Dirac inequality for highest weight Harish-Chandra modules

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Notation

- G : a connected simply connected noncompact simple Lie group of Hermitian type
- Θ: a Cartan involution
- $K = G^{\Theta}$: the group of fixed points of Θ
- $\mathfrak{g}_{\mathbf{0}}$: Lie algebra of G with Cartan involution $\theta = d\Theta$.
- $\mathfrak{g}_{\mathbf{0}} = \mathfrak{k}_{\mathbf{0}} \oplus \mathfrak{p}_{\mathbf{0}}$: Cartan decomposition corresponding to θ.

$$\mathfrak{k}_0 = \{x \in \mathfrak{g}_o \,|\, \theta(x) = x\}, \quad \mathfrak{p}_o = \{x \in \mathfrak{g}_o \,|\, \theta(x) = -x\}$$

- $\mathfrak{t}_0$ = common Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{k}_0$

Notation

- $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$: complexifications of Lie algebras $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{t}_0$
- $\Delta_{\mathfrak{k}}^+$: the set of positive compact roots
- $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$: a K–invariant decomposition, $\mathfrak{p}^\pm$ are abelian subalgebras of $\mathfrak{p}$
- ρ: the half-sum of positive roots
- $W_{\mathfrak{k}}$: Weyl group of $(\mathfrak{k}, \mathfrak{t})$
- λ^+: the unique dominant $W_{\mathfrak{k}}$–conjugate of $\lambda \in \mathfrak{t}^*$

- $N(\lambda)$: the generalized Verma module

  $$N(\lambda) = S(\mathfrak{p}^-) \otimes F_\lambda,$$

  where F_λ is the irreducible $\mathfrak{k}$–module with highest weight λ. Here λ is also $\mathfrak{g}$–highest weight of $N(\lambda)$.

- Highest weight modules are generated by a weight vector that is annihilated by the action of all positive root spaces in $\mathfrak{g}$.

- $L(\lambda)$: the irreducible quotient of $N(\lambda)$.

- $L(\lambda)$ is called unitarizable if it is equivalent to the $\mathfrak{g}$–module of $\mathfrak{k}$–finite vectors in a unitary representations of G.

- The $\mathfrak{k}$-finiteness implies that λ must be $\Delta_{\mathfrak{k}}^{+}$–dominant integral and unitarity of $L(\lambda)$ implies that λ is a real weight
- $\lambda \in \mathfrak{t}^*$: $\Delta_{\mathfrak{k}}^{+}$– dominant integral, real
- Goal: determine those $L(\lambda)$ which correspond to unitary irreducible representation of G.
- Harish-Chandra: G admits non-trivial unitary highest weight modules precisely when (G, K) is a Hermitian symmetric pair.
- That is precisely when the Lie algebra is one of the Lie algebras: $\mathfrak{sp}(2n, \mathbb{R})$, $\mathfrak{so}^*(2n)$, $\mathfrak{su}(p, q)$ $p \leq q$, $\mathfrak{so}(2, 2n-2)$, $\mathfrak{so}(2, 2n-1)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$

Dirac operator

- $U(\mathfrak{g})$ = universal enveloping algebra of $\mathfrak{g}$.
- $C(\mathfrak{p})$ = Clifford algebra of $\mathfrak{p}$ with respect to Killing form B.
- Dirac operator $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$

$$D = \sum_i b_i \otimes d_i$$

b_i - basis of $\mathfrak{p}$, d_i - dual basis of $\mathfrak{p}$ with respect to B.

- D is independent of the choice of b_i, K–invariant for adjoint action on both factors.
- S = spin module, $S = \bigwedge \mathfrak{p}^+$

$$D^2 = -(\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho\|^2) + (\mathrm{Cas}_{\mathfrak{k}_\Delta} + \|\rho_{\mathfrak{k}}^2\|)$$

- Necessary condition for unitarity: Dirac inequality

$$D^2 \geq 0$$

- The K-types of $S(\mathfrak{p}^-)$ are very well known. They are called the Schmid modules.

