Functoriality via Dirac cohomology

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Representation Theory XVII

Dubrovnik, Croatia

October 2-8, 2022

Outline

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- Functorial transfer
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Langlands functoriality conjecture

Let G', G be reductive groups defined over a global field k, and

$$\eta : {}^L G' \to {}^L G$$

a L-homomorphism between their L-groups.

Then there is a way to attach each automorphic representations

$$\pi' = \otimes_\nu \pi'_\nu \text{ of } G'$$

to a packet of automorphic representations

$$\pi = \otimes_\nu \pi_\nu \text{ of } G.$$

The automorphic functoriality is expected to be of local nature.

Functorial transfer

Suppose there are elliptic maximal tori

$$T'_{ell} \subseteq G' \text{ and } T_{ell} \subseteq G$$

with admissible embedding

$$^L T'_{ell} \subseteq {^L G'} \text{ and } {^L T_{ell}} \subseteq {^L G}$$

such that $\eta(^L T'_{ell}) \subseteq {^L T_{ell}}$.

Then we define the functorial transfer

$$\mathbf{Rep}\, G'(k_\nu) \to \mathbf{Rep}\, G(k_\nu)$$

by using induction from elliptic representations (in real case, with nonzero Dirac index).

Characters

Assume $G(F)$ is a real or p-adic group.

The character of π is a distribution

$$\Theta_\pi(f) = \operatorname{tr}\left(\int_{G(F)} f(x)\pi(x)dx\right), \quad f \in C_c^\infty(G(F)),$$

which can be identified as a function on $G(F)$,

$$\Theta_\pi(f) = \int_{G(F)} f(x)\Theta(\pi, x)dx, \quad f \in C_c^\infty(G(F)).$$

where $\Theta(\pi, x)$ is a locally integrable function on $G(F)$ and it is smooth on the open dense subset $G_{\mathrm{reg}}(F)$ of regular elements.

Elliptic representations

The normalized character $\Phi(\pi, \gamma)$ is defined to be

$$\Phi(\pi, \gamma) = |D(\gamma)|^{\frac{1}{2}} \Theta(\pi, \gamma), \ \pi \in \mathbf{Rep}(G(F)), \ \gamma \in G_{\mathrm{reg}}(F),$$

with the Weyl discriminant $D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$.
Then $\Phi(\pi, \gamma)$ is related to the Dirac index of the Harish-Chandra module of π if $G(F) = G(\mathbb{R})$.

Def. An element $\gamma \in G(F)$ is *elliptic*, if the centralizer $G_\gamma(F)$ modulo the split part $A_G(F)$ of the center $G(F)$ is compact.

Def. A representation π is called *elliptic* if $\Theta(\pi, x)$ does not vanish on the set of elliptic elements in $G_{\mathrm{reg}}(F)$.

Dirac index of elliptic representations

Suppose $G(\mathbb{R})_{ell} \neq 0$. Then $G(\mathbb{R})$ is of equal rank. The spin module S associated with $G(\mathbb{R})/K(\mathbb{R})$ splits into $S = S^+ \oplus S^-$.

Let X_π be the Harish-Chandra module of $G(\mathbb{R})$.

The spin index is define to be

$$I(X_\pi) = X_\pi \otimes S^+ - X_\pi \otimes S^-.$$

This virtue $\widetilde{K(\mathbb{R})}$-module is equal to the Dirac index of X_π.

Set $\theta_\pi = \mathrm{ch}\, I(X_\pi)$. Then $|\theta_\pi(\gamma)| = |\Phi(\pi, \gamma)|$.

Theorem

Let π be an irreducible admissible representation of $G(\mathbb{R})$ with Harish-Chandra module X_π. Then π is elliptic if and only if the Dirac index $I(X_\pi) \neq 0$.

Classification of elliptic unitary representations

Assume the infinitesimal characters of π is regular.

Theorem

Suppose π is an irreducible unitary elliptic representation of $G(\mathbb{R})$ with a regular infinitesimal character. Then the infinitesimal character of π is strongly regular, and

$$X_\pi \cong A_{\mathfrak{q}}(\lambda).$$

Theorem

(Huang-Pandžić-Vogan, Prlić)
The unitary $A_{\mathfrak{q}}(\lambda)$ of $G(\mathbb{R})$ with a regular infinitesimal character are classified by their Dirac cohomology.

