

# Hecke module structure on the orbits in double flag varieties

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## Abstract

Let  $G$  be a connected reductive algebraic group and  $K$  its symmetric subgroup. We consider a double flag variety of finite type  $\mathfrak{X} = K/B_K \times G/P$ , where  $B_K$  is a Borel subgroup of  $K$ , and  $P$  a parabolic subgroup of  $G$ . The orbit space  $\mathbb{C}\mathfrak{X}/K$  enjoys a natural Hecke module structure for the Hecke algebra  $\mathcal{H} = \mathcal{H}(K, B_K)$  of  $K$ . However, it is difficult to find out its explicit Hecke module structure. In this talk, we consider the double flag variety of type AIII, i.e., when  $G/K = \mathrm{GL}_n/\mathrm{GL}_p \times \mathrm{GL}_q$  ( $n = p + q$ ), and give an explicit action of  $\mathcal{H}$  on  $\mathbb{C}\mathfrak{X}/K$  in combinatorial way using graphs. The talk is based on the on-going joint work with Lucas Fresse in Université de Lorraine, IECL (France): [arXiv:2206.10476](https://arxiv.org/abs/2206.10476).

# Introduction

We focus on a double flag variety  $\mathfrak{X} = K/B_K \times G/P$  of finite type:

- Geometry of orbits (dimensions, closure relations, **conormal variety** etc.)
- Combinatorics based on the graphs, Young tableaux, and **RSK corresp**
- Representations of **Hecke algebras**, Weyl groups, & **Springer-Steinberg theory**

Complete results on the double flag variety of type AIII, or

$$\mathfrak{X} = \left( \mathcal{F}l(\mathbb{C}^p) \times \mathcal{F}l(\mathbb{C}^q) \right) \times \text{Gr}_r(\mathbb{C}^{p+q}) \quad \leftarrow K = \text{GL}_p \times \text{GL}_q$$

based on the (on-going) joint w with Lucas Fresse (IECL, Univ. Lorraine, France)

- Lucas Fresse and Kyo Nishiyama, *Action of Hecke algebra on the double flag variety of type AIII*, arXiv:2206.10476 [math.RT].
- —, *A Generalization of Steinberg Theory and an Exotic Moment Map*, International Mathematics Research Notices (2020), rnaa080.
- —, *Orbit embedding for double flag varieties and Steinberg map*, Contemp. Math. **768** (2021), 21–42.
- —, *On generalized Steinberg theory for type AIII*, 2021, arXiv:2103.08460.

# Double flag varieties and $K$ -orbits

Consider:

- $G$  : connected reductive algebraic group
- $K$  : symmetric subgroup
- $P \subset G$ ,  $Q \subset K$  : parabolic subgroups (psg)

The **double flag variety**  $\mathfrak{X}$  is introduced in (N-Ochiai 2011 [8])

$$K \curvearrowright \mathfrak{X} = K/Q \times G/P$$

In general  $\#\mathfrak{X}/K = \infty$ , but

$\exists$  interesting cases where  $\#\mathfrak{X}/K < \infty$ , called **finite type** (We will assume this)

even  $\exists$  classification (He-N-Ochiai-Oshima 2013 [5]) of  $\mathfrak{X}$  of finite type  
(when  $P = B_G$  or  $Q = B_K$ , i.e., one of them is a **Borel** subgrp)

Note that  $\mathfrak{X}/K \simeq Q \backslash G/P$  (preserving closure relations)

# Double flag variety of type AIII I

Take  $\mathbb{C}$  as a base field (for convenience)

In this talk, we will concentrate on the case of **symmetric sp of type AIII**:

- $G = \mathrm{GL}_n$  : general linear group
- $K = \mathrm{GL}_p \times \mathrm{GL}_q$  : block diagonal subgrp of  $G$  ( $p + q = n$ )
- $P = P_{(r, n-r)}$  : max psg in  $G$  (with 2 diag blocks of size  $r$  &  $n - r$ )
- $Q = B_K = B_p \times B_q$  : Borel subgrp in  $K$

