Principal subspaces of twisted modules for certain lattice vertex operator algebras

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Joint work with Michael Penn and Gautam Webb

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Outline

- Preliminaries and motivation

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- Lattice constructions, twisted modules, and the principal subspace

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- Presentations, recursions, and characters

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- Lattice constructions, twisted modules, and the principal subspace
- Presentations, recursions, and characters
- Some interesting examples

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Let Δ_+ denote the set of positive roots, and let x_α denote a nonzero root vector for the root α. Define also the subalgebra

$$\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathbb{C} x_\alpha$$

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V_L gives a realization of the level 1 basic $\hat{\mathfrak{g}}$-module $L(\Lambda_0)$, and $V_L e^{\lambda_i}$ gives a realization of the basic $\hat{\mathfrak{g}}$-module $L(\Lambda_i)$, in both cases with the action of $x_\alpha \otimes t^n$ given by the n-th mode of the vertex operator

$$Y(\iota(e_\alpha), x) = \sum_{n \in \mathbb{Z}} x_\alpha(n) x^{-n-1}.$$

Principal subspaces

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Let v_Λ be the highest weight vector of $L(\Lambda)$. The *principal subspace* $W(\Lambda)$ of $L(\Lambda)$ is defined by

$$W(\Lambda) = U(\bar{\mathfrak{n}}) \cdot v_\Lambda.$$

Principal subspaces were originally defined and studied by Feigin and Stoyanovsky.

Principal subspaces

The principal subspace inherits certain compatible gradings from $L(\Lambda)$. First, we have the *conformal weight grading*:

$$W(\Lambda) = \coprod_{s \in \mathbb{Z}} W(\Lambda)_{s+h_\Lambda},$$

Given a monomial

$$x_{\beta_1}(m_1) \dots x_{\beta_r}(m_r) v_\Lambda \in W(\Lambda),$$

its conformal weight is

$$-m_1 - \cdots - m_r + h_\Lambda,$$

where $h_\Lambda \in \mathbb{Q}$ is determined by Λ.
This grading is given by the Virasoro $L(0)$ operator's eigenvalues when acting on $W(\Lambda)$.

Principal subspaces

Second, $W(\Lambda)$ has λ_i-*charge gradings*:

$$W(\Lambda) = \coprod_{r_i \in \mathbb{Z}} W(\Lambda)_{r_i + \langle \lambda_i, \Lambda \rangle}$$

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These gradings are given by the eigenvalues of each $\lambda_i(0)$, $i = 1, \ldots, n$, acting on $W(\Lambda)$ and "count" the number of α_i's appearing as subscripts in each monomial.

Principal subspaces

These gradings are compatible, and we have that:

$$W(\Lambda) = \coprod_{r_1,\ldots,r_n,s\in\mathbb{Z}} W(\Lambda)_{r_1+\langle\lambda_1,\Lambda\rangle,\ldots,r_n+\langle\lambda_n,\Lambda\rangle;s+h_\Lambda}.$$

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We define the multigraded dimensions of $W(\Lambda)$ by:

$$\chi_{W(\Lambda)}(x_1,\ldots,x_n,q) = tr_{W(\Lambda)} x_1^{\lambda_1}\cdots x_n^{\lambda_n} q^{L(0)}.$$

and a modified version

$$\chi'_{W(\Lambda)}(x_1,\ldots,x_n,q) = x_1^{-\langle\Lambda,\lambda_1\rangle}\ldots x_n^{-\langle\Lambda,\lambda_n\rangle} q^{-\langle\Lambda,\Lambda\rangle/2} tr_{W(\Lambda)} x_1^{\lambda_1}\cdots x_n^{\lambda_n} q^{L(0)}$$

in order to have series with integer powers.

Principal subspaces

In a series of papers, Capparelli, Calinescu, Lepowsky, and Milas studied the principal subspaces of basic modules for all the cases mentioned above, and also studied the principal subspaces of the higher level $\widehat{\mathfrak{sl}(2)}$-modules. They constructed exact sequences among principal subspaces:

$$0 \to W(\Lambda_i) \to W(\Lambda_0) \to W(\Lambda_i) \to 0,$$

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where the maps used arise naturally from the lattice construction of $L(\Lambda)$ and intertwining operators among standard modules. They then used these to find recursions satisfied by the multigraded dimension of each $W(\Lambda_i)$:

$$\chi'_{W(\Lambda_i)}(x_1, \ldots, x_n, q) = \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} \frac{q^{\mathbf{m}M\mathbf{m}^T+2m_i}}{(q)_{m_1}\ldots(q)_{m_l}}$$

Principal subspaces

In order to prove exactness, certain natural relations arising from appropriate powers of vertex operators. In particular, for the level 1 cases we've been discussing, they needed to use the fact that

$$Y(e^{\alpha}, x)^2 = \left(\sum_{n \in \mathbb{Z}} x_{\alpha}(n) x^{-n-1} \right)^2 = 0$$

