Orbifolds and $\mathcal{W}$-algebras

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Outline

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- Permutation orbifolds
  - Free field VOAs
  - $(L_1(\mathfrak{sl}(2)))^{\otimes 3}$

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- $\mathbb{Z}_2$ orbifolds of the $N = 3$ SuperConformal algebra
  - $V_{N=3}(c, 0)^{\mathbb{Z}_2}$
  - $(V_{N=3}(c, 0) \otimes \mathcal{F})^{\mathbb{Z}_2}$

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  - $V_{N=3}(c, 0)^{\mathbb{Z}_2}$
  - $(V_{N=3}(c, 0) \otimes \mathcal{F})^{\mathbb{Z}_2}$
- $\mathcal{W}^k(\mathfrak{sl}(4), f_{\text{short}})$
  - Construction and universal structure
  - Collapsing levels
  - Coincidental isomorphisms
  - Connection to the $N = 3$ Superconformal algebra

Sources of VOAs

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  - Heisenberg VOA
  - $\beta - \gamma$ system
  - Free fermion algebra
  - symplectic fermions

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- Orbifolds: V is a VOA and $G \subset \mathrm{Aut}(V)$

$$V^G = \{v \in V | g \cdot v = v \text{ for all } g \in G\}$$

Sources of VOAs

- Cosets: V is a VOA and $W \subset V$

$$\mathrm{Comm}(W, V) = \{v \in V | v_n w = 0 \text{ for all } w \in W, n \geq 0\}$$

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- $\mathcal{W}$ algebras:
  - Start with a Lie superalgebra $\mathfrak{g}$ and a nilpotent $f \in \mathfrak{g}$.
  - Find an $\mathfrak{sl}(2)$ triple in $\mathfrak{g}$ associated to f: (h, e, f).
  - Decompose $\mathfrak{g}$ by eigenvalues of ad h.
  - Form a free field VOA, $\mathcal{F}(A_{\mathrm{ch}}) \otimes \mathcal{F}(A_{\mathrm{ne}})$, related to this decomposition.
  - Consider $\mathcal{C}(\mathfrak{g}, f) = V^k(\mathfrak{g}) \otimes \mathcal{F}(A_{\mathrm{ch}}) \otimes \mathcal{F}(A_{\mathrm{ne}})$ and a certain vertex algebra homomorphism D.
  - $\mathcal{W}^k(\mathfrak{g}, f)$ is the homology of the related complex.

Permutation Orbifolds – Outline

Given any VOA, V, S_n acts on the n-fold tensor product $V(n) = V^{\otimes n}$ by permuting the tensor factors and thus $S_n \subset \mathrm{Aut}(V(n))$.

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- $\mathcal{SF}(3)^{S_3}$
- $(L_1(\mathfrak{sl}(2)))^{\otimes 3}$
  - Connection to the $\mathcal{W}(2,3) = \mathcal{W}^k(\mathfrak{sl}(3), f_{\mathrm{princ}})$ algebra

Consider the (rank 1) Heisenberg vertex algebra

$$\mathcal{H} = \langle \alpha \rangle \ \text{ with } \alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}$$

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Set

$$\mathcal{H}(3) = \mathcal{H}^{\otimes 3} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle .$$

$\mathcal{H}(3)^{S_3}$

We can diagonalize the action of $(123) \in S_3$ with the change of basis

$$\beta_i = \frac{1}{\sqrt{3}}(\alpha_1 + \eta^i \alpha_2 + \eta^{2i} \alpha_3)$$

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where η is a primitive third root of unity. These have nontrivial OPE given by

$$\beta_0(z)\beta_0(w) \sim \frac{1}{(z-w)^2} \text{ and } \beta_1(z)\beta_2(w) \sim \frac{1}{(z-w)^2}.$$

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This allows us to "push"

$$\mathcal{H}(3)^{S_3} = \mathcal{H} \otimes \mathcal{H}(2)^{S_3},$$

where

$$\mathcal{H} = \langle \beta_0 \rangle \text{ and } \mathcal{H}(2) = \langle \beta_1, \beta_2 \rangle$$

$\mathcal{H}(3)^{S_3}$

By an argument involving the associated graded algebra and classical invariant theory, $\mathcal{H}(2)^{S_3}$ is strongly generated by

$$\omega_2(a,b) = :\partial^a\beta_1\partial^b\beta_2: + :\partial^a\beta_2\partial^b\beta_1:$$
$$\omega_3(a,b,c) = :\partial^a\beta_1\partial^b\beta_1\partial^c\beta_1: + :\partial^a\beta_2\partial^b\beta_2\partial^c\beta_2:$$

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By [Linshaw 2012], we only require quadratic generators $\omega_2(0,0)$, $\omega_2(0,2)$, $\omega_2(0,4)$, $\omega_2(0,6)$, and $\omega_2(0,8)$.