So that

$$\mathfrak{X} = \mathrm{GL}_p/B_p \times \mathrm{GL}_q/B_q \times \mathrm{GL}_n/P_{(r, n-r)}$$

$$\simeq (\mathcal{F}l(V^+) \times \mathcal{F}l(V^-)) \times \mathrm{Gr}_r(V), \quad \text{where}$$

- $V = V^+ \oplus V^-$  ( $V^+ = \mathbb{C}^p$ ,  $V^- = \mathbb{C}^q$ ) polar decomposition
- $\mathrm{Gr}_r(V)$  : **Grassmannian** of  $r$ -dim subsp's of  $V$
- $\mathcal{F}l(V^\pm)$  : **complete flag varieties**

# Double flag variety of type AIII II

## Lemma

$\#\mathfrak{X}/K < \infty$ , i.e.,  $\mathfrak{X}$  is of *finite type*

Write  $X = \text{Gr}_r(V) = G/P_{(r, n-r)} \rightsquigarrow K \curvearrowright X$ : **spherical** (i.e.,  $\#X/B_K < \infty$ )

## Lemma

$\mathfrak{X} = (\mathcal{F}l(V^+) \times \mathcal{F}l(V^-)) \times \text{Gr}_r(V)$  &  $X = \text{Gr}_r(V)$

$$\begin{array}{ccc} \mathfrak{X}/K & \xrightarrow{\simeq} & X/B_K \\ \psi & & \psi \\ K \cdot (\mathcal{F}_0^+, \mathcal{F}_0^-, [\tau]) & \longmapsto & B_K \cdot [\tau] \end{array}$$

where  $[\tau] \in \text{Gr}_r(V)$  &  $\mathcal{F}_0^\pm$ : standard flags stabilized by  $B_p$  or  $B_q$

We will often identify  $\mathfrak{X}/K \simeq X/B_K$

Let us describe what is the representative  $\{\tau\}$

# Description of $K$ orbits on $\mathfrak{X}$

**Partial permutation:**  $\tau_1 \in \mathfrak{T}_{p,r} \subset M_{p,r}$

with entries of 0 or 1, in which  $\#1 \leq 1$  in  $\forall$  rows and columns

$$\mathfrak{T} = \mathfrak{T}_{(p,q),r} := \left\{ \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathfrak{T}_{p,r} \times \mathfrak{T}_{q,r} \mid \text{rank } \tau = r \right\} \subset M_{p+q,r}$$

: pairs of partial permutations of full rank

$\mathfrak{T} \curvearrowright S_r \quad \rightsquigarrow \quad \overline{\mathfrak{T}} = \mathfrak{T}/S_r$  : **quotient** by the symmetric group action

Denote  $[\tau] := \text{Im } \tau \in \text{Gr}_r(V)$ :  $r$ -dim subsp generated by column vectors of  $\tau$

## Lemma

$\mathfrak{T} \ni \tau \mapsto [\tau] \in \text{Gr}_r(V)$  factors through to

$$\overline{\mathfrak{T}} = \mathfrak{T}/S_r \xrightarrow{\simeq} X/B_K \simeq \mathfrak{X}/K$$

so that we get  $\overline{\mathfrak{T}} \simeq \mathfrak{X}/K$

$\exists$  convenient presentation by graphs...

# Graphs I

Represent  $\tau \in \overline{\mathfrak{X}}$  by a **graph**  $\Gamma(\tau)$ :

- **signed Vertices**: **positive** vertices  $\mathcal{V}_p^+ = \{1^+, \dots, p^+\}$  & **negative** vertices  $\mathcal{V}_q^- = \{1^-, \dots, q^-\}$
- **Edges** between  $i^+ \in \mathcal{V}_p^+$  and  $j^- \in \mathcal{V}_q^-$  if  $\tau$  contains **two** 1's at the positions  $i^+$  &  $j^-$  **in the same column**
- **Marking** at the vertex  $i^+$  or  $j^-$ , with **only one** 1 at  $i^+$  or  $j^-$  in a column
- $\#(\text{Edges}) + \#(\text{Marked points}) = r$

