6A-Algebra and its Representations

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Outline

Basics

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. Let $Y(v,\ z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ denote the vertex operator of V for $v \in V$, where $v_n \in \mathrm{End}(V)$.

Definition. Let V be a Moonshine type VOA. For $u, v \in V_2$ we can define a product $u \cdot v = u_1 v$ and an inner product $u_3 v = \langle u, v\rangle \mathbf{1}$. The inner product is invariant, that is $\langle u_1 v, w\rangle = \langle v, u_1 w\rangle$for $u, v, w \in V_2$. With the product and the inner product V_2 becomes an algebra, which is called the ***Griess algebra*** of V.

Basics

Definition. *A vector $v \in V_2$ is called a **conformal vector** with central charge c_v if it satisfies $v_1 v = 2v$ and $v_3 v = \frac{c_v}{2}\mathbb{1}$. Then the operator $L^v_n := v_{n+1}$, $n \in \mathbb{Z}$ satisfy the Virasoro commutation relation*

$$[L^v_m, L^v_n] = (m-n)\, L^v_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c_v.$$

*A conformal vector $v \in V_2$ with central charge $1/2$ is called **Ising vector** if v generates the Virasoro VOA $L(1/2, 0)$.*

Moonshine VOA $V^\natural$ and the Monster group $\mathbb{M}$

The Monster simple group $\mathbb{M}$ is realized as the automorphism group of the Moonshine VOA $V^\natural$.

$\mathbb{M}$ is a 6-transposition group, i.e., $\mathbb{M}$ is generated by some $2A$ -involutions and it satifies $|\phi\psi| \leq 6$ for any two $2A$-involutions ϕ and ψ. Moreover, $\phi\psi$ must lie in one of the nine conjugacy classes $1A, 2A$, $2B$, $3A$, $3C$, $4A$, $4B$, $5A$, $6A$.

Moonshine VOA $V^\natural$ and the Monster group $\mathbb{M}$

Theorem. *[Conway, 1985]* Each $2A$-involution ϕ of the Monster simple group uniquely defines an idempotent e_ϕ called an **axis** in the Monstrous Griess algebra $V_2^\natural$. And $\langle e_\phi, e_\psi \rangle$ is uniquely determined by the conjugacy class of $\phi\psi$: $1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $4B$, $2B$, $3C$.

Moonshine VOA and the Monster simple group

Study of a VOA generated by two Ising vectors initiated in [Miyamoto, 1996].

Theorem. *[Miyamoto, 1996]* There exist involutions $\tau_e \in Aut(V)$ that are in bijection with Ising vectors $e \in V_2$. Moreover, when $V = V^\natural$, the vectors $\frac{1}{2}e$ are the $2A$-axes of the Griess algebra $V_2^\natural$ and the τ_e are the $2A$-involutions of $\mathbb{M}$.

Griess subalgebra generated by two Ising vectors

Theorem. *[Conway, 1985]* The structure of the subalgebra generated by two Ising vectors e and f in the algebra $V_2^\natural$ depends on only the conjugacy class of $\tau_e\tau_f$, and the inner product $\langle e, f\rangle$ is given by the following table:

$\langle\tau_e\tau_f\rangle^{\mathbb{M}}$	$1A$	$2A$	$3A$	$4A$	$5A$	$6A$	$3C$	$4B$	$2B$
$\langle e, f\rangle$	$\frac{1}{4}$	$\frac{1}{2^5}$	$\frac{13}{2^{10}}$	$\frac{1}{2^7}$	$\frac{3}{2^9}$	$\frac{5}{2^{10}}$	$\frac{1}{2^8}$	$\frac{1}{2^8}$	0

Griess subalgebra generated by two Ising vectors

Theorem. *[Sakuma, 2007]* Let V be an arbitrary simple VOA. The structure of the subalgebra generated by two Ising vectors in the Griess algebra V_2 of V is uniquely determined by the inner product of the two Ising vectors. The inner product of two Ising vectors again has 9 possibilities same as the case of the Moonshine VOA $V^\natural$.

Question:

What is the vertex operator subalgebra generated by two Ising vectors ?

Coset subalgebra of $V_{\sqrt{2}E_8}$

Certain vertex operator subslagebra $\mathcal{U}_{nX}$ of the lattice vertex operator algebra $V_{\sqrt{2}E_8}$ were constructed in [Lam-Yamada-Yamauchi, 2005].

$(nX = 1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, 3C\ )$

Coset subalgebra of $V_{\sqrt{2}E_8}$

Theorem. *[Lam-Yamada-Yamauchi, 2005, 2007]* The coset subalgebra $\mathcal{U}_{nX}$ always contains some conformal vectors e and f of central charge $1/2$ such that the inner product $\langle e, f \rangle$ is exactly those given in the table.

The structure and representations of these coset subalgebras are studied. $\mathcal{U}_{nX}$ are all generated by two conformal vectors of central charge $1/2$.