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By [Linshaw 2012], we only require quadratic generators $\omega_2(0,0)$, $\omega_2(0,2)$, $\omega_2(0,4)$, $\omega_2(0,6)$, and $\omega_2(0,8)$. Equations, such as

$$\begin{aligned}\omega_2(0,6) = \frac{53880}{371}\omega_2(0,4)_{-3}\mathbb{1} + \cdots + \frac{165}{371}\omega_2(1,1)_{-1}\omega_2(0,2) \\ \cdots + \frac{45}{371}\omega_2(0,0)_{-1}\omega_2(0,0)_{-1}\omega_2(1,1) \\ \cdots + \frac{60}{371}\omega_3(0,0,0)_{-1}\omega_3(0,1,1),\end{aligned}$$

allow us to remove the need for $\omega_2(0,6)$ and $\omega_2(0,8)$

Lemma

We can reduce the cubic generating set to the list

$$\omega_3(0,0,0), \omega_3(0,0,2), \text{ and } \omega_3(0,1,2).$$

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Proof.

(sketch)

Equations such as

$$\omega_3^0(0,1,3) = -\frac{1}{12}\omega_3^0(0,0,4) - \frac{1}{72}\omega_3^0(0,1,2)_{-2}\mathbb{1} + \frac{1}{72}\omega_3^0(0,0,2)_{-3}\mathbb{1},$$

$$\omega_3^0(0,2,2) = 3\omega_3^0(0,0,4) + \frac{4}{3}\omega_3^0(0,1,2)_{-2}\mathbb{1}$$
$$- \frac{1}{3}\omega_3^0(0,0,2)_{-3}\mathbb{1} - \frac{1}{3}\omega_3^0(0,0,0)_{-5}\mathbb{1},,$$

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Equations such as

$$\begin{aligned}\omega_3^0(0,3,3) = {} & \frac{5}{4}\omega_3^0(0,0,6) - \frac{1}{2}\omega_3^0(0,2,4) + \frac{1}{5}\omega_3^0(0,1,4)_{-2}\mathbb{1} \\ & + \frac{1}{20}\omega_3^0(0,0,4)_{-3}\mathbb{1} + \frac{1}{6}\omega_3^0(0,1,2)_{-4}\mathbb{1} - \frac{1}{12}\omega_3^0(0,0,2)_{-5}\mathbb{1} \\ & - \frac{1}{4}\omega_3^0(0,0,0)_{-7}\mathbb{1},\end{aligned}$$

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We can reduce the cubic generating set to the list

$$\omega_3(0,0,0), \omega_3(0,0,2), \text{ and } \omega_3(0,1,2).$$

Proof.

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allow us to write all vectors of the form $\omega_3^0(0,a,b)$ with $0 \leq a \leq b \leq 4$ can be written, using only the translation operator, in terms of our proposed generating set with the addition of the vectors $\omega_3^0(0,0,4)$, $\omega_3^0(0,1,4)$, $\omega_3^0(0,2,4)$, and $\omega_3^0(0,c,d)$ with $0 \leq c \leq d$ where $d \geq 5$.

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$$\omega_3^0(0,0,4) = -\frac{16}{15}\omega_3^0(0,1,2)_{-2}\mathbb{1} + \frac{4}{15}\omega_3^0(0,0,2)_{-3}\mathbb{1} + \frac{24}{45}\omega_3^0(0,0,0)_{-5}\mathbb{1}$$
$$+ \frac{4}{5}\omega_2^0(0,1)_{-1}\omega_3^0(0,0,1) - \frac{2}{5}\omega_2^0(1,1)_{-1}\omega_3^0(0,0,0),$$

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We can reduce the cubic generating set to the list

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and similar equations allow us to eliminate the need for $\omega_3^0(0,0,4)$, $\omega_3^0(0,1,4)$, and $\omega_3^0(0,2,4)$.

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and similar equations allow us to eliminate the need for $\omega_3^0(0,0,4)$, $\omega_3^0(0,1,4)$, and $\omega_3^0(0,2,4)$.Next, we consider expansions of the expression

$$\begin{aligned} D_5(\mathbf{a}) = {} & \omega_2(a_1,a_2)_{-1}\omega_3(a_3,a_4,a_5) - \omega_2(a_1,a_5)_{-1}\omega_3(a_2,a_3,a_4) \\ & - \omega_2(a_2,a_5)_{-1}\omega_3(a_1,a_3,a_4) - \omega_2(a_3,a_4)_{-1}\omega_3(a_1,a_2,a_5) \\ & + \omega_2(a_3,a_5)_{-1}\omega_3(a_1,a_2,a_4) + \omega_2(a_4,a_5)_{-1}\omega_3(a_1,a_2,a_3), \end{aligned}$$

Lemma

We can reduce the cubic generating set to the list

$$\omega_3(0,0,0), \omega_3(0,0,2), \text{ and } \omega_3(0,1,2).$$

Proof.

(sketch)

Next, linear combinations of $D_5(0,0,0,1,a-3)$, $D_5(0,0,0,2,a-4)$, $D_5(0,0,1,1,a-4)$, $D_5(0,0,0,3,a-5)$, $D_5(0,0,1,2,a-5)$, and $D_5(0,1,1,1,a-5)$, allow us to eliminate the need for $\omega_3^0(0,0,a)$, $\omega_3^0(0,1,a-1)$, $\omega_3^0(0,2,a-2)$, $\omega_3^0(0,3,a-3)$, $\omega_3^0(0,4,a-4)$, and $\omega_3^0(0,5,a-5)$ for $a \geq 5$. $\square$

Lemma

We can reduce the cubic generating set to the list

$$\omega_3(0,0,0), \omega_3(0,0,2), \text{ and } \omega_3(0,1,2).$$

Proof.