- The general Schmid module s is a nonnegative integer combination of the basic Schmid modules

PRW component

- F$_{\mu}, F_{\nu}$: finite-dimensional -modules with highest weights μ, ν
- What are the K–types of $F_{\mu} \otimes F_{\nu}$???
- Cartan component: $F_{\mu+\nu}$
- Parthasarthy - Ranga Rao - Varadarajan (PRV)

Theorem (Parthasarthy - Ranga Rao - Varadarajan)

Let ν^- be the lowest weight of F_{ν}, and let $\tau = (\mu + \nu^-)^+$. Then F_{τ} appears in $F_{\mu} \otimes F_{\nu}$, with multiplicity one. Moreover, for any F_{σ} appearing in $F_{\mu} \otimes F_{\nu}$,

$$\|\sigma + \rho_{\mathfrak{k}}\|^2 \geq \|\tau + \rho_{\mathfrak{k}}\|^2,$$

with equality attained if and only if $\sigma = \tau$. □

- EHW: complete classification of the unitary highest weight modules using the Dirac inequality, Jantzen's fomula and Howe's theory of dual pairs
- $L(\lambda)$ is unitarizable if and only if for all K–types F_μ in $L(\lambda)$ other than F_λ the strict Dirac inequality holds, that is

$$\|\mu+\rho\|^2 > \|\lambda+\rho\|^2$$

- The same result can be proved more directly using the Dirac inequality in a more substantial way

Dirac inequality for the PRW component

We check the Dirac inequality for the PRW component of F_λ tensored with each K–type of $S(\mathfrak{p}^-)$.

Theorem

(1) Let s_0 be a Schmid module such that the strict Dirac inequality

$$\|(\lambda - s)^+ + \rho\|^2 > \|\lambda + \rho\|^2 \tag{1}$$

holds for any Schmid module s of strictly lower level than s_0, and such that

$$\|(\lambda - s_0)^+ + \rho\|^2 < \|\lambda + \rho\|^2.$$

Then $L(\lambda)$ is not unitary.

(2) If (1) *holds for all Schmid modules s, then $N(\lambda)$ is irreducible and unitary.*

Maximal submodule

Lemma

Let $M(\lambda)$ be the maximal submodule of the generalized Verma module $N(\lambda)$, and let $F_\mu \subset M(\lambda)$ be a highest K-type of $M(\lambda)$, with highest weight μ. Then

$$D^2 = 0 \qquad \text{on} \quad F_\mu \otimes 1 \subset N(\lambda) \otimes S.$$

Equivalently,

$$\|\mu + \rho\|^2 = \|\lambda + \rho\|^2.$$

Proof (1)

- By assumption, the Dirac inequality fails for the K-type $(\lambda - s_0)^+$ of $N(\lambda)$
- We need to show that $(\lambda - s_0)^+$ is a K-type of $L(\lambda)$ and not just of $N(\lambda)$.
- Suppose $(\lambda - s_0)^+$ is a K-type of the maximal submodule $M(\lambda)$ of $N(\lambda)$.
- Then, by previous Lemma, it can not be a highest K-type of $M(\lambda)$.
- Thus it must have strictly lower level than some highest K-type of $M(\lambda)$
- On that highest K-type the Dirac inequality would be an equality, and that is a contradiction with the assumption.
- $(\lambda - s_0)^+$ can not be a K-type of $M(\lambda)$, so it must be a K-type of $L(\lambda)$

Proof (2)

- The assumption implies that

$$\|\mu + \rho\|^2 > \|\lambda + \rho\|^2$$

  for any K-type $\mu \neq \lambda$ of the generalized Verma module $N(\lambda)$.

- By previous Lemma, it follows that $N(\lambda)$ is irreducible.

- Unitarity of $N(\lambda)$ follows from the strict Dirac inequality (EHW)

- The basic Schmid -submodules of $S(\mathfrak{p}^-)$ have lowest weights $-s_i$, where
$$s_i = (\underbrace{2,\dots,2}_{i},0,\dots,0), \qquad i=1,\dots,n.$$
- The highest weight $(\mathfrak{g},K)$–modules have highest weight of the form $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$, where $\lambda_i-\lambda_j\in\mathbb{N}_0,\ i>j$, $\rho=(n,n-1,\dots,2,1)$
- Let $q\le r$ be integers in $[1,n]$ such that
$$\lambda=(\underbrace{\lambda_1,\dots,\lambda_1}_{q},\underbrace{\lambda_1-1,\dots,\lambda_1-1}_{r-q},\lambda_{r+1},\dots,\lambda_n),$$
with $\lambda_1-2\ge\lambda_{r+1}\ge\dots\ge\lambda_n$.