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They defined operators

$$R(\alpha_i, \alpha_i | t) = \sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ m_1 + m_2 = -t}} x_{\alpha_i}(m_1) x_{\alpha_i}(m_2)$$

and their truncations

$$R^0(\alpha_i, \alpha_i | t) = \sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}(m_1) x_{\alpha_i}(m_2)$$

Principal subspaces

Consider the surjection

$$F_{\Lambda_i} : U(\hat{\mathfrak{g}}) \to L(\Lambda_i) \qquad (1)$$
$$a \mapsto a \cdot v_\Lambda$$

and its restriction

$$f_{\Lambda_i} : U(\overline{\mathfrak{n}}) \to W(\Lambda_i) \qquad (2)$$
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Calinescu, Lepowsky, and Milas showed that

Theorem (Calinescu, Lepowsky, Milas)

$$Kerf_{\Lambda_0} = \textstyle\sum_{i=1}^{n} U(\bar{\mathfrak{n}})R^0(\alpha_i, \alpha_i|t) + U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$$

and

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Known results

The exact sequences yield recursions:

- In the case that $\mathfrak{g} = \mathfrak{sl}(2)$, Capparelli, Lepowsky, and Milas interpreted the Rogers-Ramanujan recursion in this context:

$$\chi_{W(\Lambda_0)}(x,q) = \chi_{W(\Lambda_0)}(xq,q) + xq\chi_{W(\Lambda_0)}(xq^2,q),$$

  and obtained

$$\chi_{W(\Lambda_0)}(x,q) = \sum_{n\geq 0} \frac{x^n q^{n^2}}{(q)_n}$$

  and

$$\chi_{W(\Lambda_1)}(x,q) = x^{1/2}q^{1/4} \sum_{n\geq 0} \frac{x^n q^{n^2+n}}{(q)_n}$$

Known results

- For higher level $\widehat{\mathfrak{sl}(2)}$-modules, Capparelli, Lepowsky, and Milas obtained the Rogers-Selberg recursions, giving the sum side of the Gordon-Andrews identities as graded dimensions. Namely, they interpreted the Rogers-Selberg recursion in this

  context, and showed that:

$$\chi_{W(i\Lambda_0+(k-i)\Lambda_1)}(x,q) = \sum_{m\geq 0} \sum_{\substack{N_1+\cdots+N_k=m \\ N_1\geq\cdots\geq N_k\geq 0}} \frac{x^{m+(k-i)/2} q^{h_{i\Lambda_0+(k-i)\Lambda_1}+N_1^2+\cdots+N_k^2+N_{i+1}+\cdots+N_k}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-1}-N_k}(q)_{N_k}}.$$

A few other extensions

- Penn studied the case where L is a positive definite even lattice of rank n whose Gram matrix has non-negative entries. In this work, he found presentations, constructed exact sequences, and obtained recursions and characters.
- Penn and Milas later constructed combinatorial bases for a more general case of this problem, namely when L is an integral lattice

In both of these works, the character of the principal subspace takes a familiar form:

$$\chi'(\mathbf{x}, q) = \sum \frac{q^{\mathbf{m}^T A \mathbf{m}}}{(q)_{m_1} \cdots (q)_{m_n}} x_1^{m_1} \cdots x_m^{m_n}$$

The aim of the work in this talk is to generalize the results found in the work by Penn to twisted modules for V_L.

Towards the twisted case

Calinescu, Lepowsky, and Milas extended their results to the principal subspace $W(\Lambda)$ of the basic $A_2^{(2)}$-module $L(\Lambda)$, and obtained (among many other results):

$$\chi'_{W(\Lambda)}(x, q) = \prod_{n\geq 1}(1 - xq^{2n+1})^{-1}$$

Specializing $x = 1$,

$$\chi'_{W(\Lambda_1)}(1, q) = \prod_{n\geq 1}(1 - q^{2n+1})^{-1}$$

gives the generating function for partitions whose parts are odd and distinct.

Towards the twisted case

In various other works, the principal subspaces of standard modules for twisted affine Lie algebras have been studied in other works:

- $A_{2n}^{(2)}$ level 1, presentations, recursions, and graded dimensions (Calinescu, Milas, Penn)

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- $A_{2n-1}^{(2)}, D_n^{(2)}, E_6^{(2)}, D_4^{(3)}$ level 1, presentations, recursions, and graded dimensions (Penn, S.)

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- $A_{2n-1}^{(2)}, D_n^{(2)}, E_6^{(2)}, D_4^{(3)}$ level $k \geq 1$, combinatorial bases and characters (Butorac, S.)
- $A_2^{(2)}$ level $k \geq 1$, presentations, some recursions, conjectured characters with computational evidence (Calinescu, Penn, S.)