Example  $((p, q) = (5, 3)$  and  $r = 4$ )

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \left( \begin{array}{cccc} \mathbf{e}_2^+ & \mathbf{e}_4^+ & \mathbf{e}_5^+ & 0 \\ \mathbf{e}_3^- & \mathbf{e}_1^- & 0 & \mathbf{e}_2^- \end{array} \right) \rightsquigarrow \Gamma(\tau) =$$

The graph  $\Gamma(\tau)$  consists of 5 positive vertices labeled  $1^+, 2^+, 3^+, 4^+, 5^+$  and 3 negative vertices labeled  $1^-, 2^-, 3^-$ . Edges connect  $2^+$  to  $3^-$ ,  $4^+$  to  $1^-$ , and  $5^+$  to  $2^-$ . There are marked points (circles with a dot) at  $2^-$  and  $5^+$ .



# Graphs II

## Lemma

- $\mathcal{G}((p, q), r)$ : graphs with vertices  $\mathcal{V}_p^+ \cup \mathcal{V}_q^-$  & exactly  $r$  edges & **marked points**
- $\overline{\mathfrak{T}} = \mathfrak{T}/S_r$  : partial permutations

$\rightsquigarrow$  The graphs classify  $K$  orbits in  $\mathfrak{X}$

$$\begin{array}{ccccc}
 \mathfrak{X}/K \simeq X/B_K & \xleftarrow{\simeq} & \overline{\mathfrak{T}} & \xrightarrow{\simeq} & \mathcal{G}((p, q), r) \\
 \psi & & \psi & & \psi \\
 B_K \cdot [\tau] & \xleftarrow{\quad} & |\tau| & \xrightarrow{\quad} & \Gamma(\tau)
 \end{array}$$

# Orbital invariants: $a^\pm(\tau)$ , $b(\tau)$ , $c(\tau)$ & $R(\tau) = (r_{i,j}(\tau))$

For the graph  $\Gamma(\tau)$  we define:

- **degree** of vertices:  $\deg i^\pm := 0, 1, 2$  (NO [edges/marks], **edges**, **marked**)
  - $a^\pm(\tau) := \#\{(i^\pm, j^\pm) \mid i < j \text{ \& } \deg(i^\pm) < \deg(j^\pm)\}$
  - $b(\tau) := \#\{\text{edges}\}$        $c(\tau) := \#\{\text{crossings}\}$
  - $r_{i,j}(\tau) := \#\text{Edges} + \#\text{Marks}$   
     within vertices among  $1^+ \leq k^+ \leq i^+ \text{ \& } 1^- \leq \ell^- \leq j^-$
- $R(\tau) := (r_{i,j}(\tau))_{0 \leq i \leq p, 0 \leq j \leq q} \in M_{p+1, q+1}$  : the **rank** matrix

- decomposition  $\mathcal{V}_p^+ = \{1, \dots, p\} = I \sqcup L \sqcup L'$ :  
      $\{i \in \{1, \dots, p\} \mid i^+ \text{ is a vertex of degree } 1 \text{ (resp. } 2, 0)\}$
- Similar decomp  $\mathcal{V}_q^- = \{1, \dots, q\} = J \sqcup M \sqcup M'$ :  
      $\{j \in \{1, \dots, q\} \mid j^- \text{ is a vertex of degree } 1 \text{ (resp. } 2, 0)\}$
- $\sigma : J \rightarrow I$ : **bijection** defined by  $\sigma(j) = i$  if  $(i^+, j^-)$  is an edge in  $\Gamma(\tau)$ .

