	$H = \bigoplus_{w \in W} \mathbb{Z}[v^{\pm 1}]\delta_w$
• Give M_{LV} structure of H-module via my conventions → $(v^{-1}\delta_s =)$ $T_s : M_{LV} \longrightarrow M_{LV}$ defined by either 1) explicit formulas on basis 2) geometric push-pull	⟷	$H \circlearrowleft H$ by right multiplication
• construct a partial order on D	⟷	Bruhat order
• prove existence of involution $D : M_{LV} \longrightarrow M_{LV}$ satisfying appropriate properties	⟷	Kazhdan-Lusztig involution

① What is the LV module?

Theorem (Lusztig-Vogan '83): For each $\delta \in D$, there is a unique *"self-dual" (i.e. $D(C_\delta) = v^{2\ell(\delta)} C_\delta$) element $C_\delta \in M_{LV}$ s.t.

$$C_\delta = \sum_{\gamma \le \delta} P_{\gamma,\delta}(v)\, \gamma$$

Kazhdan-Lusztig-Vogan polynomials

with $P_{\gamma,\delta} \in \mathbb{Z}[v]$ and $P_{\delta,\delta} = 1$.

Theorem (Vogan '83): For $\delta = (\mathcal{O}, \mathcal{L}),\ \gamma = (\mathcal{O}', \mathcal{L}') \in D$,

$$[M(\mathcal{O}, \mathcal{L}) : L(\mathcal{O}', \mathcal{L}')] = P_{\gamma,\delta}(1)$$

The punchline: KLV polynomials are defined as a change-of-basis matrix in the Lusztig-Vogan module, and their value at 1 determines multiplicities of $L(\mathcal{O}, \mathcal{L})$'s in $M(\mathcal{O}, \mathcal{L})$'s.

② M_{LV} through the lens of categorification

THE IDEA: View the H-module M_{LV} as a shadow of something happening on the level of categories.

$$\mathcal{M}_{LV} \circlearrowleft \mathcal{H} \xrightarrow[\text{groups}]{\text{take Grothendieck}} M_{LV} \circlearrowleft H$$

LV category Hecke category LV module Hecke algebra

- **Note:** This is essentially what [LV83] does in the original definition, but the results aren't exactly stated in the language of categorical actions.

Utility of this perspective:

- gives access to results about categorical actions
- can extract information about module category from information about monoidal category that acts

② M_{LV} through the lens of categorification

k = field of char 0

First step: a categorical upgrade of the Hecke algebra

- found within the B-equivariant derived category of constructible sheaves of k-vector spaces on G/B:

$$D^b_B(G/B)$$

  - triangulated category with 6 functor formalism
  - monoidal category under convolution $*$
  - contains B-equivariant intersection cohomology sheaves of Schubert varieties $\overline{X}_w$:

$$\{IC_w := IC(\overline{X}_w)\}_{w\in W} \subset D^b_B(G/B)$$

- define the geometric Hecke category

$$\mathcal{H} := \langle IC_w \mid w \in W \rangle_{\oplus, [-]} \subset D^b_B(G/B)$$

Semisimple complexes in $D^b_B(G/B)$

  - additive monoidal category (Decomposition Theorem)
  - categorifies the Hecke algebra: $[\mathcal{H}]_{\oplus} \simeq H$ $\left(\begin{array}{c}\mathbb{Z}[v^{\pm 1}]\text{-algebra via} \\ v\cdot[\mathcal{F}] := [\mathcal{F}[1]]\end{array}\right)$

split Grothendieck group $\nearrow$

② M_{LV} through the lens of categorification

Next: A categorical upgrade of M_{LV}

(**Rmk:** In [LV83], they define a category $\mathcal{C}$ of constructible K-equ't $\overline{\mathbb{Q}}_\ell$-sheaves on G/B over $\overline{\mathbb{F}}_p$ + sauce ("Frobenius") and identify $[\mathcal{C}] = M_{LV}$. They work in this setting to use Deligne's proof of the Weil conjectures. We will work in a different geometric world.)