$1A, 2A, 4A, 4B, 2B$: Tensor product of Virasoro VOAs, or extension of them

$3A$: Studied in *[Sakuma-Yamauchi, 2003] , denote it by* $\mathcal{V}$

$5A, 3C$: Irreducible modules for $5A$ and $3C$ are classified in *[Lam-Yamada-Yamauchi, 2007]*

$6A$: not well understood yet.

$6A$-Algebra $\mathcal{U}_{6A}$

The 3A-algebra $\mathcal{V}$: *[Sakuma-Yamauchi, 2003]* Rationality, C_2-cofiniteness, classification of irreducible modules, fusions rules were obtained.

$6A$-algebra:

$$\mathcal{U}_{6A} \cong \mathcal{V} \otimes L\left(\frac{25}{28}, 0\right) \oplus \mathcal{V}\left(\frac{1}{7}\right) \otimes L\left(\frac{25}{28}, \frac{34}{7}\right) \oplus \mathcal{V}\left(\frac{5}{7}\right) \otimes L\left(\frac{25}{28}, \frac{9}{7}\right)$$
$$= P_1 \otimes Q_1 \oplus P_2 \otimes Q_2 \oplus P_3 \otimes Q_3$$

Where $\mathcal{V}\left(\frac{1}{7}\right)$, $\mathcal{V}\left(\frac{5}{7}\right)$ are irreducible $\mathcal{V}$-modules and

$$P_1 = \mathcal{V},\ P_2 = \mathcal{V}\left(\frac{1}{7}\right),\ P_3 = \mathcal{V}\left(\frac{5}{7}\right),$$

$$Q_1 = L\left(\frac{25}{28}, 0\right),\ Q_2 = L\left(\frac{25}{28}, \frac{34}{7}\right),\ Q_3 = L\left(\frac{25}{28}, \frac{9}{7}\right).$$

$6A$-Algebra $\mathcal{U}_{6A}$

Fusion rules of P_i, Q_i's are as follows:

P_1	P_2	P_3
P_2	$P_1 + P_3$	P_2+P_3
P_3	$P_2 + P_3$	$P_1 + P_2+P_3$

Q_1	Q_2	Q_3
Q_2	$Q_1 + Q_3$	$Q_2 + Q_3$
Q_3	$Q_2 + Q_3$	$Q_1 + Q_2 + Q_3$

Denote

$$U^i = P_i \otimes Q_i, i = 1, 2, 3.$$

Then

$$\mathcal{U}_{6A} \cong U^1 \oplus U^2 \oplus U^3$$

is an extension of a rational, C_2-cofinite VOA $U^1 = \mathcal{V} \otimes L\left(\frac{25}{28}, 0\right)$ by two irreducible U^1-modules U^2 and U^3 which are not simple current modules.

Main difficulty: Uniqueness of VOA structure on $\mathcal{U}_{6A}$.

Rough Idea: Let $(\mathcal{U}_{6A},\ Y)$ be a vertex operator algebra structure on $\mathcal{U}_{6A}$. We fix a basis $\overline{\mathcal{Y}}_{a,b}^{c} \in I_{Q_1}\binom{Q_c}{Q_a\ Q_b}$ as in [Felder-Fröhlich-Keller, 1989] and choose an arbitrary basis of $\mathcal{Y}_{a,b}^{c} \in I_{P_1}\binom{P_c}{P_a\ P_b}$. Then $\mathcal{I}_{a,b}^{c} = \mathcal{Y}_{a,b}^{c} \otimes \overline{\mathcal{Y}}_{a,b}^{c}$ is a basis of $I_{U^1}\binom{U^c}{U^a\ U^b}$. Then for any $u^k, v^k \in U^k$, $k = 1, 2, 3$,

$$Y\left(u^2, z\right) u^1 = \mathcal{I}_{2,1}^{2}(u^2, z)u^1;$$
$$Y\left(u^3, z\right) u^1 = \mathcal{I}_{3,1}^{3}\left(u^3, z\right) u^1;$$
$$Y\left(u^2, z\right) v^2 = \left(\mathcal{I}_{2,2}^{1}\left(u^2, z\right) + \mathcal{I}_{2,2}^{3}\left(u^2, z\right)\right) v^2;$$
$$Y\left(u^2, z\right) v^3 = \left(\mathcal{I}_{2,3}^{2}\left(u^2, z\right) + \mathcal{I}_{2,3}^{3}\left(u^2, z\right)\right) v^3;$$
$$\cdots\cdots$$

Assume that there is another VOA structure $(\mathcal{U}_{6A}, \overline{Y})$, then for any $u^i, v^i \in U^i$, $i = 1, 2, 3$, we have

$$\overline{Y}\left(u^2, z\right) u^1 = \lambda_{2,1}^2 \cdot \mathcal{I}_{2,1}^2(u^2, z)u^1,$$

$$\overline{Y}\left(u^3, z\right) u^1 = \lambda_{3,1}^3 \cdot \mathcal{I}_{3,1}^3\left(u^3, z\right) u^1,$$

$$\overline{Y}\left(u^2, z\right) v^2 = \left(\lambda_{2,2}^1 \cdot \mathcal{I}_{2,2}^1\left(u^2, z\right) + \lambda_{2,2}^3 \cdot \mathcal{I}_{2,2}^3\left(u^2, z\right)\right) v^2,$$

$$\cdots\cdots$$

for some $\lambda_{a,b}^c$ $a, b, c \in \{1, 2, 3\}$.