## Example

Let  $\tau$  be as in (4):

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_2^+ & \mathbf{e}_4^+ & \mathbf{e}_5^+ & 0 \\ \mathbf{e}_3^- & \mathbf{e}_1^- & 0 & \mathbf{e}_2^- \end{pmatrix} \rightsquigarrow \Gamma(\tau) =$$

$$a^+(\tau) = 7, \quad a^-(\tau) = 1, \quad b(\tau) = 2, \quad c(\tau) = 1, \quad R(\tau) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$I = \{2, 4\}, \quad L = \{5\}, \quad L' = \{1, 3\},$$

$$J = \{1, 3\}, \quad M = \{2\}, \quad M' = \emptyset, \quad \sigma = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \in \text{Bij}(J, I).$$

# Dimensions and closure relations of Orbits

Recall the based point  $(\mathcal{F}_0^+, \mathcal{F}_0^-, [\tau])$  in  $\mathfrak{X} = (\mathcal{Fl}(V^+) \times \mathcal{Fl}(V^-)) \times \text{Gr}_r(V)$

## Theorem

Denote a  $K$  orbit in  $\mathfrak{X}$  by  $\mathbb{O}_\tau := K \cdot (\mathcal{F}_0^+, \mathcal{F}_0^-, [\tau])$

$$\textcircled{1} \dim \mathbb{O}_\tau = \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + a^+(\tau) + a^-(\tau) + \frac{b(\tau)(b(\tau)+1)}{2} + c(\tau)$$

$$\textcircled{2} \mathbb{O}_\tau = \{(\mathcal{F}^+, \mathcal{F}^-, W) \mid \dim W \cap (\mathcal{F}_i^+ + \mathcal{F}_j^-) = r_{i,j}(\tau)$$

$$\forall (i, j) \in \{0, \dots, p\} \times \{0, \dots, q\}\}$$

$$\textcircled{3} \overline{\mathbb{O}_\tau} \subset \overline{\mathbb{O}_{\tau'}} \iff r_{i,j}(\tau) \geq r_{i,j}(\tau') \quad \forall (i, j) \in \{0, \dots, p\} \times \{0, \dots, q\}$$

Can describe the **cover relation** of closure of orbits, **which is not given today**

See our recent preprint Fresse-N [arXiv:2103.08460](https://arxiv.org/abs/2103.08460) [3] for details.

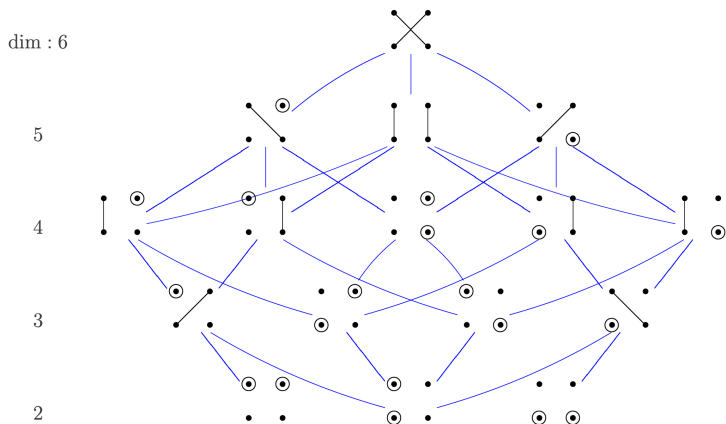
However, we notice

## Corollary

$$\mathbb{O}_{\tau'} \text{ covers } \mathbb{O}_\tau \implies \dim \mathbb{O}_{\tau'} = \dim \mathbb{O}_\tau + 1$$

# Example: Hasse Diagram

Figure: Closure relations of  $K$  orbits for  $p = q = r = 2$



# Hecke algebra action

$\exists$  standard way to define **Hecke algebra action** of  $\mathcal{H} = \mathcal{H}(K, B_K)$  on the DFV  $\mathfrak{X}$  by the **convolution product** (cf. Chriss-Ginzburg [1, § 2.7]):

$$\begin{array}{ccc}
 & K/B_K \times K/B_K \times G/P & \\
 & \swarrow p_{12} & \searrow p_{23} \\
 K/B_K \times K/B_K & & K/B_K \times G/P = \mathfrak{X}
 \end{array}$$

However, we prefer a simpler picture

$$\begin{array}{ccc}
 & K \times_{B_K} G/P & \\
 & \swarrow p_1 & \searrow p_2 \\
 K/B_K & & G/P = X
 \end{array}$$

More generally, if  $X$  is a spherical  $K$ -variety, Hecke algebra/Weyl group actions are considered by Mars-Springer [7] and Knop [6].