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Finally,for $5 \leq a < b$, $D_5(b+2, a-1, 0, 0, 0)$ can be used to construct

$$\begin{aligned} \omega_3^0(0, a+1, b) = {} & A\omega_3^0(0, a-1, b+2) \\ & + B\omega_3^0(0, 0, a+b+1) \\ & + \Psi, \end{aligned}$$

where Ψ is a vertex algebraic polynomial and A and B depend on a and b can be inductively used to finish our argument. $\square$

$\mathcal{H}(3)^{S_3}$

Theorem (Milas-P-Shao)

(i) The vertex operator algebra $\mathcal{H}(3)^{S_3}$ *is simple of type* $(1,2,3,4,5,6^2)$, *i.e. it is strongly generated by seven vectors whose conformal weights are:* $1,2,3,4,5,6,6$. *This generating set is minimal.*
(ii) $\mathcal{H}(3)^{S_3}$ *is isomorphic to* $\mathcal{H}(1) \otimes W$, *where* W *is of type* $(2,3,4,5,6^2)$.
(iii) $\mathcal{H}(3)^{S_3}$ *is not freely generated (by any set of generators).*

The cyclic subgroup $\mathbb{Z}_3 \cong \langle (1 \;\; 2 \;\; 3) \rangle \subset S_3$

We can perform a similar analysis of the orbifold $\mathcal{H}(3)^{\mathbb{Z}_3}$, where our initial generating set is

$$\begin{aligned}
\omega_1(a) &= \beta_0(-1-a)\mathbb{1} \\
\omega_2(a,b) &= \beta_1(-1-a)\beta_2(-1-b)\mathbb{1} \\
\omega_{3,1}(a,b,c) &= \beta_1(-1-a)\beta_1(-1-b)\beta_1(-1-c)\mathbb{1} \\
\omega_{3,2}(a,b,c) &= \beta_2(-1-a)\beta_2(-1-b)\beta_2(-1-c)\mathbb{1}
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\omega_{3,2}(a,b,c) &= \beta_2(-1-a)\beta_2(-1-b)\beta_2(-1-c)\mathbb{1}
\end{aligned}$$

The minimal strong generating set is

$$\begin{aligned}
&\omega_1(0) \\
&\omega_2(0,0), \omega_2(0,1), \omega_2(0,2), \omega_2(0,3) \\
&\omega_{3,1}(0,0,0), \omega_{3,1}(0,0,2) \\
&\omega_{3,2}(0,0,0), \omega_{3,2}(0,0,2)
\end{aligned}$$

Theorem

(i) The vertex operator algebra $\mathcal{H}(3)^{\mathbb{Z}_3}$ *is simple of type* $(1,2,3^3,4,5^3)$*, i.e. it is strongly generated by nine vectors whose conformal weights are:* $1,2,3,3,3,4,5,5,5$*. This generating set is minimal.*

(ii) $\mathcal{H}(3)^{\mathbb{Z}_3}$ *is isomorphic to* $\mathcal{H}(1)\otimes W$*, where* W *is of type* $(2,3^3,4,5^3)$*.*

(iii) $\mathcal{H}(3)^{\mathbb{Z}_3}$ *is not freely generated (by any set of generators).*

Characters

From [Bantay-1997], it follows that

$$\mathrm{ch}[\mathcal{H}(3)^{S_3}](\tau) = \frac{q^{-1/8}}{6}\left(\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^3}\right.$$
$$\left.+3\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}+2\prod_{n=1}^{\infty}\frac{1}{(1-q^{3n})}\right).$$

and

$$\mathrm{ch}[\mathcal{H}(3)^{\mathbb{Z}_3}](\tau) = \frac{q^{-1/8}}{3}\left(\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^3}+2\prod_{n=1}^{\infty}\frac{1}{(1-q^{3n})}\right).$$

Modular Invariance Properties

We have

$$\mathrm{ch}[\mathcal{H}(3)^{S_3}]\left(-\frac{1}{\tau}\right) = \sum_{i=1}^{3}\int_{\mathbb{R}^i} S_{\mathcal{H}(3)^{S_3},M_{\lambda_i}}\mathrm{ch}[M_{\lambda_i}(\tau)]d\lambda_i,$$

where $\lambda_i \in \mathbb{R}^i$ parameterize certain $\mathcal{H}(3)^{S_3}$-modules, M_{λ_i}.

$\mathcal{F}(3)^{S_3}$

Consider the (rank 1) free fermion vertex algebra

$$\mathcal{F} = \langle \alpha \rangle \text{ with } \varphi(z) = \sum_{n \in \mathbb{Z}} \varphi(n) z^{-n-\frac{1}{2}}$$

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Set

$$\mathcal{F}(3) = \mathcal{F}^{\otimes 3} = \langle \varphi_1, \varphi_2, \varphi_3 \rangle .$$

$\mathcal{F}(3)^{S_3}$

We can take the following as an initial generating set.

$$\omega_1(a) = \sum_{i=1}^{3} \partial^a \varphi_i$$

$$\omega_2(a,b) = \sum_{i=1}^{3} {}^{\circ}_{\circ} \partial^a \varphi_i \partial^b \varphi_i {}^{\circ}_{\circ}$$

$$\omega_3(a,b,c) = \sum_{i=1}^{3} {}^{\circ}_{\circ} \partial^a \varphi_i \partial^b \varphi_i \partial^c \varphi_i {}^{\circ}_{\circ}$$

Reduced generators for $\mathcal{F}(3)^{S_3}$

We employ the same strategy of:

- diagonalizing the action of the 3-cycle.
- Writing down generators in the changed basis.
- Looking for classical relations, which are easier to find due to the anti-commutativity.
- Lifting the classical relations to reduce the generating set.