Setting

Consider a rank D positive-definite even lattice

$$L = \mathbb{Z}\alpha_1 \oplus \cdots \mathbb{Z}\alpha_D$$

equipped with a symmetric, nondegenerate, bilinear form $\langle \cdot, \cdot, \rangle$, and its Gram matrix

$$A = (a_{i,j}) = (\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq D}.$$

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Running Example: Consider the lattice

$$L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4$$

with Gram matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Setting

Let

$$L_+ = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \cdots \mathbb{Z}_{\geq 0}\alpha_D.$$

- Consider an isometry $\nu : L \to L$ such that $\nu(L_+) \subset L_+$.
- It's easy to show that ν is a permutation of the α_i.
- We realize ν as a permutation, decomposed into l_d disjoint cycles:

$$(1, 2, \ldots, l_1)(l_1+1, l_1+2, \ldots, l_2) \cdots (l_1+l_2+\cdots+l_{d-1}+1, \ldots, l_d).$$

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Running Example: Consider the automorphism $\nu : L \to \nu$ defined by:

$$\nu(\alpha_1) = \alpha_1$$

$$\nu(\alpha_2) = \alpha_3, \quad \nu(\alpha_3) = \alpha_4$$

Which can be realized by the permutation:

$$(1)(2, 3, 4).$$

$$(1, 2, \ldots, l_1)(l_1+1, l_1+2, \ldots, l_2) \cdots (l_1+l_2+\cdots+l_{d-1}+1, \ldots, l_d).$$

Setting

We relabel the elements of our basis of L to interact more nicely with the isometry ν. Define:

$$\alpha_1^{(r)} = \alpha_{l_1+l_2+\cdots+l_{r-1}+1}$$

and

$$\alpha_1^{(r)} = \alpha_{l_1+l_2+\cdots+l_{r-1}+j} = \nu^j \alpha_1^{(r)}.$$

For simplicity, set

$$\alpha^{(r)} = \alpha_1^{(r)},$$

the first element of each orbit, for $1 \leq r \leq d$.

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Running Example: Define

$$\alpha_1^{(1)} = \alpha_1$$

and

$$\alpha_1^{(2)} = \alpha_2$$
$$\alpha_2^{(2)} = \nu\alpha_2$$
$$\alpha_3^{(2)} = \nu^2\alpha_2$$

In particular, we have

$$\alpha^{(1)} = \alpha_1, \quad \alpha^{(2)} = \alpha_2.$$

Setting

Now, let k be twice the order of ν, and let η be a k-th root of unity. We consider the two central extentions of L by $\langle \nu \rangle$:

$$1 \longrightarrow \langle \eta \rangle \longrightarrow \hat{L} \xrightarrow{\bar{}} L \longrightarrow 0$$

and

$$1 \longrightarrow \langle \eta \rangle \longrightarrow \hat{L}_\nu \xrightarrow{\bar{}} L \longrightarrow 0$$

with commutator maps

$$C_0(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$$

and

$$C(\alpha, \beta) = \prod_{j=0}^{k-1} (-\eta^j)^{\langle \nu^j \alpha, \beta \rangle}$$

respectively.

Setting

Let $e : L \to \hat{L}$ be a normalized section such that:

$$e_0 = 1 \text{ and } \overline{e_\alpha} = \alpha \text{ for all } \alpha \in L.$$

Let ϵ_{C_0} be the normalized cocycle defined by:

$$\epsilon_{C_0}(\alpha_i, \alpha_j) = \begin{cases} 1 & \text{if } i \le j \\ (-1)^{\langle \alpha_i, \alpha_j \rangle} & \text{if } i > j. \end{cases}$$

under which we have

$$e_\alpha e_\beta = \epsilon_{C_0}(\alpha, \beta) e_{\alpha+\beta}.$$

Setting

We lift ν to an automorphism $\hat{\nu}$ of $\hat{L}$ such that

$$\overline{\hat{\nu} a} = \nu a$$

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For $\alpha \in \{\alpha_j^{(r)} | 1 \leq j \leq l_r\}$, we define this lifting by:

$$\hat{\nu}e_\alpha = \begin{cases} e_{\nu\alpha} & \text{if } l_r \text{ is odd} \\ e_{\nu\alpha} & \text{if } l_r \text{ is even and } \left\langle \nu^{l_r/2}\alpha, \alpha \right\rangle \in 2\mathbb{Z} \\ \eta_{2l_r} e_{\nu\alpha} & \text{if } l_r \text{ is even and } \left\langle \nu^{l_r/2}\alpha, \alpha \right\rangle \notin 2\mathbb{Z}, \end{cases}$$

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We say that $\alpha_i \in L$ satisfies the **evenness condition** if $\alpha_i = \alpha_j^{(r)}$ for some $0 \leq j \leq l_r - 1$ and one of the following holds

1. l_r is a positive even integer and $\langle \alpha_i, \nu^{l_r/2}\alpha_i \rangle \in 2\mathbb{Z}$
2. l_r is a positive odd integer.