# Orbit multiplications

As in the above,  $\tau \in \mathfrak{T}$  is identified with a **graph** with vertices  $\mathcal{V}_p^+ \sqcup \mathcal{V}_q^-$  which are equipped with several markings and edges.

## Lemma

A double coset  $B_K s_i B_K$  generates at most two  $B_K$  orbits on the Grassmannian  $X = \text{Gr}_r(\mathbb{F}^n)$ . Namely we have

$$B_K s_i B_K \cdot \tau = \begin{cases} B_K s_i \tau = B_K \tau & \text{if } s_i \tau = \tau & \text{case (I)} \\ B_K s_i \tau \cup B_K \tau & \text{if } s_i \tau \neq \tau \text{ and } \tau \text{ is in case (II)} \\ B_K s_i \tau & \text{if } s_i \tau \neq \tau \text{ and } \tau \text{ is in case (III)} \end{cases}$$

where

- case (I):  $i, i + 1$  are both of degree 0 (isolated) or both of degree 2 (marked)
- case (II):  $\deg_\tau(i) < \deg_\tau(i + 1)$  or  $i, i + 1$  are end points of edges with crossing
- case (III):  $\deg_\tau(i) > \deg_\tau(i + 1)$  or  $i, i + 1$  are end points of edges without crossing

# Hecke algebra actions

## Theorem

The action of the generators in  $\mathcal{H} = \mathcal{H}(K, B_K)$  is given by ( $\mathbf{q}$ : indeterminate)

$$T_i * \xi_\tau = \begin{cases} \mathbf{q} \xi_\tau & (s_i \tau = \tau) & \text{Case (I)} \\ (\mathbf{q} - 1) \xi_\tau + \mathbf{q} \xi_{s_i \tau} & (s_i \tau \neq \tau) & \text{Case (II)} \\ \xi_{s_i \tau} & (s_i \tau \neq \tau) & \text{Case (III)} \end{cases}$$

they satisfy the Hecke algebra relations:

$$(T_s + 1)(T_s - \mathbf{q}) = 0 \quad (3.1)$$

$$T_{ww'} = T_w T_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w') \quad (3.2)$$

If we specialize  $\mathbf{q} = 1$ , then the Hecke alg rep goes down to that of the Weyl group  $W_K = S_p \times S_q$ . It's just a permutation on the vertices of the graphs. Thus we get



# Weyl group representations

## Corollary

The representation of  $W_K = S_p \times S_q$  on  $\mathbb{C}[\mathfrak{X}/K]$  is isomorphic to

$$\bigoplus_{(k,s,t)} \text{Ind}_{H_{k,s,t}}^{S_p \times S_q} \mathbf{1}$$

where  $(k, s, t)$  moves over

$$p \geq k + s, \quad q \geq k + t, \quad r = k + s + t$$

and

$$H_{k,s,t} \simeq \Delta S_k \times S_s \times S_{s'} \times S_t \times S_{t'},$$

with  $s' = p - (k + s)$  &  $t' = q - (k + t)$ .

We can describe the basis of the representation by pairs of special Young tableaux through **generalized/exotic Steinberg map**

(Fresse-N, 2020 IMRN and Contemp. Math. [2, 4]).

↪ Springer type theorem associated to nilpotent orbits

# Generalized/Exotic Steinberg theory I

Recall the **double flag variety** (in a broader context)

$$\mathfrak{X} = \mathcal{B}_K \times \mathcal{P}_G, \quad \mathcal{B}_K = K/B_K, \quad \mathcal{P}_G = G/P$$

Identify  $\mathcal{P}_G = G/P$  with the set of parabolic subalgs  $\mathfrak{p}_1 \overset{\text{conj}}{\sim} \mathfrak{p} = \text{Lie}(P)$