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Proposition

The orbifold $\mathcal{F}(3)^{S_3}$ is minimally strongly generated by

$$\omega_1(0), \omega_2(0,1), \omega_2(0,3), \text{ and } \omega_3(0,1,2).$$

Structure of $\mathcal{F}(3)^{S_3}$

Theorem (Milas-P-Wauchope)

(i) The vertex operator algebra $\mathcal{F}(3)^{S_3}$ is simple of type $(\frac{1}{2}, 2, 4, \frac{9}{2})$, i.e. it is strongly generated by four vectors whose conformal weights are: $(\frac{1}{2}, 2, 4, \frac{9}{2})$. This generating set is minimal.
(ii) $\mathcal{F}(3)^{S_3}$ is isomorphic to $\mathcal{F}(1) \otimes W$, where W is of type $(2, 4, \frac{9}{2})$.
(iii) $\mathcal{F}(3)^{S_3}$ is not freely generated (by any set of generators).

Structure of $\mathcal{F}(3)^{\mathbb{Z}_3}$

Theorem (Milas-P-Wauchope)

(i) The vertex operator algebra $\mathcal{F}(3)^{\mathbb{Z}_3}$ *is simple of type* $(\frac{1}{2}, 1, \frac{9}{2}^2)$*, i.e. it is strongly generated by five vectors whose conformal weights are:* $(\frac{1}{2}, 1, \frac{9}{2}, \frac{9}{2})$*. This generating set is minimal.*
(ii) $\mathcal{F}(3)^{\mathbb{Z}_3}$ *is isomorphic to* $\mathcal{F}(1) \otimes W$*, where* W *is of type* $(1, \frac{9}{2}^2)$*.*
(iii) $\mathcal{F}(3)^{\mathbb{Z}_3}$ *is not freely generated (by any set of generators).*

Another realization of $\mathcal{F}(3)^{\mathbb{Z}_3}$ and $\mathcal{F}(3)^{S_3}$

Theorem (Milas-P-Wauchope)

- $\mathcal{F}(3)^{\mathbb{Z}_3} \cong \mathcal{F} \otimes V_{3\mathbb{Z}}$
- $\mathcal{F}(3)^{S_3} \cong \mathcal{F} \otimes V_{3\mathbb{Z}}^{+}$
- $V_{3\mathbb{Z}}^{+} \cong SComm\left(V_2(\mathfrak{so}(9)), V_1(\mathfrak{so}(9)) \otimes V_1(\mathfrak{so}(9))\right)$, *where* $SComm(-)$ *denotes a simple current extension of* $Comm(-)$.

Another realization of $\mathcal{F}(2)^{S_3}$

By [Genra-2013], we know

$$\mathcal{W}^k(\mathfrak{osp}(1|8), f_{\mathrm{reg}})$$

is of type $\left(2, 4, 6, 8, \frac{9}{2}\right)$ with central charge

$$c = -\frac{9(14k+55)(16k+65)}{4(2k+9)}.$$

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Solving $c = 1$ gives us $k = -\frac{63}{16}$ or $-\frac{73}{18}$. At this level the weight 6 and 8 generators are singular. Further,

$$\mathcal{F}(2)^{S_3} \cong \mathcal{W}_{-\frac{63}{16}}(\mathfrak{osp}(1|8), f_{\mathrm{reg}}).$$

The character of $\mathcal{F}(3)^{S_3}$ and $\mathcal{F}(3)^{\mathbb{Z}_3}$

We have

$$\chi_{\mathcal{F}(3)^{S_3}}(q) = \frac{q^{1/16}}{6}\left(\prod_{n\geq 1}(1+q^{n/2})^3 + 3\prod_{n\geq 1}(1+q^{n/2})(1-q^n)\right.$$
$$\left. +2\prod_{n\geq 1}(1+q^{3n/2})\right).$$

and

$$\chi_{\mathcal{F}(3)^{\mathbb{Z}_3}}(q) = \frac{q^1/16}{3}\left(\prod_{n\geq 1}(1+q^{n/2})^3 + 2\prod_{n\geq 1}(1+q^{3n/2})\right)$$

$\mathcal{F}(2)^{\mathbb{Z}_3}$- "higher" Boson-Fermion correspondence

Comparing characters for $\mathcal{F}(2)^{\mathbb{Z}_3}$ and $V_{3\mathbb{Z}}$ leads us to the following q-series identity:

$$\frac{1}{3}\prod_{n\geq 1}(1+q^{n-\frac{1}{2}})^2+\frac{2}{3}\prod_{n\geq 1}\frac{(1+q^{3(n-\frac{1}{2})})}{(1+q^{n-\frac{1}{2}})}=\frac{\sum_{n\in\mathbb{Z}} q^{\frac{9n^2}{2}}}{(q;q)_\infty}.$$

$\mathcal{SF}(3)^{S_3}$ and $\mathcal{SF}(3)^{\mathbb{Z}_3}$

Theorem (Milas-P.)

