A non-vanishing criterion for Dirac cohomology

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Dirac operator

- Paul A. M. Dirac, “Quantum theory of the electron", 1928.
- Atiyah’s remark: $-\Delta = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})^2$.
- Harish-Chardra, a PhD student of Dirac graduated in 1947, classified discrete series in 1960’s.
- Parthasarathy introduced Dirac operator for semisimple Lie groups in 1972, and realized most discrete series in the kernel of the Dirac operator. This project was finished by Atiyah and Schmid in 1976.
- Discrete series played an important role in the Langlands classification of the irreducible admissible dual of real reductive Lie groups.

Paul Dirac

Figure 1: Paul Dirac (1902–1984) in 1933.

Let G be a real reductive Lie group (e.g., $GL(n,\mathbb{R})$).

- Let $\mathcal{H}$ be a Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle$.
- A *unitary representation* is a continuous map

$$\pi : G \times \mathcal{H} \to \mathcal{H}$$

  such that

$$\langle \pi(g)u, \pi(g)v\rangle = \langle u, v\rangle, \quad \forall g \in G, \forall u, v \in \mathcal{H}.$$

- Equip G with the Haar measure, then $L^2(G)$ is a unitary representation of G. In $L^2(G)$ lives *discrete series* and other tempered representations.
- Unitary representations go beyond $L^2(G)$.

Fundamental problem: determine the **unitary dual** $\widehat{G}$ of G.

Dirac operators in Lie theory

- Let θ be the Cartan involution of G. Assume that $K := G^\theta$ is a maximal compact subgroup of G. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition on the Lie algebra level. Then $G = K\exp(\mathfrak{p}_0)$ is the Cartan decomposition on the group level.
- e.g., $GL(n, \mathbb{R})$, $\theta(g) = (g^{-1})^t$; the polar decomposition.
- Let $B(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ be the Killing form on $\mathfrak{g}$, $\mathfrak{p}$ etc.
- Let $\{Z_i\}_{i=1}^n$ be an o.n.b. of $\mathfrak{p}_0$ w.r.t. $B(\cdot, \cdot)$. The **Dirac operator** is defined by Parthasarathy as:

$$D := \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

- Note that we have

$$D^2 = -(\Omega_{\mathfrak{g}} \otimes 1 + \|\rho\|^2) + (\Omega_{\mathfrak{k}_\Delta} + \|\rho_c\|^2).$$

Parthasarathy's Dirac operator inequality

- Let X be an $(\mathfrak{g}, K)$-module with infnitesimal character Λ.
- Let S be a Spin module for $C(\mathfrak{p})$.
- Using the double cover $\mathrm{Pin}(\mathfrak{p}_0) \to SO(\mathfrak{p}_0)$, one constructs the pin covering group $\widetilde{K}$ of K.
- Let γ be any highest weight of any $\widetilde{K}$-type occurring in $X \otimes S$.
- If X is unitarizable, then $D^2 \geq 0$ on $X \otimes S$. Thus,

$$\|\gamma + \rho_c\| \geq \|\Lambda\|.$$

- Parthasarathy's Dirac operator inequality is very effective in detecting **non-unitarity** since most irreducible representations are non-unitary.

Dirac cohomology

- Let X be a $(\mathfrak{g}, K)$-module. Then

$$D : X \otimes S \to X \otimes S,$$

and in the 1997 MIT Lie groups seminar, Vogan introduced the **Dirac cohomology** of X to be

$$H_D(X) = \text{Ker}\, D/(\text{Ker}\, D \cap \text{Im}\; D).$$

- Moreover, Vogan conjectured that when $H_D(X)$ is nonzero, it should reveal the infinitesimal character of X.
- For any $z \in Z(\mathfrak{g})$, there is a unique $\zeta(z) \in Z(\mathfrak{k}_\Delta)$ and there are some $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ such that

$$z \otimes 1 = \zeta(z) + Da + bD.$$

HP Theorem

Theorem (Huang-Pandžić, JAMS, 2002)

Let X be an irreducible admissible $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Assume that $H_D(X) \neq 0$ and that γ is the highest weight of a $\widetilde{K}$-type occurring in $H_D(X)$. Then Λ is conjugate to $\gamma + \rho_c$ under $W(\mathfrak{g}, \mathfrak{h})$.

- Thus as an invariant of X, Dirac cohomology (whenever non-zero) is finer than the infinitesimal character.
- For X unitary, we have $H_D(X) = \mathrm{Ker}\, D = \mathrm{Ker}\, D^2$.

The classification problem

Classify $\widehat{G}^{\mathrm{d}}$— the set of all the equivalence classes of irreducible unitary representations with non-zero Dirac cohomology.

- Note that $\widehat{G}^{\mathrm{d}} \subset \widehat{G}$.
- These representations are extreme ones among the unitary dual in the following sense: they are exactly the ones on which Parthasarathy's Dirac inequality becomes equality.
- Thus we expect $\widehat{G}^d$ to cut out an interesting part of $\widehat{G}$.
- As suggested by Huang, we call $\widehat{G}^{\mathrm{d}}$ the **Dirac series** for G.
- Working with a unitary representation X, classifying the Dirac series amounts to solving the equation $D^2 = 0$ among its K-types.
- Discrete series are all Dirac series.