The VOA V_L

Consider the lattice VOA V_L characterized by the linear isomorphism

$$V_L \cong S(\widehat{\mathfrak{h}}^-) \otimes \mathbb{C}[L]$$

where

$$\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}, \text{ and } \widehat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}]$$

with the vertex operators

$$Y(h, x) = \sum_{n \in \mathbb{Z}} h(n) x^{-n-1}$$

for $h \in \mathfrak{h}$ and

$$Y(\iota(e_\alpha), x) = E^-(-\alpha, x) E^+(-\alpha, x) e_\alpha x^\alpha.$$

and vacuum and conformal vectors

$$\mathbb{1} = 1 \otimes 1, \quad \omega = \frac{1}{2} \sum_{i=1}^{D} u_i(-1)^2 \mathbb{1}$$

respectively.

The twisted module V_L^T

We now construct the twisted module we call V_L^T for the VOA V_L. For $n \in \mathbb{Z}$, consider

$$\mathfrak{h}_{(n)} := \{h \in \mathfrak{h} | \nu h = \eta^n h\}$$

so that

$$\mathfrak{h} = \coprod_{n \in \mathbb{Z}/k\mathbb{Z}} \mathfrak{h}_{(n)}.$$

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We also project each $h \in \mathfrak{h}$ onto $\mathfrak{h}_{(n)}$ via the map

$$P_n : \mathfrak{h} \to \mathfrak{h}_{(n)}$$

given by $P_n(h) = \frac{1}{k}(h + \eta^{-n}\nu h + \eta^{-2n}\nu^2 h + \cdots + \eta^{-(k-1)n}\nu^{k-1} h)$ for $h \in \mathfrak{h}$, and we call this projection simply $h_{(n)}$

The twisted module V_L^T

Running Example: We will be primarily concerned with the 0-th projection, and as such we have: In our example, we have that

$$\mathfrak{h}_{(0)} = \mathbb{C}\alpha_1 + \mathbb{C}(\alpha_2 + \alpha_3 + \alpha_4)$$

and

$$\alpha_{1(0)} = \alpha_1$$

$$\alpha_{2(0)} = \frac{1}{3}(\alpha_2 + \alpha_3 + \alpha_4)$$

In general:

$$(\alpha_j^{(r)})_{(0)} = \frac{1}{l_r}\left(\alpha_1^{(r)} + \cdots + \alpha_{l_r}^{(r)}\right),$$

for $1 \leq r \leq d$ and $1 \leq j \leq l_r$.

The twisted module V_L^T

Form the affine Lie algebra

$$\hat{\mathfrak{h}}[\nu] = \coprod n \in \frac{1}{k}\mathbb{Z}\mathfrak{h}_{(kn)} \otimes t^n \oplus \mathbb{C}\mathbf{k}$$

with

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m\delta_{m+n,0}\mathbf{k}$$

for $\alpha \in \mathfrak{h}_{(km)}$, $\beta \in \mathfrak{h}_{(kn)}$, $m, n \in \frac{1}{k}\mathbb{Z}$ and $\mathbf{k}$ is central.
Form the induced module

$$S[\nu] = U\left(\hat{\mathfrak{h}}[\nu]\right) \otimes_{U\left(\coprod_{n\geq 0} \mathfrak{h}_{(kn)} \otimes t^n \oplus \mathbb{C}\mathbf{k}\right)} \mathbb{C}, \tag{3}$$

where $\coprod_{n\geq 0} \mathfrak{h}_{(kn)} \otimes t^n$ acts trivially on $\mathbb{C}$ and $\mathbf{k}$ acts as 1. We will make use of the fact that this is linearly isomorphic to $S\left(\hat{\mathfrak{h}}[\nu]^-\right)$.

The twisted module V_L^T

Following the construction in [L] and the work of Calinescu, Lepowsky, and Milas, we define:

$$N = \{\alpha \in L | \langle \alpha, \mathfrak{h}_{(0)} \rangle = 0\}$$

$$M = (1 - \nu)L \subset N$$

$$R = \{\alpha \in N | \eta^{\sum_{j=0}^{k-1} \langle j\nu^j \alpha, \beta \rangle} = 0\}$$

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Using a theorem of [L], we assume that what we call the "twisted Gram matrix":

$$A_L^\nu = (\langle \alpha_{(0)}^{(i)}, \alpha_{(0)}^{(j)} \rangle)_{i,j=1}^d$$

of our lattice is invertible, to ensure that $R = M = N$, which gives us a unique irreducible twisted module for V_L, which we call V_L^T, characterized by the linear isomorphism

$$V_L^T \cong S\left(\hat{\mathfrak{h}}[\nu]^-\right) \otimes \mathbb{C}[L/N]$$

Importantly, we have a twisted vertex operator, and we focus on

$$Y^{\hat{\nu}}(\iota(e_\alpha), x) = k^{-\langle\alpha,\alpha\rangle/2}\sigma(\alpha)E^-(-\alpha, x)E^+(-\alpha, x)e_\alpha x^{\alpha_{(0)}+\langle\alpha_{(0)},\alpha_{(0)}\rangle/2-\langle\alpha,\alpha\rangle/2}, \tag{4}$$

where

$$E^{\pm}(-\alpha, x) = \exp\left(\sum_{n\in\pm\frac{1}{k}\mathbb{Z}_+} \frac{-\alpha_{(kn)}(n)}{n} x^{-n}\right). \tag{5}$$