Then the **cotangent bundle over  $\mathcal{P}_G$**  :

$$T^*\mathcal{P}_G = \{(\mathfrak{p}_1, x) \mid \mathfrak{p}_1 \in \mathcal{P}_G, x \in \mathfrak{u}_{\mathfrak{p}_1}\} \simeq G \times_P \mathfrak{u}_{\mathfrak{p}},$$

where  $\mathfrak{u}_{\mathfrak{p}} = \text{nilradical}(\mathfrak{p})$

The **moment map**:

$$\begin{array}{ccc} \mu_{\mathcal{P}_G} : T^*\mathcal{P}_G & \longrightarrow & \mathcal{N}_{\mathfrak{g}} = (\text{nilpotent variety}) \quad (2\text{nd proj}) \\ \downarrow \Psi & & \downarrow \Psi \\ (\mathfrak{p}_1, x) & \longmapsto & x \end{array}$$

wrt a standard symplectic structure on  $T^*\mathcal{P}_G$ .

Similarly, we have the moment map

$$\begin{aligned} \mu_{\mathcal{B}_K} : T^*\mathcal{B}_K = \{(\mathfrak{q}_1, y) \mid \mathfrak{q}_1 \in \mathcal{B}_K, y \in \mathfrak{u}_{\mathfrak{q}_1}\} &\rightarrow \mathcal{N}_{\mathfrak{k}} = (\text{nilpotent var for symm sp}), \\ \mu_{\mathcal{B}_K}(\mathfrak{q}_1, y) &= y \end{aligned}$$

# Generalized/Exotic Steinberg theory II

## Definition

$\mathcal{Y} := T^*\mathcal{B}_K \times_{\mathcal{N}_{\mathfrak{k}}} T^*\mathcal{P}_G$  : fiber product over **nilpotent var**  $\mathcal{N}_{\mathfrak{k}}$ :

$$\begin{array}{ccc}
 \mathcal{Y} = T^*\mathcal{B}_K \times_{\mathcal{N}_{\mathfrak{k}}} T^*\mathcal{P}_G & \xrightarrow{p_1} & T^*\mathcal{P}_G \ni (\mathfrak{p}_1, x) \\
 \downarrow p_2 & \searrow \varphi^\theta & \downarrow \mu_{\mathcal{P}_G} \\
 & & \mathcal{N}_{\mathfrak{g}} \ni x \\
 & & \downarrow \text{pr}_{\mathfrak{k}} \\
 (q_1, y) \in T^*\mathcal{B}_K & \xrightarrow{-\mu_{\mathcal{B}_K}} & \mathcal{N}_{\mathfrak{k}} \ni -y = x^\theta
 \end{array}$$

We call  $\mathcal{Y} = \mathcal{Y}_{\mathfrak{x}}$  the **conormal variety** for the double flag variety  $\mathfrak{X}$ .

# Generalized/Exotic Steinberg theory III

## Fact

- ① Let  $\mu_{\mathfrak{X}} : T^*\mathfrak{X} \rightarrow \mathfrak{k}$  be the moment map on  $T^*\mathfrak{X}$ .  
Then  $\mathcal{Y} \simeq \mu_{\mathfrak{X}}^{-1}(0)$  is the **null fiber**.
- ②  $\mathcal{Y}$  is a disjoint union of the conormal bundles:  $\mathcal{Y} = \coprod_{\mathbb{O} \in \mathfrak{X}/K} T_{\mathbb{O}}^*\mathfrak{X}$
- ③  $\mathcal{Y} = \bigcup_{\mathbb{O} \in \mathfrak{X}/K} \overline{T_{\mathbb{O}}^*\mathfrak{X}}$  gives the **irred decomp**

the diagonal map in the fiber product:  $\varphi^\theta : \mathcal{Y} \rightarrow \mathcal{N}_{\mathfrak{k}}$ :

$$\varphi^\theta((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) = x^\theta = -y \quad \text{for } ((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) \in \mathcal{Y}.$$

we need another map

$$\varphi^{-\theta}((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) = x^{-\theta} = x + y \quad \text{for } ((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) \in \mathcal{Y}.$$