Three infinities

Towards the classification of the Dirac series of G, there are three infinities to overcome.

- Typically an irreducible unitary representation X is infinite-dim'l.
- HP theorem allows us to focus on finitely many K-types of X (called **spin-lowest K-types** by the speaker in his 2011 HKUST thesis).
- The unitary dual $\widehat{G}$ has infinitely many members.
- A result of the speaker allows us to focus on finitely members of $\widehat{G}$.
- A family of classical Lie groups has infinitely many members.
- Sometimes the unitary dual is known; if unknown, we can still do something.

A finiteness result

Theorem (D., IMRN 2020)

Let $G(\mathbb{R})$ be a real reductive Lie group. For all but finitely many exceptions, any member π in $\widehat{G(\mathbb{R})}^{\mathrm{d}}$ is cohomologically induced from a member $\pi_{L(\mathbb{R})}$ in $\widehat{L(\mathbb{R})}^{\mathrm{d}}$ which is in the good range. Here $L(\mathbb{R})$ is the Levi factor of a proper θ-stable parabolic subalgebra of $G(\mathbb{R})$.

- We can pin down the infinite set $\widehat{G(\mathbb{R})}^{\mathrm{d}}$ via finite calculations.
- In particular, we can determine $\widehat{G(\mathbb{R})}^{\mathrm{d}}$ when $G(\mathbb{R})$ is fixed.

The linear split G_2: string representations

Let a, b be nonnegative integers.

$\#x$	λ	ν	spin LKTs
0	$[a, b]$	$[0, 0]$	$[a+3b+2, a+b]$
1	$[a, b]$	$[0, 0]$	$[2a+3b+3, b-1]$, $b \geq 1$
2	$[a, b]$	$[0, 0]$	$[a-1, a+2b+1]$, $a \geq 1$
3	$[a, 1]$	$[-\frac{3}{2}, 1]$	$[a+2, a+2]$
4	$[1, b]$	$[1, -\frac{1}{2}]$	$[3b+4, b]$

There are five strings in total: the first three strings consist of tempered representations (including all discrete series), the last two strings come from the trivial representation of $SL(2, \mathbb{R})$.

The linear split G_2: FS-scattered representations

$\#x$	λ	ν	spin LKTs	mult	u-small	mult
8	$[3, 0]$	$[1, 0]$	$[3, 1]$	1	Yes	1
9	$[1, 1]$	$[1, 0]$	$[3, 1]$	1	Yes	1
9	$[1, 1]$	$[1, 1]$	$[0, 0]$	1	Yes	1

- A remarkable Dirac series X of $F_{4(4)}$ (the linear split F_4, which is quaternionic at the same time): $H_D(X) \neq 0$ but $\mathrm{DI(X)} = 0$. Note that **Dirac index** of X is defined as

$$\mathrm{DI}(X) = H_D^+(X) - H_D^-(X).$$

- Therefore, Dirac series go beyond elliptic representations, and we disprove a conjecture in 2015.

An enhanced version of a conjecture of Huang

- **Conjecture.** Assume that G is linear and equal rank. Let π be any irreducible unitary $(\mathfrak{g}, K)$ module such that $H_D(\pi)$ is non-zero. Then $\mathrm{DI}(\pi) = 0$ if $\mathrm{Hom}_{\widetilde{K}}(H_D^+(\pi), H_D^-(\pi)) \neq 0$.
- It asserts that there should be *dichotomy* among the spin LKTs whenever cancellation happens.

The `atlas` software

- J. Adams, M. van Leeuwen, P. Trapa and D. Vogan, *Unitary representations of real reductive groups*, **Astérisque 417** (2020).
- Atlas of Lie Groups and Representations, version 1.1, July 2022.
- http://www.liegroups.org
- http://math.mit.edu/~dav/atlassem/
- For the study of Dirac series, other than `atlas`, we need some codes which have been built up over the past decade.

Dirac series for complex classical Lie groups

Theorem (Barbasch, D., Wong, Adv. Math., 2022)

Let G be a complex classical Lie group. Any Dirac series of G is of the form

$$\pi := \mathrm{Ind}_{MN}^{G}((\mathbb{C}_\xi \otimes \pi_u) \otimes \mathbf{1}),$$

where $P = MN$ is a parabolic subgroup of G with Levi factor M, and $\mathbb{C}_\xi$ is a unitary character on M, π_u is either the trivial representation, or a unipotent representation with non-zero Dirac cohomology. Moreover, π has a unique spin-lowest K-type which occurs exactly once.