We define the following modes of the twisted vertex operators

$$Y^{\hat{\nu}}(\iota(e_\alpha), x) = \sum_{m\in\frac{1}{k}\mathbb{Z}} (e^\alpha)^{\hat{\nu}}_m x^{-m-\frac{\langle\alpha,\alpha\rangle}{2}}.$$

The principal subspace W_L^T

We assume now that the Gram matrix of L contains only non-negative entries. We define the *principal subalgebra* of V_L to be

$$W_L = \langle e^{\alpha_1}, \dots, e^{\alpha_D} \rangle,$$

the smallest vertex subalgebra of V_L containing $e^{\alpha_1}, \dots, e^{\alpha_D}$.

We define the *principal subspace* of W_L^T by

$$W_L^T = W_L \cdot 1_T$$

where $1_T = 1 \otimes 1 \in V_L^T$.

The principal subspace W_L^T

V_L^T, and thus W_L^T is $\frac{1}{k}\mathbb{Z}$-graded by the eigenvalues of $L^{\hat{\nu}}(0)$, given by

$$Y^{\hat{\nu}}(\omega, x) = \sum_{n\in\mathbb{Z}} L^{\hat{\nu}}(n)x^{-n-2}.$$

We call this the grading by *conformal weight*. In particular, we have

$$\begin{aligned} & L^{\hat{\nu}}(0)(e_{\alpha_{i_1}})^{\hat{\nu}}_{m_1}\cdots(e_{\alpha_{i_r}})^{\hat{\nu}}_{m_r}\cdot 1_T \\ = \quad & \left(-(m_1+\cdots+m_r+r)+\mathrm{wt}(1_T)\right)(e_{\alpha_{i_1}})^{\hat{\nu}}_{m_1}\cdots(e_{\alpha_{i_r}})^{\hat{\nu}}_{m_r}\cdot 1_T \end{aligned}$$

In particular, we will use $kL^{\hat{\nu}}(0)$ to ensure that our weights are integers.

Setting

We also have d gradings by *charge*, given by the eigenvalues of $l_i(\lambda^{(i)})_{(0)}$ for $1 \leq i \leq d$. We call the sum of these charges the *total charge*.

In essence, the $(\lambda^{(i)})_{(0)}$-charge counts the number of α_i appearing in a monomial in W_L^T.

Setting

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In essence, the $(\lambda^{(i)})_{(0)}$-charge counts the number of α_i appearing in a monomial in W_L^T.

Running Example: The element

$$(e_{\alpha^{(1)}})^{\hat{\nu}}_{-3}(e_{\alpha^{(1)}})^{\hat{\nu}}_{-1}(e_{\alpha^{(2)}})^{\hat{\nu}}_{-\frac{1}{3}} \cdot 1_T$$

has

- conformal weight $= 9 + 3 + 1 + wt(1_T)$
- $(\lambda^{(1)})_{(0)}$-charge $= 2$
- $(\lambda^{(2)})_{(0)}$-charge $= 1$.

The principal subspace W_L^T

Now that we have endowed W_L^T with $(d+1)$-gradings, define the homogeneous graded components

$$\left(W_L^T\right)_{(n,\mathbf{m})} = \{v \in W_L^T | \text{wt } v = n, \text{ch } v = \mathbf{m}\}.$$

and the multigraded dimension

$$\chi(q;\mathbf{x}) = \mathrm{tr}|_{W_L^T} x_1^{l_1(\lambda^{(1)})_{(0)}} \cdots x_d^{l_d(\lambda^{(d)})_{(0)}} q^{k\hat{L}^{\hat{\nu}}(0)},$$

where $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_d^{m_d}$. We also define the shifted multigraded dimensions

$$\chi'(q;\mathbf{x}) = q^{-\mathrm{wt}(1_T)}\chi(q;\mathbf{x}) = \sum_{\substack{n\in\mathbb{Z}_{\geq 0}\\ \mathbf{m}\in(\mathbb{Z}_{\geq 0})^d}} \dim \left(W_L^T\right)_{(n,\mathbf{m})} q^n \mathbf{x}^{\mathbf{m}}$$

so that the powers of $x_1, \ldots, x_d$ and q are all integers.