- $\varphi^\theta$  the **generalized Steinberg map** and  $\text{Im } \varphi^\theta \subset \mathcal{N}_{\mathfrak{k}}$
- $\varphi^{-\theta}$  the **exotic Steinberg map** and  $\text{Im } \varphi^{-\theta} \subset \mathcal{N}_{\mathfrak{s}}$

## Generalized/Exotic Steinberg theory IV

$\pi : T^*\mathfrak{X} \rightarrow \mathfrak{X} : \text{bundle map} \rightsquigarrow K\text{-equiv double fibration}$

$$\begin{array}{ccc}
 \mathcal{Y} = T^*\mathcal{B}_K \times_{\mathcal{N}^\theta} T^*\mathcal{P}_G & & \\
 \pi \swarrow & & \searrow \varphi^{\pm\theta} \\
 \mathfrak{X} = \mathcal{B}_K \times \mathcal{P}_G & & \mathcal{N}^{\pm\theta}
 \end{array}$$

Here  $\mathcal{N}^\theta = \mathcal{N}_\mathfrak{k}$ ,  $\mathcal{N}^{-\theta} = \mathcal{N}_\mathfrak{s}$ . Using this diagram, we define orbit maps

$$\begin{array}{ccc}
 \Phi^{\pm\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}^{\pm\theta}/K & \text{by} & \overline{\varphi^{\pm\theta}(\pi^{-1}(\mathbb{O}))} = \overline{\mathcal{O}}, \\
 \psi \downarrow & & \downarrow \psi \\
 \mathbb{O} \longmapsto & \longrightarrow & \mathcal{O}
 \end{array}$$

where  $\mathbb{O}$  is a  $K$ -orbit in  $\mathfrak{X}$  and  $\mathcal{O}$  is a nilpotent  $K$ -orbit in  $\mathcal{N}^{\pm\theta}$ .

$$\text{Since } \pi^{-1}(\mathbb{O}) = T_0^*\mathfrak{X}, \quad \Phi^{\pm\theta}(\mathbb{O}) = \mathcal{O} \iff \overline{\varphi^{\pm\theta}(T_0^*\mathfrak{X})} = \overline{\mathcal{O}}.$$

Thus  $\Phi^{\pm\theta}$  is a map from irred compnts of  $\mathcal{Y}$  to nilpotent  $K$  orbits (in  $\mathfrak{k}$  or  $\mathfrak{s}$ ).

# Generalized/Exotic Steinberg theory V

$\pi : T^*\mathfrak{X} \rightarrow \mathfrak{X} : \text{bundle map} \rightsquigarrow K\text{-equiv double fibration}$

$$\begin{array}{ccc}
 & \mathcal{Y} = T^*\mathcal{B}_K \times_{\mathcal{N}^\theta} T^*\mathcal{P}_G & \\
 \pi \swarrow & & \searrow \varphi^{\pm\theta} \\
 \mathfrak{X} = \mathcal{B}_K \times \mathcal{P}_G & & \mathcal{N}^{\pm\theta}
 \end{array}$$

Here  $\mathcal{N}^\theta = \mathcal{N}_\mathfrak{k}$ ,  $\mathcal{N}^{-\theta} = \mathcal{N}_\mathfrak{s}$ . Using this diagram, we define orbit maps

$$\begin{array}{ccc}
 \Phi^{\pm\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}^{\pm\theta}/K & \text{by} & \overline{\varphi^{\pm\theta}(\pi^{-1}(\mathbb{O}))} = \overline{\mathcal{O}}, \\
 \downarrow \psi & & \downarrow \psi \\
 \mathbb{O} \longmapsto & \longrightarrow & \mathcal{O}
 \end{array}$$

where  $\mathbb{O}$  is a  $K$ -orbit in  $\mathfrak{X}$  and  $\mathcal{O}$  is a nilpotent  $K$ -orbit in  $\mathcal{N}^{\pm\theta}$ .

- $\Phi^\theta$  called the **generalized Steinberg map** and
- $\Phi^{-\theta}$  called the **exotic Steinberg map**.

# Generalized RS correspondence for type AIII

Recall  $\overline{\mathfrak{S}} = \mathfrak{S}/S_r \simeq \mathfrak{X}/K$  : pairs of partial permutations classifying  $K$  orbits on  $\mathfrak{X}$ .

