Relative Weil algebra and primitive invariants (corrected version)

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Representation Theory XVII
4th of October, 2022.

Supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

EUROPSKI STRUKTURNI I INVESTICIJSKI FONDOVI

Operativni program KONKURENTNOST I KOHEZIJA

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Setting

$\mathfrak{g}$ = complex semisimple Lie algebra

$\mathfrak{h}$ = Cartan subalgebra

B = nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$

$C(V), C(V;B), \bigwedge V, S(V) \dots$

$$W(\mathfrak{g}) = S(\mathfrak{g}) \otimes \bigwedge \mathfrak{g}$$

Weil algebra

B extends to $\bigwedge \mathfrak{g}$:

- $\bigwedge^j \mathfrak{g} \perp \bigwedge^k \mathfrak{g}$ for $j \neq k$
- $B(a_1 \wedge \dots \wedge a_k, b_1 \wedge \dots \wedge b_k) = \det[B(a_i, b_j)]$

History

$$J = \left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}} = \{x \in \bigwedge \mathfrak{g} : \mathrm{ad}_y(x) = 0, \forall y \in \mathfrak{g}\}$$

$$J^+ = \sum_{k \geq 1} J^k \quad \text{augmentation ideal}$$

$$P = (J^+ \wedge J^+)^\perp \quad \text{primitive invariants}$$

History

Theorem (Hopf-Koszul-Samelson)

$\dim P = \operatorname{rk} \mathfrak{g}$ *and* $P \hookrightarrow J$ *extends to an isomorphism of algebras*

$$\bigwedge P \to J$$

$q : \bigwedge \mathfrak{g} \to C(\mathfrak{g})$ Chevalley/quantization map

$$q(z_{i_1} \wedge z_{i_2} \wedge \cdots \wedge z_{i_k}) = z_{i_1} \cdot z_{i_2} \cdot \cdots \cdot z_{i_k}$$

z_i orthonormal basis for $\mathfrak{g}$

q is $\mathfrak{g}$-equivariant

Theorem (Kostant)

$$q(J) \cong C(P, B_0)$$

History

$$\alpha : U(\mathfrak{g}) \to C(\mathfrak{g}) \quad \alpha(x) = \frac{-1}{4} \sum_{i=1}^{n} [x, b_i] d_i, \; x \in \mathfrak{g}$$

b_i, d_i dual bases of $\mathfrak{g}$

$$E = \alpha(U(\mathfrak{g}))$$

Theorem (Kostant, ρ-decomposition)

$$E \cong \operatorname{End} V_\rho, \quad C(\mathfrak{g}) = E \otimes q(J)$$

History

$\iota_x : \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{g}, \quad \iota(x)y = B(x,y),\ x, y \in \mathfrak{g}$ contraction
(extended as a derivation by the Leibniz rule)

Lemma (Kostant)

$$q(\iota_x p) \in E, \forall x \in \mathfrak{g}, p \in P$$

Theorem (Kostant)

$$\forall x \in \mathfrak{g} \quad x = \sum_{i=1}^{\operatorname{rk}\mathfrak{g}} (\iota_x p_i) q_i$$

p_i, q_i : B_0-*dual bases of* P

History

$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ triangular decomposition

$$C(\mathfrak{g}) = C(\mathfrak{h}) \oplus (\mathfrak{n}^+ C(\mathfrak{g}) + C(\mathfrak{g})\mathfrak{n}^-)$$

Harish-Chandra map:

$$\mu : C(\mathfrak{g}) \to C(\mathfrak{h})$$

Theorem (Bazlov)

$\mu : C(\mathfrak{g})^{\mathfrak{h}} \to C(\mathfrak{h})$ *is an algebra isomorphism*

Theorem (Bazlov)

μ *is a linear bijection between* $\mathfrak{q}(P)$ *and* $\mathfrak{h}$

Filtration on P:

$$P^{(k)} = \bigoplus_{i \leq k} P^i$$

$P \subset \bigwedge \mathfrak{g} \rightsquigarrow$ grading P^i

! q respects filtration

Bazlov $\rightsquigarrow$

$$\mathfrak{h}^{(k)} := \mu(q(P^{(k)}))$$

$\check{\mathfrak{g}}$ = LA defined by the dual root system, $\check{\mathfrak{h}}$ corresponding CSA

$(\check{e}, \check{h}, \check{f}) \subset \check{\mathfrak{g}}$ principal $\mathfrak{sl}_2$-triple

$\rho \in \mathfrak{h}^*$ as an element of $\check{\mathfrak{h}}$ coincides with $\check{h}$ (Kostant)

$\check{\mathfrak{h}}^* \cong \mathfrak{h} \rightsquigarrow$

$$\mathfrak{h}_{(m)} := \{x \in \mathfrak{h} : (\mathrm{ad}^*_{\check{e}})^{m+1} x = 0\}$$