Properties of W_L^T

Following [L], we have that

$$Y^{\hat{\nu}}(\hat{\nu}^r v, x) = \lim_{x^{1/k} \to \eta^{-r} x^{1/k}} Y^{\hat{\nu}}(v, x).$$

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From this, we have n fact, if l_j is odd or if l_j is even and $\langle a^{(j)}, \nu^{\frac{l_j}{2}} \alpha^{(j)} \rangle \in 2\mathbb{Z}$ we have

$$\begin{aligned} Y^{\hat{\nu}}(e^{\alpha^{(j)}}, x) &= \sum_{n \in \frac{1}{l_j}\mathbb{Z}} (e^{\alpha^{(j)}})_n^{\hat{\nu}} x^{-n-\frac{<\alpha^{(j)}, \alpha^{(j)}>}{2}} \\ &\in (\text{End } V_L^T)[[x^{1/l_j}, x^{-1/l_j}]] \subset (\text{End } V_L^T)[[x^{1/k}, x^{-1/k}]] \end{aligned} \quad . \quad (6)$$

Further, if l_j is even and $\langle \alpha^{(j)}, \nu^{\frac{l_j}{2}} \alpha^{(j)} \rangle \notin 2\mathbb{Z}$ we have

$$\begin{aligned} Y^{\hat{\nu}}(e^{\alpha^{(j)}}, x) &= \sum_{n \in \frac{1}{2l_j} + \frac{1}{l_j}\mathbb{Z}} (e^{\alpha^{(j)}})_n^{\hat{\nu}} x^{-n-\frac{<\alpha^{(j)}, \alpha^{(j)}>}{2}} \\ &\in (\text{End } V_L^T)[[x^{1/2l_j}, x^{-1/2l_j}]] \subset (\text{End } V_L^T)[[x^{1/k}, x^{-1/k}]]. \end{aligned} \quad (7)$$

Running Example: In our example, we have

$$Y^{\hat{\nu}}(e^{\alpha^{(1)}}, x) = \sum_{n \in \mathbb{Z}} (e^{\alpha^{(1)}})^{\hat{\nu}}_n x^{-n-1}$$

and

$$Y^{\hat{\nu}}(e^{\alpha^{(2)}}, x) = \sum_{n \in \frac{1}{3}\mathbb{Z}} (e^{\alpha^{(2)}})^{\hat{\nu}}_n x^{-n-1}$$

since we satisfy the evenness condition in all cases.

Properties of W_L^T

Our twisted properties satisfy several other conditions. Namely, we have that:

- $$(e^{\nu^r \alpha^{(i)}})_n^{\hat{\nu}} = \eta_{l_i}^{rnl_i} (e^{\alpha^{(i)}})_n^{\hat{\nu}},$$

  if $\alpha^{(i)}$ satisfies the evenness condition

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- $$(e^{\nu^r \alpha^{(i)}})_n^{\hat{\nu}} = \eta_{2l_i}^{2rnl_i - 1} (e^{\alpha^{(i)}})_n^{\hat{\nu}},$$

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- We have
  $$(e^{\alpha})_n^{\hat{\nu}} 1_T = 0$$
  for all $n > -\frac{\langle \alpha_{(0)}, \alpha_{(0)} \rangle}{2}$. This is the first of our relations in W_L^T.

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- All of the operators which occur as modes of the twisted vertex operator commute, due to the fact that the Gram matrix of L contains only non-negative entries.

Relations

Our operators satisfy a certain natural set of relations, namely:

$$\frac{1}{(m-1)!}\left(\frac{\partial}{\partial x}\right)^{m-1}\left(Y^{\hat{\nu}}(e^{\alpha_i},x)\right)Y^{\hat{\nu}}(e^{\alpha_j},x)=0. \tag{8}$$

It follows that for all $1 \leq m \leq \left\langle \nu^r \alpha^{(i)}, \alpha^{(j)} \right\rangle$ we have

$$\frac{1}{(m-1)!}\left(\frac{\partial}{\partial x}\right)^{m-1}\left(Y^{\hat{\nu}}(e^{\nu^r\alpha^{(i)}},x)\right)Y^{\hat{\nu}}(e^{\alpha^{(j)}},x)=0. \tag{9}$$

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Extracting appropriate coefficients of the formal variable x from (**??**) we have the expressions

$$R(i,j,r,m|t)=\sum_{\substack{n_1+n_2=-t\\ n_1\in Z_i^-\\ n_2\in Z_j^-}}\eta_{L_i}^{rn_1L_i}\binom{-n_1-\frac{\left\langle\alpha^{(i)},\alpha^{(i)}\right\rangle}{2}}{m-1}(e^{\alpha^{(i)}})_{n_1}^{\hat{\nu}}(e^{\alpha^{(j)}})_{n_2}^{\hat{\nu}}, \tag{10}$$

which act as 0 on 1_T.

Presentation of W_L^T

We define

$$U_L^T = \mathbb{C}\left[x_{\alpha^{(i)}}^{\hat{\nu}}(n) \middle| 1 \leq i \leq d, n \in Z_i\right].$$

We have a projection

$$f_L^T : U_L^T \to W_L^T$$

$$x_{\alpha^{(i_1)}}^{\hat{\nu}}(n_{i_1}) \cdots x_{\alpha^{(i_j)}}^{\hat{\nu}}(n_{i_j}) \mapsto (e^{\alpha^{(i_1)}})_{n_{i_1}}^{\hat{\nu}} \cdots (e^{\alpha^{(i_j)}})_{n_{i_j}}^{\hat{\nu}} \cdot 1_T,$$

(recall, the modes of the twisted vertex operator commute due to our restriction on the Gram matrix of L).
By presentation, we mean to find the generators of $\ker f_L^T$.