Cohomologically induced modules

Assume the inducing $(\mathfrak{l}, L\cap K)$-module Z has inf. char. $\lambda_L \in i\mathfrak{t}_{f,0}^*$ which is dominant for $\Delta^+(\mathfrak{l}\cap, \mathfrak{t}_f)$. Assume that Z is *weakly good*, namely,

$$\langle \lambda_L + \rho(\mathfrak{u}), \alpha^\vee \rangle \geq 0, \quad \forall \alpha \in \Delta(\mathfrak{u}, \mathfrak{t}_f).$$

Then we have that (Zuckerman, Knapp-Vogan 1995):

- $\mathcal{L}_j(Z) = \mathcal{R}^j(Z) = 0$ for $j \neq S := \dim(\mathfrak{u}\cap\mathfrak{k})$.
- $\mathcal{L}_S(Z) \cong \mathcal{R}^S(Z)$ as $(\mathfrak{g}, K)$-modules.
- if Z is irreducible, then $\mathcal{L}_S(Z)$ is either zero or an irreducible $(\mathfrak{g}, K)$-module with inf. char. $\lambda_L + \rho(\mathfrak{u})$.
- if Z is unitary, then $\mathcal{L}_S(Z)$, if nonzero, is a unitary $(\mathfrak{g}, K)$-module.
- if Z is in **good range**, then $\mathcal{L}_S(Z)$ is nonzero, and it is unitary if and only if Z is unitary.

Dirac cohomology of coho. induced modules

- Huang, Kang and Pandžić studied Dirac cohomology of admissible $A_{\mathfrak{q}}(\lambda)$ modules (2009, Transformation Groups).
- Pandžić, *Dirac cohomology and the bottom layer K-types*, Glas. Mat. **45** (2010), no.2, 453-460.
- Using the previous setting, we have that (D.-Huang 2015 AJM)

$$H_D(\mathcal{L}_S(Z)) \cong \mathcal{L}_S^{\widetilde{K}}(H_D(Z) \otimes \mathbb{C}_{-\rho(\mathfrak{u}\cap\mathfrak{p})})$$

- As a consequence, if $H_D(\mathcal{L}_S(Z))$ is non-zero, then $H_D(Z)$ must be non-zero.

Huang-Pandžić condition

- Conversely, if $H_D(Z)$ is non-zero, must $H_D(\mathcal{L}_S(Z))$ be non-zero?
- Pavle's idea in 2019.
- A weight $\Lambda \in \mathfrak{h}_f^*$ is said to satisfy the **Huang-Pandžić condition** if

$$\{\delta - \rho_n^{(j)}\} + \rho_c = w\Lambda,$$

where δ is any highest weight of some K-type, $0 \leq j \leq s-1$, and $w \in W(\mathfrak{g}, \mathfrak{t}_f)$.

A non-vanishing criterion

Theorem (D., Transform. Groups, online)

Let G be a simple **linear** real Lie group in the Harish-Chandra class. Let Z be an irreducible unitary $(\mathfrak{l}, L \cap K)$-module with inf. char. $\lambda_L \in i\mathfrak{t}_{f,0}^*$ which is $\Delta^+(\mathfrak{l} \cap ,\mathfrak{t}_f)$-dominant. Assume that $\lambda_L + \rho(\mathfrak{u})$ is good. Assume moreover that $\lambda_L + \rho(\mathfrak{u})$ satisfies the HP condition. Then $H_D(\mathcal{L}_S(Z))$ is non-zero if and only if $H_D(Z)$ is non-zero.

- If $\lambda_L + \rho(\mathfrak{u})$ does not satisfy the HP condition, then $H_D(\mathcal{L}_S(Z))$ must be zero in view of HP Theorem.
- The good range condition in the above theorem can *not* be weakened, say, to be weakly good.

The number of strings in $\widehat{G}^d$

If we can verify **two conjectures** for an equal rank group G with rank l, we can count the number of strings in $\widehat{G}^d$ as follows:

- Let S be any *proper* subset of the simple roots of $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$.
- Collect the dominant integral HP infinitesimal characters Λ whose coordinates are 0 or 1 on the digits corresponding to S, and whose coordinates outside S are 1 by $\Omega(S)$.
- Denote by $N(S)$ the number of Dirac series representations with infinitesimal character in $\Omega(S)$ and support S.
- Put

$$N_i = \sum_{\#S=i} N(S).$$

- Then $N_0 + N_1 + \cdots + N_{l-1}$ is the number of strings in $\widehat{G}^d$.

Two conjectures

- **Conjecture 2.6 of [D., Transform. groups, online]** Let π be any irreducible unitary $(\mathfrak{g}, K)$ module whose inf. char. Λ meets the HP condition. Then $\pi_{L(x)}$ must be unitary.
- **Conjecture 1.4 of [Ding, D., He, J Algebra 2022]** Let $\Lambda = \sum_{i=1}^{l} n_i \zeta_i \in \mathfrak{h}_f^*$ be the inf. char. of any fully supported representation in $\widehat{G}$, where each $2n_i \in \mathbb{Z}_{\geq 0}$. Then we must have $n_i = 0, \frac{1}{2}$, or 1 for every $1 \leq i \leq l$.
- Conjecture 1.4 above is related to Conjecture 5.7' of [Salamanca-Riba and Vogan, 1998 Ann. Math.]. It can be viewed as the integral version of Vogan's FPP conjecture. See also Daniel Wong's talk.

Thank you for listening!

- Let me thank the organizers sincerely.
- I miss the stay, foods and conversations of the 2018 Zagreb conference.