The vertex operator algebra $\mathcal{SF}(3)^{S_3}$ *is isomorphic to* $\mathcal{SF}(1) \otimes W$ *where* W *is of type* $(2, 3^3, 4^3, 5^5, 6^4)$. *Further, there are even generators of weight* $2, 3^3, 4, 5^3$ *and odd generators of weight* $1^2, 4^2, 5^2, 6^4$.

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$(L_1(\mathfrak{sl}(2))^{\otimes 3})^{S_3}$

Set

$$L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3$$

and consider the lattice VOA

$$V_L \cong L_1(\mathfrak{sl}(2))^{\otimes 3}.$$

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Similar to a result by [Dong-Lam-Wang-Yamada]:

Proposition

We have

$$V_L = L(3\Lambda_0) \otimes \left(L(\frac{1}{2},0) \otimes L(\frac{7}{10},0) \oplus L(\frac{1}{2},\frac{1}{2}) \otimes L(\frac{7}{10},\frac{3}{2}) \right)$$
$$\oplus L(\Lambda_0 + 2\Lambda_1) \otimes \left(L(\frac{1}{2},\frac{1}{2}) \otimes L(\frac{7}{10},\frac{1}{10}) \oplus L(\frac{1}{2},0) \otimes L(\frac{7}{10},\frac{3}{5}) \right).$$

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Proposition

We have

$$\begin{aligned} V_L = {} & L(3\Lambda_0) \otimes M \\ & \oplus L(\Lambda_0 + 2\Lambda_1) \otimes M'. \end{aligned}$$

$(L_1(\mathfrak{sl}(2))^{\otimes 3})^{S_3}$

By [Dong-Lam-Tanabe-Yamada-Yokoyama] we have

$$M^{\mathbb{Z}_3} \cong W_{3,\frac{6}{5}},$$

the simple quotient of the universal $\mathcal{W}(2,3)$ algebra of central charge $\frac{6}{5}$.

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By [Ali-Linshaw], the $\mathbb{Z}_2$ orbifold of the universal $\mathcal{W}(2,3)$ algebra of central charge $\frac{6}{5}$ is of type $(2,6,8,10,12)$.We can check that the primary weight 12 generator is singular and thus

$$W_{3,\frac{6}{5}}^{\mathbb{Z}_2}$$

is of type $(2,6,8,10)$.

$(L_1(\mathfrak{sl}(2))^{\otimes 3})^{S_3}$

Theorem (Milas-P.)

$$V_L^{S_3} = L(3\Lambda_0) \otimes W_{2,\frac{6}{5}}^{\mathbb{Z}_2} \oplus L(\Lambda_0 + 2\Lambda_1) \otimes W,$$

where W is a $W_{2,\frac{6}{5}}^{\mathbb{Z}_2}$-module of conformal weight $2 + \frac{2}{5}$.

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where W *is a* $W_{2,\frac{6}{5}}^{\mathbb{Z}_2}$*-module of conformal weight* $2 + \frac{2}{5}$*.Further,* $L(\Lambda_0 + 2\Lambda_1) \otimes W$ *has highest weight vector*

$$Z = \sum_{\sigma \in S_3} \sigma \cdot \widetilde{Z},$$

where

$$\begin{aligned}\widetilde{Z} =& 4\alpha_1(-2)e^{\alpha_2} - 2\alpha_1(-1)\alpha_2(-1)\left(e^{\alpha_1} + e^{\alpha_2} + 2e^{\alpha_3}\right) \\ &+ \alpha_1(-1)^2\left(-2e^{\alpha_1} + 5e^{\alpha_2} + e^{\alpha_3}\right) - 16e^{\alpha_1+\alpha_2-\alpha_3}.\end{aligned}$$

thus $V_L^{S_3}$ *is of type* $(1^3, 2, 3, 6, 8, 10)$.

$(L_{Vir}(c,0)^{\otimes 3})^{S_3}$ and $(L_{Vir}(c,0)^{\otimes 3})^{\mathbb{Z}_3}$

Using similar techniques to the above cases, we expect

Conjecture

For generic c, the orbifold $(L_{Vir}(c,0)^{\otimes 3})^{S_3}$ is of type $(2,4,6^2,8^2,9,10^2,11,12^3,14)$ and the orbifold $(L_{Vir}(c,0)^{\otimes 3})^{\mathbb{Z}_3}$ is of type $(2,4,5,6^3,7,8^3,9^3,10^2)$.

Consider the $N=3$ SuperConformal algebra of central charge c, $V_{N=3}(c,0)$, which is generated by

$$j^0, j^+, j^-, L, G^0, G^+, G^-,$$

where j^0, j^+, j^- generate a sub VOA isomorphic to $V^{\frac{1}{3}(2c+1)}(\mathfrak{sl}(2))$.