We use results in [Huang, 1995, 1996, 2000], [Knizhnik-Zamolodchikov, 1984], [Tsuchiya- Kanie, 1988] and [Felder-Fröhlich-Keller, 1989] to prove certain entries in the braiding matrices for $L\left(\frac{25}{28}, 0\right)$-modules are nonzero and hence obtain relations among these coefficients $\lambda_{a,b}^c$ and then prove the uniqueness of the VOA structure.

Main Results-Uniqueness of the VOA structure

Theorem. *[Dong-Jiao-Y., 2019]* The VOA structure on $\mathcal{U}_{6A}$ over $\mathbb{C}$ is unique.

Main Results-Classification of irreducible modules

First we construct 14 irreducible $\mathcal{U}_{6A}$-modules $M^0,\ M^1, \cdots, M^{13}$ from decomposition of an even lattice VOA

Use $[h_1,\ h_2]$ to denote the module $\mathcal{V}(h_1) \otimes L(\frac{25}{28},\ h_2)$.

$$M^0 = [0,0] \oplus \left[\frac{1}{7}, \frac{34}{7}\right] \oplus \left[\frac{5}{7}, \frac{9}{7}\right],$$

$$M^1 = \left[0, \frac{3}{4}\right] \oplus \left[\frac{1}{7}, \frac{45}{28}\right] \oplus \left[\frac{5}{7}, \frac{1}{28}\right],$$

$$\vdots$$

$$M^{13} == \left[\frac{2}{5}, \frac{165}{32}\right] \oplus \left[\frac{19}{35}, \frac{3}{224}\right] \oplus \left[\frac{39}{35}, \frac{323}{224}\right].$$

Main Results-Classification of irreducible modules

Recall: Let V be a vertex operator algebra with finitely many inequivalent irreducible modules $M^0, \cdots, M^d$. The *global dimension* of V is defined as

$$\mathrm{glob}\,(V) = \sum_{i=0}^{d} \left(q\dim_V M^i\right)^2.$$

Main Results-Classification of irreducible modules

We say a vertex operator algebra V is "good" if V is a rational and C_2-cofinite simple vertex operator algebra of CFT type with $V \cong V'$. Let $M^0, M^1, \cdots, M^d$ be all the inequivalent irreducible V-modules with $M^0 \cong V$. The corresponding conformal weights λ_i satisfy $\lambda_i > 0$ for $0 < i \leq d$.

Theorem. *[Abe-Buhl-Dong, 2004; Huang-Kirillov-Lepowsky, 2015, Ai-Dong-Jiao-Ren, 2018]* Let V be a "good" vertex operator algebra. Let U be a simple vertex operator algebra which is an extension of V. Then U is also "good" and

$$\mathrm{glob}\,(V) = \mathrm{glob}\,(U) \cdot (q\dim_V(U))^2 .$$

Main Results-Classification of irreducible modules

In particular,

$$\operatorname{glob}\left(\mathcal{U}_{3A}\otimes L\left(\frac{25}{28},0\right)\right)=\left(q\dim_{\mathcal{V}_{3A}\otimes L\left(\frac{25}{28},0\right)}\mathcal{U}_{6A}\right)^2\cdot\operatorname{glob}\mathcal{U}_{6A},$$

It turns out

$$\operatorname{glob}\mathcal{U}_{6A}=\sum_{i=0}^{13}\left(q\dim_V M^i\right)^2$$

Main Results-Classification of irreducible modules

Theorem. *[Dong-Jiao-Y., 2019]* $\mathcal{U}$ has exactly 14 inequivalent irreducible modules $M^0, M^1, \cdots, M^{13}$.

Main Results-Fusion Rules

For modules $M^i,\ i = 0, 1, \cdots, 13$, we denote the summands of each M^i by $M^i_1,\ M^i_2,\ M^i_3$ from left to right.

Theorem. *[Dong-Jiao-Y., 2019]* All fusion rules for irreducible $\mathcal{U}$-modules are given by

$$\dim_{\mathcal{U}} \binom{M^k}{M^i, M^j} = \dim_{U^1} \binom{M^k_1}{M^i_1, M^j_1}$$

where $i, j, k = 0, 1, \cdots, 13$.

Thank you !