- Consider the K-equivariant derived category

$$D^b_K(G/B) \supset \{ IC(\overline{O},\mathcal{L}) := i_{O!*}(\mathcal{L})[\dim O] \mid (O,\mathcal{L}) \in D \}$$

- convolution gives a right action $D^b_K(G/B) \circlearrowleft D^b_B(G/B)$

- **Fact:** For $\mathcal{G} \in D^b_K(G/B)$, $\mathcal{G} * IC_s = \pi_s^* \pi_{s*} \mathcal{G}[1]$ where $\pi_s : G/B \longrightarrow G/P_s$

- Define

$$\mathcal{M}^{ss}_{LV} := \langle IC(\mathcal{O},\mathcal{L}) \mid (O,\mathcal{L}) \in D \rangle_{\oplus,[-]} \overset{*}{\circlearrowleft} \mathcal{H}$$

$$\cup$$

$$\mathcal{M}^{\circ}_{LV} := \langle IC_Q * \mathcal{H} \mid Q \text{ closed} \rangle_{\oplus,\ominus,[-]} \overset{*}{\circlearrowleft} \mathcal{H}$$

semisimple complexes in $D^b_K(G/B)$

submodule gen'd by closed orbits

(Fact $\Rightarrow \mathcal{M}^{ss.}_{LV}$ is $\mathcal{H}$-stable)

(Here IC_Q are IC's corresponding to trivial local systems on closed orbits.)

② M_{LV} through the lens of categorification

Example: $SL(2,\mathbb{R})$

$$\begin{tikzcd}
\{0\} \arrow[dr, bend left, "i_0"] & & \\
& \mathbb{C}\mathbb{P}^1 & \mathbb{C}^\times \arrow[l, "j"'] \\
\{\infty\} \arrow[ur, bend right, "i_\infty"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}\mathbb{P}^1 \arrow[d, "\pi"] \\
pt
\end{tikzcd}$$

Four K-equt IC's:

$IC_0 := i_{0!} k_0 = i_{0*} k_0$

$IC_\infty := i_{\infty!} k_\infty = i_{\infty *} k_\infty$

$IC_{triv} := j_{!*} k_{\mathbb{C}^\times}[1] = k_{\mathbb{P}^1}[1]$

$IC_{sgn} := j_{!*}\mathcal{L} = j_!\mathcal{L} = j_*\mathcal{L}$

$\uparrow \quad \uparrow$ $\mathcal{L}$ is clean

$$\mathcal{M}^{ss}_{LV} = \langle IC_0, IC_\infty, IC_{triv}, IC_{sgn} \rangle_{\oplus,[-]}$$

Action: For $y \in \mathcal{M}^{ss}$, $[y]\cdot b_s = [\pi^*\pi_* y[1]]$ on a basis of $[\mathcal{M}^{ss}]$:

$$[IC_0]\cdot b_s = [\pi^*\pi_* i_{0*} k_{pt}[1]]$$
$$= [\pi^* k_{pt}[1]]$$
$$= [k_{\mathbb{P}^1}[1]] = [IC_{triv}]$$

same computation

$$[IC_\infty]\cdot b_s \overset{\searrow}{=} [IC_{triv}]$$

$$[IC_{sgn}]\cdot b_s = [\pi^* \underbrace{\pi_* j_* \mathcal{L}[2]}_{H^*(\mathbb{C}^\times,\mathcal{L})=0}]$$
$$= 0$$

$$[IC_{triv}]\cdot b_s = [\pi^*\pi_* k_{\mathbb{P}^1}[2]]$$
$$= [\pi^*(k_{pt} \oplus k_{pt}[2])]$$
$$= [k_{\mathbb{P}^1}] + [k_{\mathbb{P}^1}[2]]$$
$$= v^{-1}[IC_{triv}] + v[IC_{triv}]$$

FORMULAS:

$$\left.\begin{array}{l} m_0 \cdot b_s = m_{triv} \\ m_\infty \cdot b_s = m_{triv} \\ m_{triv}\cdot b_s = (v+v^{-1})\, m_{triv} \end{array}\right\} \text{block of trivial rep'n} = \text{submodule gen'd by closed orbits}$$

$$m_{sgn}\cdot b_s = 0$$

② M_{LV} through the lens of categorification

Some facts about $\mathcal{M}^{\circ}_{LV} = \langle IC_Q * \mathcal{H} \mid Q \text{ closed} \rangle_{\oplus, \ominus, [\cdot]}$:

- If the K-orbits are equivariantly simply connected (i.e. no nontrivial K-eqvt local systems exist), then $\mathcal{M}^{\circ}_{LV} = \mathcal{M}^{ss}_{LV}$
- $IC(\text{open}, \text{trivial}) \in \mathcal{M}^{\circ}_{LV}$ always. This is the IC corresponding to the trivial rep'n.
  (saw this in $SL(2,\mathbb{R})$ example.)