History

Theorem

$$\mu(q(P^{(2m+1)})) = \mathfrak{h}_{(m)}$$

conjectured: Kostant

proved: Joseph, Alekseev–Moreau

Transgression

$$W(\mathfrak{g}) = S(\mathfrak{g}) \otimes \bigwedge \mathfrak{g}$$

Weil algebra

$$W(\mathfrak{g}) = \bigwedge \mathfrak{g} \oplus S^1(\mathfrak{g})W(\mathfrak{g})$$

$$\pi : W(\mathfrak{g}) \to \bigwedge \mathfrak{g}$$

$d_W = d_{CE} + d_K =$ Chevalley–Eilenberg + Koszul

$C_p \in W(\mathfrak{g})$ such that $d_W C_p = p$

Transgression

$$t : (S^+(\mathfrak{g}))^{\mathfrak{g}} \to \bigwedge \mathfrak{g}, \quad p \mapsto \pi(C_p)$$

Relative case

$$W_{\mathfrak{g},\mathfrak{t}} = W_{\mathfrak{k},\mathfrak{t}} \qquad \text{e.g. } \mathfrak{g} = \mathfrak{sl}(2n+1,\mathbb{R}), \mathfrak{k} = \mathfrak{so}(2n+1,\mathbb{R})$$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition

$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ fundamental Cartan subalgebra

$\mathfrak{t} = \mathfrak{k} \cap \mathfrak{h}, \quad \mathfrak{a} = \mathfrak{p} \cap \mathfrak{h}$

Theorem (Panyushev, Han)

$C(\mathfrak{p})$ *admits the* ρ*-decomposition*

$$C(\mathfrak{p}) = E \otimes J = \alpha(U(\mathfrak{k})) \otimes C(\mathfrak{p})^{\mathfrak{k}}.$$

Present

$$C(\mathfrak{p}) = C(\mathfrak{a}) \oplus ((\mathfrak{n}^+ \cap \mathfrak{p})C(\mathfrak{p}) + C(\mathfrak{p})(\mathfrak{n}^- \cap \mathfrak{p}))$$

$$\mu : C(\mathfrak{p}) \to C(\mathfrak{a})$$

Theorem (G., Krutov, Pandžić)

$\mu : C(\mathfrak{p})^{\mathfrak{k}} \to C(\mathfrak{a})$ *is an algebra isomorphism*

Future?

We want

$$\forall x \in \mathfrak{p}, p \in P(\mathfrak{p}) \quad q(\iota_x p) \in E$$

$\Rightarrow \mu$ is a (vector space) isomorphism between $P(\mathfrak{p})$ and $\mathfrak{a}$

Clifford algebra conjecture for $C(\mathfrak{p})$

Progress

relative Weil algebra

$$W(\mathfrak{g},\mathfrak{k}) = \left(S(\mathfrak{g}) \otimes \bigwedge \mathfrak{p}\right)^{\mathfrak{k}} = W(\mathfrak{g})_{\mathfrak{k}-\mathrm{bas}}$$

d_W restricts to $W(\mathfrak{g},\mathfrak{k})$ from $W(\mathfrak{g})$

$$\mathrm{pr}_S : S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{k})^{\mathfrak{k}}$$

$$T := \{x \in S(\mathfrak{g})^{\mathfrak{g}} \subset W(\mathfrak{g},\mathfrak{k}) : \mathrm{pr}_S(x) = 0\}$$

$$p = d_W(C_p)$$

Transgression

$$t : T \to \left(\bigwedge \mathfrak{p}\right)^{\mathfrak{k}}, \quad p \mapsto \pi(C_p)$$

Progress

$$t(T^2) = 0$$

$$\lambda_{\mathfrak{p}} : S(\mathfrak{k}) \to \bigwedge \mathfrak{p}, \; \lambda_{\mathfrak{p}}(x) = \frac{-1}{4} \sum_i^{(\mathfrak{p})} [x, e_i] \wedge f_i, \; x \in \mathfrak{k}$$

$\mu \in \mathfrak{g} \rightsquigarrow \widehat{\mu}$ generators for $S(\mathfrak{g})$

$$t(\widehat{\xi}_1 \widehat{\xi}_2 \ldots \widehat{\xi}_j \widehat{\psi}) = \text{const.} \cdot \lambda_{\mathfrak{p}}(\xi_1 \xi_2 \ldots \xi_j)\psi$$

$$t(p) = \text{const.} \sum_i^{(\mathfrak{p})} e_i \wedge \lambda_{\mathfrak{p}}(\iota^S_{f_i} p)$$

Conjecture

$$\ker \mathrm{t} = \mathsf{T}^2$$

$$\mathrm{im}\,\mathrm{t} = \text{primitive invariants in } \bigwedge \mathfrak{p}$$

THANK YOU!
HVALA!
DANKE SCHÖN!