$\exists$  generalized RS correspondence to pairs of standard tableaux **with decorations**.

## Notation

Define  $\lambda' \subset \lambda \iff \lambda' \subset \lambda$  & the skew tableau  $\lambda/\lambda'$  is *column strip*  
 $\mathcal{P}(n) := \{\lambda \vdash n\}$ : the set of partitions of  $n$

## Theorem (gen RS correspondence)

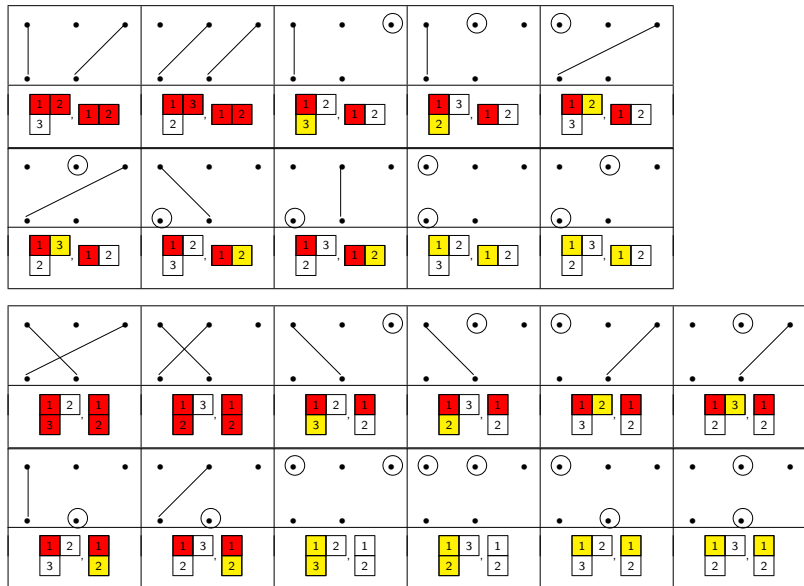
$\exists$  combinatorial bijection between the pairs of partial permutations and pairs of decorated standard tableaux:

$$\overline{\mathfrak{S}} \xrightarrow{\cong} \bigsqcup_{(\lambda, \mu) \in \mathcal{P}(p) \times \mathcal{P}(q)} \mathcal{T}_{\lambda, \mu}$$

where  $\mathcal{T}_{\lambda, \mu} = \{(T_1, T_2; \lambda', \mu'; \nu) \text{ satisfying } (*) \text{ \& } (**) \text{ below}\}$

(\*)  $(T_1, T_2) \in \text{STab}(\lambda) \times \text{STab}(\mu)$

(\*\*)  $\nu \subset \lambda' \subset \lambda, \nu \subset \mu' \subset \mu$  &  $|\lambda'| + |\mu'| = |\nu| + r$ .

Examples of gen RSK corr:  $(p, q, r) = (3, 2, 2)$  ( $[3]$ )









Get a commutative diagram

$$\begin{array}{ccc}
 \mathfrak{X}/K \simeq \mathfrak{T}_{(p,q;r)}/S_r & \xrightarrow[\text{genRS}]{\sim} & \coprod_{(\lambda,\mu) \in \mathcal{P}(p) \times \mathcal{P}(q)} \mathcal{T}_{\lambda,\mu} \ni (T_1, T_2; \lambda', \mu'; \nu) \\
 & \searrow \Phi^\theta & \downarrow \\
 & & \mathcal{P}(p) \times \mathcal{P}(q) \ni (\lambda, \mu) \\
 & & \downarrow \\
 & & (\lambda, \mu)
 \end{array}$$

This diagram actually commutes with the Weyl group representations i.e.,

the fiber of a nilpotent orbit  $\mathcal{O}_{(\lambda,\mu)} \subset \mathcal{N}_{\mathfrak{k}}$  inherits a structure of Weyl group representations (**Springer correspondence**)

However, we don't know a rigorous geometric reason of this phenomenon

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