Orthogonality relations

Define elliptic pairing $(\Theta_\pi, \Theta_{\pi'})_{ell}$:

$$|W(G(\mathbb{R}), T_{ell}(\mathbb{R}))|^{-1} \int_{T_{ell}(\mathbb{R})/A_G(\mathbb{R})} |D(\gamma)|\Theta_\pi(\gamma)\overline{\Theta_{\pi'}(\gamma)}d\gamma,$$

where $W(G(\mathbb{R}), T_{ell}(\mathbb{R}))$ is the Weyl group,
$d\gamma$ is the normalized Haar measure on $T_{ell}(\mathbb{R})/A_G(\mathbb{R})$.
The bilinear form is extended to characters of admissible representations,

$$(\Theta_\pi, \Theta_{\pi'})_{ell} = (\theta_\pi, \theta_{\pi'}).$$

Theorem

If π_1, π_2 are irreducible tempered elliptic representations, then either $(\Theta_{\pi_1}, \Theta_{\pi_2})_{ell} = 0$ or $\Phi_{\pi_1} = \pm\Phi_{\pi_2}$.

Remark The orthogonality relation is extended to all tempered representations with a generalized elliptic pairing equal to the sum of $(\ ,\ )_{M,ell}$ over Levi subgroups M of cuspidal parabolics P.

Functorial transfer by L-packets

Given any admissible irreducible representation σ of $H(\mathbb{R})$.

The L-transfer: $\sigma \mapsto \sigma_G$, with image in the Grothendieck group of virtual representations of $G(\mathbb{R})$.

Let ϕ be the Langlands parameter for σ.

Let Π_ϕ be the L-packet of the admissible irreducible representations of $H(\mathbb{R})$ corresponding to a Langlands parameter ϕ.

Let $\Pi_{\eta\circ\phi}$ be the L-packet of representations of $G(\mathbb{R})$ corresponding to $\eta \circ \phi$ (maybe empty if this parameter is not relevant for G).

Langlands Functoriality Principle asserts a map from Grothendieck group of virtual representations of $H(\mathbb{R})$ to that of $G(\mathbb{R})$.

Orbital integral

For general functorial transfer, we use the geometric transfer.

The geometric transfer $f \mapsto f^H$ is dual of a transfer for representations.

The orbital integrals are parameterized by the set of regular semisimple conjugacy classes in G. Recall for such a γ the orbital integral is defined as

$$\mathcal{O}_\gamma(f) = \int_{G/G_\gamma} f(x^{-1}\gamma x)dx, \quad f \in C_c^\infty(G),$$

and the stable orbital integral is defined as

$$S\mathcal{O}_\gamma(f) = \sum_{\gamma' \in S(\gamma)} \mathcal{O}_{\gamma'}(f),$$

where $S(\gamma)$ is the stable conjugacy class.

Dirac index of $\mathbb{1}$

Let $\mathbb{1}$ denote the trivial representation of G and $\theta_{\mathbb{1}}$ the character of the Dirac index of the trivial representation. That is

$$\theta_{\mathbb{1}} = \operatorname{ch} H_D^+(\mathbb{1}) - \operatorname{ch} H_D^-(\mathbb{1}) = \operatorname{ch} S^+ - \operatorname{ch} S^-.$$

We note that

$$\overline{\theta_{\mathbb{1}}} = (-1)^q(\operatorname{ch} S^+ - \operatorname{ch} S^-) = (-1)^q \theta_{\mathbb{1}},$$

where $q = \frac{1}{2} \dim G(\mathbb{R})/K(\mathbb{R})$.

Pseudo-coefficients of discrete series

Recall that θ_π denotes the character of the Dirac index of π. If π is the discrete series representation with Dirac cohomology E_μ, then

$$\theta_\pi = (-1)^q \chi_\mu.$$

Labesse showed that there exists a function f_π so that for any admissible representations π',

$$\mathrm{tr}\, \pi'(f_\pi) = \int_K \Theta_{\pi'}(k)\overline{\theta_{\mathbb{1}} \cdot \theta_\pi} dk = (\theta_{\pi'}, \theta_\pi)$$

The orbital integrals of the pseudo-coefficient f_π are easily computed for γ regular semisimple:

$$\mathcal{O}_\gamma(f_\pi) = \begin{cases} \Theta_\pi(\gamma^{-1}) & \text{if } \gamma \text{ is elliptic} \\ 0 & \text{if } \gamma \text{ is not elliptic.} \end{cases}$$