$V_{N=3}(c,0)^{\mathbb{Z}_2}$

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where j^0, j^+, j^- generate a sub VOA isomorphic to $V^{\frac{1}{3}(2c+1)}(\mathfrak{sl}(2))$. Later, we will also consider this algebra tensored with, $\mathcal{F} = \langle \varphi \rangle$, such that

$$\varphi(z)\varphi(w) \sim \frac{\frac{1}{2}(2c+1)}{z-w}.$$

ie.

$$V_{N=3}(c,0) \otimes \mathcal{F}.$$

$V_{N=3}(c,0)^{\mathbb{Z}_2}$

The $N=3$ SuperConformal algebra of central charge c, $V_{N=3}(c,0)$, described above admits an automorphism

$$G^0 \mapsto -G^0, G^+ \mapsto -G^+, \text{ and } G^- \mapsto -G^-.$$

$V_{N=3}(c,0)^{\mathbb{Z}_2}$

The $N=3$ SuperConformal algebra of central charge c, $V_{N=3}(c,0)$, described above admits an automorphism

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The building blocks for the strong generators of our orbifold are the following elements, which are quadratic in the vectors G^i for $i \in \{0,+,-\}$,

$$\begin{aligned}
W^{0,0}_{a,b} &= {:}\partial^a G^0 \partial^b G^0{:},\\
W^{+,-}_{a,b} &= {:}\partial^a G^+ \partial^b G^-{:},\\
W^{+,0}_{a,b} &= {:}\partial^a G^+ \partial^b G^0{:},\\
W^{-,0}_{a,b} &= {:}\partial^a G^- \partial^b G^0{:},\\
W^{+,+}_{a,b} &= {:}\partial^a G^+ \partial^b G^+{:},\\
W^{-,-}_{a,b} &= {:}\partial^a G^- \partial^b G^-{:}.
\end{aligned}$$

$V_{N=3}(c,0)^{\mathbb{Z}_2}$

We have lowest weight relations for $W^{+,-}_{a,0}$ and $W^{0,0}_{a,0}$ given by

$$
\begin{aligned}
{}^\circ_\circ W^{+,-}_{0,0} W^{+,-}_{0,0} {}^\circ_\circ &= -\frac{2}{3} W^{0,0}_{3,0} + \frac{1}{108}\left(-2c^2 + c + 73\right) W^{+,-}_{3,0} + P^0 \\
{}^\circ_\circ W^{+,0}_{0,0} W^{-,0}_{0,0} {}^\circ_\circ &= \frac{1}{216}\left(2c^2 - c - 73\right) W^{0,0}_{3,0} + \frac{1}{216}\left(-2c^2 + c + 217\right) W^{+,-}_{3,0} \\
&\quad + P^{\pm},
\end{aligned}
$$

where P^0 and $P^{\pm}$ are normally ordered polynomials in lower weight vectors from the orbifold. These equations allow for us to solve for the generators $W^{0,0}_{3,0}$ and $W^{+,-}_{3,0}$ in terms of lower weight members fo the orbifold for all $c \notin \{\frac{1}{4}(1 \pm 9i\sqrt{7}), \frac{1}{4}(1 \pm 3\sqrt{129})\}$, which we refer to as the excluded set.

Next, we have the equations

$$
\begin{aligned}
{}^{\circ}_{\circ} W^{+,0}_{0,0} W^{+,-}_{0,0} {}^{\circ}_{\circ} &= \frac{1}{216}\left(-2c^2 + c - 71\right) W^{+,0}_{3,0} + P^{+},\\
{}^{\circ}_{\circ} W^{-,0}_{0,0} W^{+,-}_{0,0} {}^{\circ}_{\circ} &= \frac{1}{216}\left(2c^2 - c + 71\right) W^{-,0}_{3,0} + P^{-},\\
{}^{\circ}_{\circ} W^{+,0}_{0,0} W^{+,0}_{0,0} {}^{\circ}_{\circ} &= \frac{1}{216}\left(-2c^2 + c - 71\right) W^{+,+}_{3,0} + \widehat{P}^{+},\\
{}^{\circ}_{\circ} W^{-,0}_{0,0} W^{-,0}_{0,0} {}^{\circ}_{\circ} &= \frac{1}{216}\left(-2c^2 + c - 71\right) W^{-,-}_{3,0} + \widehat{P}^{-},
\end{aligned}
$$

where P^{+}, P^{-}, $\widehat{P}^{+}$, and $\widehat{P}^{-}$ are normally ordered polynomials in lower weight vectors from the orbifold. These equation allows us to solve for the generators $W^{+,0}_{3,0}$, $W^{+,-}_{3,0}$, $W^{+,+}_{3,0}$, and $W^{-,-}_{3,0}$ in terms of lower weight vectors for all $c \neq \frac{1}{4}(1 \pm 9i\sqrt{7})$.