Consider the following expressions

$$R(i,j,r,m|t) = \sum_{\substack{n_1+n_2=-t \\ n_1 \in Z_i^- \\ n_2 \in Z_j^-}} \eta_{L_i}^{rn_1 L_i} \binom{-n_1 - \frac{\langle \alpha^{(i)}, \alpha^{(i)} \rangle}{2}}{m-1} x_{\alpha^{(i)}}^{\hat{\nu}}(n_1) x_{\alpha^{(j)}}^{\hat{\nu}}(n_2), \tag{11}$$

and let J_L^T be the left ideal generated by these expressions. Also, define left ideal

$$U_L^{T+} = U_L^T \mathbb{C}\left[x_{\alpha^{(i)}}^{\hat{\nu}}(n) \middle| 1 \leq i \leq d, n \in Z_i^+ \right]. \tag{12}$$

Presentation of W_L^T

Theorem (Penn, S., Webb)

We have that

$$Ker\ f_L^T = J_L^T + U_L^{T+}.$$

Presentation of W_L^T

Theorem (Penn, S., Webb)

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Idea of the proof: The fact that $J_L^T + U_L^{T+} \subset \text{Ker } f_L^T$ is true by the above slides. For the reverse inclusion, we consider an element

from Ker f_L^T which is not in $J_L^T + U_L^{T+}$. We consider a homogeneous element with respect to all gradings with smallest positive total charge. Among these elements, we choose one which has lowest conformal weights.

Idea of the proof: We use certain shifting maps to move our element and eventually end up with $a \in I_L^T$, a contradiction. The most important ingredients in the proof are the maps: For each $\lambda^{(i)}$, $1 \leq i \leq d$, define:

$$\begin{aligned} \tau_{\lambda^{(i)}} &: U_L^T \to U_L^T \\ x_{\alpha^{(j)}}^{\hat{\nu}}(n) &\mapsto x_{\alpha^{(j)}}^{\hat{\nu}}\left(n + \left\langle \left(\alpha^{(i)}\right)_{(0)}, \lambda^{(i)} \right\rangle\right), \end{aligned} \tag{13}$$

where $n \in Z_i$.

Presentation of W_L^T

At the level of W_L^T, we have analogous maps, which are twisted analogues of maps originally introduced in the work of Haisheng Li:

$$\Delta^T(\lambda^{(i)}, -x) = \left(\prod_{j=0}^{l_i-1} (-\eta_{l_i}^j)^{-\nu^j \lambda^{(i)}}\right) x^{\lambda_{(0)}^{(i)}} E^+(-\lambda^{(i)}, x). \qquad (14)$$

At the level of W_L^T, we have analogous maps, which are twisted analogues of maps originally introduced in the work of Haisheng Li:

$$\Delta^T(\lambda^{(i)}, -x) = (\prod_{j=0}^{l_i-1}(-\eta_{l_i}^j)^{-\nu^j\lambda^{(i)}})x^{\lambda^{(i)}_{(0)}}E^+(-\lambda^{(i)}, x). \tag{14}$$

Taking the constant term of this map and calling it $\Delta_c^T(\lambda^{(i)}, -x)$, we have that

$$\begin{aligned}\Delta_c^T(\lambda^{(i)}, -x) : W_L^T &\to W_L^T \\ a \cdot 1_T &\mapsto \tau_{\lambda^{(i)}}(a) \cdot 1_T,\end{aligned} \tag{15}$$

where $a \in U_L^T$. We note here that the map $\tau_{\lambda^{(i)}} : U_L^T \to U_L^T$ is a lifting of the map $\Delta_c^T(\lambda^{(i)}, -x)$

Presentation of W_L^T

Finally, we also need to show a few more relations hold. Namely, we need:
For all $i, j, s, t \in \mathbb{Z}$ such that $1 \leq i, j \leq d$, $s, t \geq 0$, and $s + t \leq l_i \left\langle \alpha_{(0)}^{(i)}, \alpha^{(j)} \right\rangle - 1$, we have

$$x_{\alpha^{(i)}}^{\hat{\nu}} \left(-\frac{\left\langle \alpha_{(0)}^{(i)}, \alpha_{(0)}^{(i)} \right\rangle}{2} - \frac{s}{l_i} \right) x_{\alpha^{(j)}}^{\hat{\nu}} \left(-\frac{\left\langle \alpha_{(0)}^{(j)}, \alpha_{(0)}^{(j)} \right\rangle}{2} - \frac{t}{l_i} \right) \in I_L^T.$$

These types of relations only appeared in earlier work by Milas and Penn, but were not needed in the affine Lie algebra cases (both untwisted and twisted).