Endoscopic transfer factors

Assume $\eta\colon {}^L H \to {}^L G$ is an admissible embedding. Then for any pseudo-coefficent f of a discrete series of G, there is a linear combination f^H of pseudo-coefficents of discrete series of H such that for $\gamma = j(\gamma_H)$ regular in $T(\mathbb{R})$, one has

$$\mathcal{SO}_{\gamma_H}(f^H) = \Delta(\gamma_H, \gamma_G)\mathcal{O}^{\kappa}_{\gamma_G}(f), \quad (1)$$

where the transfer factor

$$\Delta(\gamma_H, \gamma_G) = (-1)^{q(G)-q(H)}\chi_{G,H}(\gamma)\Delta_B(\gamma^{-1})\Delta_{B_H}(\gamma_H^{-1})^{-1}. \quad (2)$$

The transfer $f \mapsto f^H$ of the pseudo-coefficents of discrete series can be extended to all of functions in $C_c^\infty(G(\mathbb{R}))$ with extension of the correspondence $\gamma \mapsto \gamma_H$, so that the above identity (1) holds for all f.

Induction by Dirac cohomology

Assume $T^H_{ell}(\mathbb{R})$ of $H(\mathbb{R})$ is contained in $T^G_{ell}(\mathbb{R})$ of $G(\mathbb{R})$.

Recall Bott Induction for finite-dimensional modules:

$$\mathrm{Ind}(\sigma) = \sum_{\pi,\text{finite-dim}} \dim \mathrm{Hom}_{H(\mathbb{R})}(\pi|_{H(\mathbb{R})}, \sigma)\pi.$$

Dirac Induction for finite-dimensional modules:

$$\mathrm{Ind}(\sigma) = \sum_{\pi,\text{finite-dim}} \dim \mathrm{Hom}_{H(\mathbb{R})}(H_D(\pi), \sigma)\pi.$$

Dirac Induction for discrete series and tempered elliptic representations:

$$\mathrm{Ind}(\sigma) = \sum_{\pi,\text{tempered}} (\theta_\pi|_{T^H_{ell}(\mathbb{R})}, \theta_\sigma)\pi.$$

General transfer: discrete series

Suppose there are elliptic maximal tori

$$T'_{ell} \subseteq G' \text{ and } T_{ell} \subseteq G$$

with admissible embedding

$${}^L T'_{ell} \subseteq {}^L G' \text{ and } {}^L T_{ell} \subseteq {}^L G$$

such that $\eta({}^L T'_{ell}) \subseteq {}^L T_{ell}$.

Transfer of discrete series (and their limits) is defined by Dirac inductions.

The transfer factor

$$\Delta(\gamma, \gamma') = \frac{(\mathrm{ch}\, S^+(\mathfrak{g}/\mathfrak{t}) - \mathrm{ch}\, S^-(\mathfrak{g}/\mathfrak{t}))\,(\gamma)}{(\mathrm{ch}\, S^+(\mathfrak{g}'\mathfrak{t}') - \mathrm{ch}\, S^-(\mathfrak{g}'/\mathfrak{t}'))\,(\gamma')}.$$

General transfer: tempered representations

Suppose there are Levi subgroups $M' \subseteq G'$ and $M \subseteq G$, with

$$T'_{ell,M'} \subseteq M' \text{ and } T_{ell,M} \subseteq M$$

and admissible embedding

$$\eta({}^L M') \subseteq {}^L M, \text{ and } \eta({}^L T'_{ell,M'}) \subseteq {}^L T_{ell,M}.$$

Suppose a tempered representation π' is parabolically induction from M'.

Transfer of π is defined by Dirac induction from M' to M followed by parabolic induction for G.

General transfer: non-tempered representations

Suppose there are θ-stable parabolics $(L', \mathfrak{q}')$ for G' and $(L, \mathfrak{q})$ for G, with

$$T'_{ell,L'} \subseteq L' \text{ and } T_{ell,L} \subseteq L$$

and admissible embedding

$$\eta({}^L L') \subseteq {}^L L, \text{ and } \eta({}^L T'_{ell,L'}) \subseteq {}^L T_{ell,L}.$$

Let σ be an $A_{\mathfrak{q}}(\lambda)$-module from $(L', \mathfrak{q}')$ for G'.

$$\sigma_G = \mathrm{Tran}(\sigma) := \sum \epsilon_j \pi_j,$$

with $\epsilon_j = \pm 1$ and π_j is $A_{\mathfrak{q}}(\lambda)$-module from $(L, \mathfrak{q})$ for G, so that

$$\sigma_G(f) = \sigma(f'), \text{ for all geometric transfer } f \mapsto f'.$$

Thank You!