$\mathfrak{sp}(2n, \mathbb{R})$, the basic Dirac inequality

$$(\lambda - s_1)^+ = (\underbrace{\lambda_1, \dots, \lambda_1}_{q-1}, \underbrace{\lambda_1 - 1, \dots, \lambda_1 - 1}_{r-q}, \lambda_1 - 2, \lambda_{r+1}, \dots, \lambda_n)$$
$$= \lambda - (\epsilon_q + \epsilon_r).$$

- The basic necessary condition for unitarity is the Dirac inequality
$$\|(\lambda - s_1)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2.$$
- $\lambda_1 \leq -n + \frac{r+q}{2}$.

The Dirac inequality for $s_i,\ i \in \{2, \ldots, q\}$

- $(\lambda - s_i)^+ = (\underbrace{\lambda_1}_{q-i}, \underbrace{\lambda_1 - 1}_{i}, \underbrace{\lambda_1 - 1}_{r-q-i}, \underbrace{\lambda_1 - 2}_{i}, \lambda_{r+1}, \ldots, \lambda_n)$
- $\lambda = (\underbrace{\lambda_1}_{q-i}, \underbrace{\lambda_1}_{i}, \underbrace{\lambda_1 - 1}_{r-q-i}, \underbrace{\lambda_1 - 1}_{i}, \lambda_{r+1}, \ldots, \lambda_n)$
- $\|(\lambda - s_i)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2$
- $\lambda_1 \leq -n + \frac{r+q-i+1}{2}$

$\mathfrak{sp}(2n,\mathbb{R})$, general Schmid module

Lemma

1. *If for some integer $i \in [1,q]$*

$$\lambda_1 < -n + \frac{r+q-i+1}{2},$$

   then the Dirac inequality holds strictly for any Schmid module $s = (2b_1, \ldots, 2b_n),\ b_j \in \mathbb{Z},\ b_1 \geq \cdots \geq b_n \geq 0$ with at most i nonzero components, i.e.

$$\|(\lambda - s)^+ + \rho\|^2 > \|\lambda + \rho\|^2. \tag{2}$$

2. *If*

$$\lambda_1 < -n + \frac{r+1}{2},$$

   then the Dirac inequality holds strictly for any Schmid module s.

Let $\lambda = (\underbrace{\lambda_1,\dots,\lambda_1}_{q},\underbrace{\lambda_1-1,\dots,\lambda_1-1}_{r-q},\lambda_{r+1},\dots,\lambda_n)$. Then:

1. If
$$\lambda_1 > -n + \frac{r+q}{2},$$
then $L(\lambda)$ is not unitary.
2. If for some integer $i \in [1, q-1]$
$$-n + \frac{r+q-i}{2} < \lambda_1 < -n + \frac{r+q-i+1}{2},$$
then $L(\lambda)$ is not unitary.
3. If
$$\lambda_1 < -n + \frac{r+1}{2},$$
then $L(\lambda) = N(\lambda)$ and it is unitary.

- We consider roots as 8–tuples which have the same sixth and seventh coordinate, and the eight coordinate is equal to minus the sixth coordinate.
- The positive compact roots are

$$\epsilon_i \pm \epsilon_j, \quad 5 \geq i > j.$$

and the positive noncompact roots are

$$\frac{1}{2}\left(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^{5}(-1)^{n(i)}\epsilon_i\right), \quad \sum_{i=1}^{5} n(i) \text{ even.}$$

- In this case

$$\rho = (0, 1, 2, 3, 4, -4, -4, 4).$$

- The set of simple roots is

$$\begin{aligned}\Pi &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \\ &= \{\frac{1}{2}\left(\epsilon_8 - \epsilon_7 - \epsilon_6 - \epsilon_5 - \epsilon_4 - \epsilon_3 - \epsilon_2 + \epsilon_1\right), \\ &\epsilon_2 + \epsilon_1, \epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2, \epsilon_4 - \epsilon_3, \epsilon_5 - \epsilon_4\}.\end{aligned}$$