Exact Sequences

Knowing this presentation, it is easy to construct exact sequences:

Theorem (Penn, S., Webb)

For each $i = 1, \ldots, d$, we have the following short exact sequences

$$0 \to W_L^T \xrightarrow{e_{\alpha^{(i)}}} W_L^T \xrightarrow{\Delta_c^T(\lambda^{(i)}, -x)} W_L^T \to 0$$

Importantly, we have the

Corollary

We have the following short exact sequences for $i = 1, \ldots, d$:

$$0 \to \left(W_L^T\right)_{\left(\mathbf{m}-\epsilon_i, n+k\frac{\left\langle \alpha_{(0)}^{(i)}, \alpha_{(0)}^{(i)} \right\rangle}{2} - k\sum_{j=1}^{d} m_j \left\langle \alpha_{(0)}^{(i)}, \alpha_{(0)}^{(j)} \right\rangle\right)} \xrightarrow{e_{\alpha^{(i)}}} \left(W_L^T\right)_{(\mathbf{m},n)} \xrightarrow{\Delta_c^T(\lambda^{(i)}, -x)} \left(W_L^T\right)_{\left(\mathbf{m}, n-\frac{k}{l_i}m_i\right)} \to 0.$$

Corollary

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$$0 \to \left(W_L^T\right)_{\left(\mathbf{m}-\epsilon_i, n+k\frac{\left\langle \alpha_{(0)}^{(i)}, \alpha_{(0)}^{(i)} \right\rangle}{2} - k\sum_{j=1}^{d} m_j \left\langle \alpha_{(0)}^{(i)}, \alpha_{(0)}^{(j)} \right\rangle\right)} \xrightarrow{e_{\alpha^{(i)}}} \left(W_L^T\right)_{(\mathbf{m},n)} \xrightarrow{\Delta_c^T(\lambda^{(i)}, -x)} \left(W_L^T\right)_{\left(\mathbf{m}, n-\frac{k}{l_i}m_i\right)} \to 0.$$

Moreover, we have recursions for $i = 1, \ldots, d$ of the form:

$$\begin{aligned} & \chi'(\mathbf{x}; q) \\ = \; & \chi'(x_1, \ldots, x_{i-1}, q^{\frac{k}{l_i}} x_i, x_{i+1}, \ldots, x_d; q) \\ & + x_i q^{k\frac{\left\langle \alpha_{(0)}^{(i)}, \alpha_{(0)}^{(i)} \right\rangle}{2}} \chi'\left(q^{k\left\langle \alpha_{(0)}^{(1)}, \alpha_{(0)}^{(i)} \right\rangle} x_1, \ldots, q^{k\left\langle \alpha_{(0)}^{(d)}, \alpha_{(0)}^{(i)} \right\rangle} x_d; q\right). \end{aligned} \tag{16}$$

Character of W_L^T

Finally, we solve this recursion to obtain:

Corollary

We have

$$\chi^{'}(\mathbf{x};q) = \sum_{\mathbf{m}\in(\mathbb{Z}^d_{\geq 0})} \frac{q^{\frac{\mathbf{m}^t A \mathbf{m}}{2}}}{(q^{\frac{k}{l_1}};q^{\frac{k}{l_1}})_{m_1}\cdots(q^{\frac{k}{l_d}};q^{\frac{k}{l_d}})_{m_d}} x_1^{m_1}\cdots x_d^{m_d}$$

where A is the $(d \times d)$-matrix defined by

$$A_{i,j} = k\left\langle \alpha^{(i)}_{(0)}, \alpha^{(j)}_{(0)} \right\rangle .$$

Running Example: In our example from earlier, we obtain:

$$\chi'(1,1,q) = \sum_{m_1,m_2\geq 0} \frac{q^{3m_1^2+3m_1m_2+m_2^2}}{(q^3;q^3)_{m_1}(q;q)_{m_2}}. \tag{17}$$

We note that this is the analytic sum-side for the Kanade-Russell Conjecture I_1, found by Kursungoz. Namely, one form of the Kanade-Russell Conjecture I_1 is:

$$\sum_{m_1,m_2\geq 0} \frac{q^{3m_1^2+3m_1m_2+m_2^2}}{(q^3;q^3)_{m_1}(q;q)_{m_2}} = \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty}, \tag{18}$$

Character of W_L^T

In a similar example, we can delete the last row and last column of our Gram matrix in our example to obtain a 3×3 Gram matrix (and rank 3 lattice), and applying the above theory we obtain:

$$\chi'(1,1,q) = \sum_{m_1,m_2 \geq 0} \frac{q^{2m_1^2+2m_1m_2+m_2^2}}{(q^2;q^2)_{m_1}(q;q)_{m_2}}, \tag{19}$$

which can be interpreted as the generating function of partitions of n in which no part appears more than twice and no two parts differ by 1 (Bressoud).

Beyond this, this example doesn't generalize since the Gram matrix is no longer positive definite if we increase the rank of the lattice.

Thank you!