We similarly have relations at arbitrary weight for $W^{+,-}_{a,0}$ and $W^{0,0}_{a,0}$ given by given by

$$1296\binom{a+3}{3} {}^{\circ}_{\circ} W^{+,0}_{a,0} W^{-,0}_{0,0} {}^{\circ}_{\circ} = b_1 W^{0,0}_{a+3,0} + b_2 W^{+,-}_{a+3,0} + P_1$$
$$1296\binom{a+3}{3} {}^{\circ}_{\circ} W^{-,0}_{a,0} W^{+,0}_{0,0} {}^{\circ}_{\circ} = b_3 W^{0,0}_{a+3,0} + b_4 W^{+,-}_{a+3,0} + P_2$$
$$1296\binom{a+3}{3} {}^{\circ}_{\circ} W^{+,-}_{a,0} W^{+,-}_{0,0} {}^{\circ}_{\circ} = b_5 W^{0,0}_{a+3,0} + b_6 W^{+,-}_{a+3,0} + P_3$$

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where

$$\begin{aligned} b_1 = {} & \left(6a^2 + 18a + 12\right) c^2 \\ & + \left(18(-1)^a a^2 + 15a^2 + 72(-1)^a a - 9a + 54(-1)^a - 60(-1)\right) c \\ & - 18(-1)^a a^2 - 75a^2 - 99(-1)^a a - 306a - 135(-1)^a - 303, \end{aligned}$$

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\end{aligned}
$$

where

$$
\begin{aligned}
b_2 = {} & -26a^3 - 30a^2 + \left(-2a^3 - 12a^2 - 22a - 12\right) c^2 \\
& + \left(-17a^3 - 66a^2 - 36(-1)^a a - 79a - 54(-1)^a - 48\right) c \\
& + 144(-1)^a a + 362a + 351(-1)^a + 573,
\end{aligned}
$$

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\end{aligned}
$$

where

$$
\begin{aligned}
b_3 = {} & 3\left(2(-1)^a a^2 + 6(-1)^a a + 4(-1)^a\right) c^2 \\
& + 3\left(5(-1)^a a^2 - 3(-1)^a a + 6(a+3)a - 20(-1)^a + 6(a+3)\right) c \\
& + 3\left(-25(-1)^a a^2 - 102(-1)^a a - 6(a+3)a - 101(-1)^a\right. \\
& \qquad \left. - 15(a+3)\right),
\end{aligned}
$$

$V_{N=3}(c,0)^{\mathbb{Z}_2}$

We similarly have relations at arbitrary weight for $W^{+,-}_{a,0}$ and $W^{0,0}_{a,0}$ given by given by

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where

$$\begin{aligned} b_4 = {} & 2(-1)^{a+1}(a+1)(a+2)(a+3)c^2 \\ & + ((-1)^a(a(a+42)+119)a - 36a + 60(-1)^a - 54)\, c \\ & + 951(-1)^a + 144a + (-1)^a a(a(37a+348)+1055) + 351, \end{aligned}$$

$V_{N=3}(c,0)^{\mathbb{Z}_2}$

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\end{aligned}$$

where

$$b_5 = -108a^2 - 54\left(3(-1)^a + 9\right)a - 54\left(7(-1)^a + 9\right),$$

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where

$$b_6 = 37a^3 + 333a^2 + \left(-2a^3 - 18a^2 - 40a - 24\right) c^2 \\ + \left(a^3 + 9a^2 + 20a + 12\right) c + 54(-1)^a a + 902a + 162(-1)^a + 714.$$

Now if we consider the pair of matrices

$$B_1 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} b_1 & b_2 \\ b_5 & b_6 \end{pmatrix} \tag{1}$$

we can show that for all $c \in \mathbb{C}$ and $a \geq 1$, det B_1 and det B_2 are never simultaneously zero. As such, we may solve for $W^{0,0}_{a+3,0}$ and $W^{+,-}_{a+3,0}$ for all $a \geq 1$ in terms of lower weight terms from the orbifold.

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We make a similar argument to reduce the remaining generators.

Theorem (P)

For all $c \in \mathbb{C}$ outside of the excluded set the orbifold $V_{N=3}(c,0)^{\mathbb{Z}_2}$ is minimally strongly generated by j^0 ,j^+, j^-, L, $W^{0,0}_{1,0}$, $W^{+,+}_{1,0}$, $W^{-,-}_{1,0}$, $W^{+,0}_{0,0}$, $W^{+,0}_{1,0}$, $W^{+,0}_{2,0}$, $W^{-,0}_{0,0}$, $W^{-,0}_{1,0}$, $W^{-,0}_{2,0}$, $W^{+,-}_{0,0}$, $W^{+,-}_{1,0}$, and $W^{+,-}_{2,0}$. Thus it is of type $(1^3, 2, 3^3, 4^6, 5^3)$.

$(V_{N=3}(c,0) \otimes \mathcal{F})^{\mathbb{Z}_2}$

For

$$(V_{N=3}(c,0) \otimes \mathcal{F})^{\mathbb{Z}_2}$$

we retain the generators for $V_{N=3}(c,0)^{\mathbb{Z}_2}$ with the addition of

$$\begin{aligned}
\omega_{a,b} &= {}^\circ_\circ \partial^a \varphi \partial^b \varphi {}^\circ_\circ \\
w^+_{a,b} &= {}^\circ_\circ \partial^a G^+ \partial^b \varphi {}^\circ_\circ \\
w^-_{a,b} &= {}^\circ_\circ \partial^a G^- \partial^b \varphi {}^\circ_\circ \\
w^0_{a,b} &= {}^\circ_\circ \partial^a G^0 \partial^b \varphi {}^\circ_\circ .
\end{aligned}$$