Theorem: As right H-modules,

(i) $[\mathcal{M}^{ss}_{LV}]_{\oplus} \simeq M_{LV}$ i.e. $\mathcal{M}^{ss}$ is a categorification of the LV module

(ii) $[\mathcal{M}^{\circ}_{LV}]_{\oplus} \simeq$ block of trivial rep'n in M_{LV} i.e. $\mathcal{M}^{geom}_{LV}$ is a categorification of the principal block of the LV module

③ An algebraic incarnation

You may be wondering: What is the point of defining $\mathcal{M}^\circ_{LV}$ if $\mathcal{M}^{ss}_{LV}$ categorifies M_{LV}?

Answer: It's exactly the piece of $\mathcal{M}^{ss}_{LV}$ that can be studied by taking hypercohomology.

Hypercohomology:

complicated geometric category — its algebraic skeleton

$$\mathbb{H}^\bullet_K(-) := \mathrm{Hom}^\bullet_{D^b_K(G/B)}(k_{G/B}, -) : D^b_K(G/B) \longrightarrow H^\bullet_K(G/B)\text{-mod}$$

- A priori, $\mathbb{H}^\bullet_K(y)$ is a graded vector space. But it gains the structure of a graded $H^\bullet_K(G/B) = \mathrm{Hom}^\bullet_{D^b_K(G/B)}(k_{G/B}, k_{G/B})$-module via precomposition.

(ASSUME: $K = K^\circ$ is the identity component of G^θ.)

- Lemma: $H^\bullet_K(G/B) = P^K \otimes_{R^W} R$, where $P := S(\mathfrak{t}^*_K)$, $P^K := P^{W_K}$, $R = S(\mathfrak{t}^*)$

(Here $T_K \subset K$ max'l torus, $\mathfrak{t}_K = \mathrm{Lie}\, T_K$, W_K = Weyl group of $K \subset W$)

- The quotient $P^K \otimes R \longrightarrow P^K \otimes_{R^W} R$ gives a functor $P^K \otimes_{R^W} R\text{-mod} \hookrightarrow P^K \otimes R\text{-mod}$

- Composing, we have
$$\boxed{\mathbb{H}^\bullet_K(-) : D^b_K(G/B) \longrightarrow P^K\text{-mod-}R}$$
category of graded (P^K, R)-bimodules

The point: $\mathcal{M}^\circ_{LV}$ is exactly the piece of $\mathcal{M}^{ss}_{LV}$ where $\mathbb{H}^\bullet_K(-)$ doesn't vanish!

③ An algebraic incarnation

We can construct an algebraic version of $\mathcal{M}^\circ_{LV}$ inside P^K-mod-R

- Generating objects: $\mathbb{H}^\bullet_K(IC_Q) = H^\bullet_K(Q) = P_w$ "Standard bimodule"
  - $P_w \simeq P_v$ whenever both corresponding T-fixed points are in Q
- The algebraic version of $\mathcal{H}$: Soergel bimodules
  - For $\underline{w} = (s_1, \dots, s_n)$ expression of $w \in W$, define Bott-Samelson bimodule

$$BS(\underline{w}) := R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R\,(\ell(\underline{w})) \in R\text{-mod-}R$$

  - Define

$$\mathbb{S}Bim := \langle BS(\underline{w}) \mid w \in W \rangle_{\oplus,\ominus,(-)} \subset R\text{-mod-}R$$

  - monoidal category via $\otimes_R$, categorifies H: $[\mathbb{S}Bim]_\oplus = H$

Define:

$\left\{\begin{matrix}\text{closed} \\ \text{orbits}\end{matrix}\right\} \longleftrightarrow W_K \backslash W^\theta$

$$\boxed{\mathcal{N}^\circ_{LV} := \langle P_w \otimes_R \mathbb{S}Bim \mid w \text{ coset rep in } W_K \backslash W^\theta \rangle_{\oplus,\ominus,(-)}}$$

our algebraic LV category

③ An algebraic incarnation

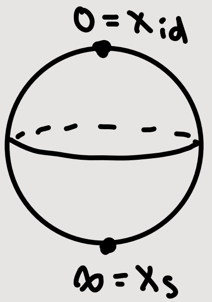
Example: $SL(2,\mathbb{R})$

$$T_K = T = K = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right\} \rightsquigarrow P = R = S(\mathfrak{t}^*) \simeq \mathbb{R}[\overset{\text{deg } 2}{\downarrow}\alpha]$$

polynomial ring in one variable

$$W_K = \mathrm{id} \subset W = \{1, s\} \circlearrowright R \quad \text{and} \quad P^K = R$$