- The numbering of simple roots in the Dynkin diagram is given by

$$\begin{pmatrix} 13456 \\ 2 \end{pmatrix}.$$

$\mathfrak{e}_6$, strongly orthogonal roots

- The strongly orthogonal non-compact positive roots are

$$\begin{aligned}\beta_1 &= \binom{12321}{2} = \alpha_1 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_2 \\ &= \frac{1}{2}\left(1,1,1,1,1,-1,-1,1\right), \\ \beta_2 &= \binom{11111}{0} = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\ &= \frac{1}{2}\left(-1,-1,-1,-1,1,-1,-1,1\right).\end{aligned}$$

The basic Schmid $\mathfrak{k}$–modules in $S(\mathfrak{p}^-)$ have lowest weight $-s_i$, $i = 1, 2$, where

$$s_1 = \beta_1 = \frac{1}{2}\left(1, 1, 1, 1, 1, -1, -1, 1\right),$$
$$s_2 = \beta_1 + \beta_2 = \left(0, 0, 0, 0, 1, -1, -1, 1\right).$$

The highest weight $(\mathfrak{g}, K)$–modules have highest weights of the form

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_6, -\lambda_6), \quad |\lambda_1| \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5,$$
$$\lambda_i - \lambda_j \in \mathbb{Z},\ 2\lambda_i \in \mathbb{Z}, \quad i, j \in \{1, 2, 3, 4, 5\}$$

In this case

$$\rho = (0, 1, 2, 3, 4, -4, -4, 4).$$

The Dirac inequality for s_1 and s_2

- $||(\lambda - s_i)^+ + \rho||^2 \geq ||\lambda + \rho||^2, \quad i \in \{1, 2\}$
- **Case 1:** $\lambda_i = 0, \quad i \in \{1, 2, 3, 4, 5\}$.
  The basic Dirac inequality can be written as
  $$\lambda_6 \geq 0.$$
  The Dirac inequality for the second basic Schmid module is equivalent to
  $$\lambda_6 \geq 2.$$

The Dirac inequality for s_1 and s_2

- **Case 2:** $\lambda_i = 0, \quad i \in \{1,2,3,4\}, \quad \lambda_5 \neq 0$.
  In this case the basic Dirac inequality can be written as

  $$\lambda_5 + 8 \leq 3\lambda_6.$$

  The Dirac inequality for the second basic Schmid module is equivalent to

  $$\lambda_5 + 14 \leq 3\lambda_6.$$

- **Case 3:** $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \neq (0,0,0,0)$. The Dirac inequality for the second basic Schmid module is automatically satisfied if the basic Dirac inequality holds.

Theorem

(Case 1) Let λ be the highest weight of the form $\lambda = (0,0,0,0,0,\lambda_6,\lambda_6,-\lambda_6)$.

1. *If $\lambda_6 > 2$ then λ satisfies the strict Dirac inequality*

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 > \|\lambda + \rho\|^2 \quad \forall a,b \in \mathbb{N}_0, a+b \neq 0.$$

2. *If $0 < \lambda_6 < 2$ then*

$$\|(\lambda - s_2)^+ + \rho\|^2 < \|\lambda + \rho\|^2$$

   and the strict Dirac inequality holds for any Schmid module of strictly lower level than s_2.

3. *If $\lambda_6 < 0$ than the basic Dirac inequality fails.*

Theorem

Let λ *be the highest weight of the form* $\lambda = (0,0,0,0,\lambda_5,\lambda_6,\lambda_6,-\lambda_6)$

1. *If* $3\lambda_6 - \lambda_5 > 14$ *than* λ *satisfies the strict Dirac inequality*

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 > \|\lambda + \rho\|^2 \quad \forall a,b \in \mathbb{N}_0, a+b \neq 0.$$

2. *If* $8 < 3\lambda_6 - \lambda_5 < 14$ *then*

$$\|(\lambda - s_2)^+ + \rho\|^2 < \|\lambda + \rho\|^2$$

   and the strict Dirac inequality holds for any Schmid module of strictly lower level than s_2.

3. *If* $3\lambda_6 - \lambda_5 < 8$ *than the basic Dirac inequality fails.*

Theorem

(Case 3) Let λ be the highest weight as in Case 3, i.e., $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \neq (0,0,0,0)$ such that strict basic Dirac inequality holds. Then

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a, b \in \mathbb{N}_0, (a,b) \neq (0,0).$$

Thank you for your attention!