Closed orbits are T-fixed points:



$\rightsquigarrow$ $H_T^\bullet(x_{id}) = R$ as vector spaces

$H_T^\bullet(x_s) = R$ + left R-modules

right R-actions twisted by id, s

$\Rightarrow$ generating R-bimodules:

$$R_{id} \quad , \quad R_s$$

$\mathbb{S}Bim = \langle R, B_s \rangle_{\oplus, \ominus, (-)}$ acts via:

$$R_{id} \underset{R}{\otimes} B_s = R \underset{R}{\otimes} R \underset{R^s}{\otimes} R(1) = R \underset{R^s}{\otimes} R(1) = B_s$$

$$R_s \underset{R}{\otimes} B_s = R_s \underset{R^s}{\otimes} R(1) \simeq R \underset{R^s}{\otimes} R(1) = B_s$$

$\rightsquigarrow$ $\mathcal{N}^\circ_{LV} = \langle R_{id}, R_s, B_s \rangle_{\oplus, \ominus, (-)}$

$$B_s \underset{R}{\otimes} B_s = R \underset{R^s}{\otimes} R \underset{R^s}{\otimes} R(2) \simeq R \underset{R^s}{\otimes} (R^s \overset{\uparrow}{\oplus} R^s(-2)) \underset{R^s}{\otimes} R(2)$$

R splits into s-invariants and s-antiinvariants

$$\simeq \underbrace{B_s(1) \oplus B_s(-1)}_{\downarrow}$$

compare to $m_{triv} \cdot b_s = v\, m_{triv} + v^{-1} m_{triv}$

(Compare to:
$\mathcal{M}^\circ_{LV} = \langle I(0, I(\infty, I(triv \rangle_{\oplus, \ominus, (-)}$)

④ Main result + discussion

Main Theorem (Larson-R., Bezrukavnikov-Vilonen)

$$\mathbb{H}_K^\bullet : \mathcal{M}^\circ_{LV} \longrightarrow \mathcal{N}^\circ_{LV} \quad \text{is an equivalence of categories.}$$

Some Remarks:

- essential surjectivity is basically by construction
- fully faithfulness requires two main tools:

  ① $K \curvearrowright G/B$ satisfies necessary conditions to use techniques of parity sheaves (in the sense of Juteau-Mautner-Williamson) and objects in $\mathcal{M}^{geom}_{LV}$ are parity.

  → gives critical Hom-vanishing results

  → $H^n_K(\mathcal{O};\mathcal{L})=0$ for n odd

  ② localisation to torus fixed points

  - key fact: $G/B^{T_K} = G/B^{T} \longleftrightarrow W$ T-fixed points and T_K-fixed points agree
  - For $\mathcal{G} \in \mathcal{M}^{geom}_{LV}$, $H^\bullet_{T_K}(X;\mathcal{G}) \xhookrightarrow{i^*_{T_K}} H^\bullet_{T_K}(X^{T_K}; i^*_{T_K}\mathcal{G})$ localisation to T_K-fixed points is injective

- If we specialize to $G \overset{\Delta}{\subset} G \times G$, get alternate proof of a classical result of Soergel ("Kategorie $\mathcal{O}$, Perverse Garben, ..." '90) which does not rely on known facts about dimensions of modules in $\mathcal{O}$.

④ Main result + discussion

THE TAKEAWAY:

- This strategy opens up new tools (Soergel bimodules) in the study of real groups.

- The module categories we get are interesting from a Soergel bimodule perspective
  - Recent work of Tubbenhauer + ___ classifies module categories over $\mathbb{S}\mathrm{Bim}$. Where do the LV categories fit into this classification? (Ongoing project w/ Tubbenhauer)
  - Can the module categories be described diagrammatically, as $\mathbb{S}\mathrm{Bim}$ can?
  - Do "exotic" LV modules attached to "exotic" involutions (e.g. non-crystallographic Weyl groups) exist? Can we classify indecomposables w/o geometry?

- Hypercohomology vanishes outside of $\mathcal{M}^{\circ}_{LV}$, so this strategy doesn't work for non-principal blocks of M_{LV}, unfortunately

- In special cases this categorifies one side of Vogan duality. Useful approach to Soergel's conjecture that Koszul duality categorifies Vogan duality?

(See also Bezrukavnikov-Vilonen "Koszul duality for quasi-split real groups.")

THANKS FOR LISTENING!

More details can be found in our paper "A categorification of the Lusztig-Vogan module" arXiv 2203.09007