$(V_{N=3}(c,0)\otimes\mathcal{F})^{\mathbb{Z}_2}$

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we retain the generators for $V_{N=3}(c,0)^{\mathbb{Z}_2}$ with the addition of
$$\begin{aligned}\omega_{a,b} &= {:}\partial^a\varphi\partial^b\varphi{:}\\ w^+_{a,b} &= {:}\partial^a G^+\partial^b\varphi{:}\\ w^-_{a,b} &= {:}\partial^a G^-\partial^b\varphi{:}\\ w^0_{a,b} &= {:}\partial^a G^0\partial^b\varphi{:}.\end{aligned}$$
Equations such as
$$\begin{aligned}W^{0,0}_{1,0} = \frac{1}{432(2c+1)}\big(&{:}w^0_{0,0}w^0_{0,0}{:} + 27{:}j^0j^0\omega_{1,0}{:}\\ &-36(1+2c){:}L\omega_{1,0}{:} + 2(1+1-2c^2)\omega_{3,0}\big)\end{aligned}$$
allow us to eliminate the need for most of the remaining generators of the form $W^{i,j}_{a,0}$.

$(V_{N=3}(c,0) \otimes \mathcal{F})^{\mathbb{Z}_2}$

Theorem (P)

For $c \neq -\frac{1}{2}$, the orbifold $(V_{N=3}(c,0) \otimes \mathcal{F})^{\mathbb{Z}_2}$ is minimally generated by the fields $j^0, j^{\pm}, L, W^{+,0}_{0,0}, W^{-,0}_{0,0}$ $W^{+,-}_{0,0}$, $\omega_{1,0}$, $w^0_{0,0}$, $w^0_{1,0}$, $w^{\pm}_{0,0}$, and $w^{\pm}_{0,0}$ and is of type $(1^3, 2^4, 3^6)$.

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Proof.

(sketch)

This follows from the explicit decoupling relations above. $\square$

$(V_{N=3}(c,0) \otimes \mathcal{F})^{\mathbb{Z}_2}$

We now consider the special case when $c = -2$, which will be important later in the talk. In this case, we can check that $T = L - L_{\text{sug}}$, where

$$L_{\text{sug}} = \frac{3}{16} {}^{\circ}_{\circ} j^0 j^0 {}^{\circ}_{\circ} + \frac{3}{4} {}^{\circ}_{\circ} j^+ j^- {}^{\circ}_{\circ} - \frac{3}{8} \partial j^0,$$

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is singular and thus in the maximal ideal. Further, equations such as

$$\begin{aligned} W^{+,+}_{0,1} = &-2T_1 W^{+,+}_{0,1} - \frac{3}{2} {:}j^+ W^{+,0}_{0,0}{:} - \frac{3}{32} {:}\partial j^0 j^+ j^+{:} \\ &+ \frac{1}{4} {:}\partial j^+ \partial j^+{:} + \frac{1}{8} {:}\partial^2 j^+ j^+{:} \end{aligned}$$

allow us to eliminate all remaining vectors of the form $W^{i,j}_{0,a}$ for $i,j \in \{0,+,-\}$ from the strong generating set.

Now, equations such as

$$w^{+}_{1,0} = T_0 w^{+}_{0,0} + \frac{3}{4} {:}j^0 w^{+}_{0,0}{:} - \frac{3}{4} {:}j^{+} w^{0}_{0,0}{:},$$

will allow us to remove the fields $w^{i}_{1,0}$ for $i \in \{0, +, -\}$. Observe that we may replace the generator $\omega_{1,0}$ with the field

$$\tilde{L} = L_{\text{sug}} + 3\omega_{1,0},$$

setting up the following

Theorem (P)

$(L_{N=3}(c,0) \otimes \mathcal{F})^{\mathbb{Z}_2}$ is minimally strongly generated by j^0, j^+, j^-, $\tilde{L}$, $w^0_{0,0}$, $w^+_{0,0}$, $w^-_{0,0}$ and is of type $(1,1,1,2,2,2,2)$.

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

Consider the Lie algebra $\mathfrak{sl}(4) = A_3$, with simple roots $\alpha_1, \alpha_2, \alpha_3$. The positive roots are given by

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

and all roots are given by $\Delta = \Delta_+ \cup \Delta_-$.

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and all roots are given by $\Delta = \Delta_+ \cup \Delta_-$. We decompose

$$\mathfrak{sl}(4) = \mathfrak{h} \bigoplus_{\alpha\in\Delta} \mathbb{C}x_\alpha$$

and consider the nilpotent element

$$f = x_{-\alpha_1-\alpha_2} + x_{-\alpha_2-\alpha_3},$$

which is completed into an $\mathfrak{sl}(2)$ triple with

$$e = x_{\alpha_1+\alpha_2} + x_{\alpha_2+\alpha_3} \text{ and } h = \frac{1}{2}(\alpha_1 + 2\alpha_2 + a_3).$$

Proposition

The algebra $\mathcal{W}^k(\mathfrak{sl}(4), f)$ is of type $(1,1,1,2,2,2,2)$ and is strongly generated by

$$J^0, J^+, \text{ and } J^-$$

which generate a sub VOA isomorphic to $V^{2k+4}(\mathfrak{sl}(2))$, L, which is a Virasoro vector of central charge

$$c = -\frac{12k^2 + 41k + 32}{k+4},$$

and three remaining vectors of weight 2: H, E, and F.

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

Some levels collapse to the $\mathfrak{sl}(2)$ sub-VOA.

Proposition

We have

$$\mathcal{W}_{-\frac{8}{3}}(\mathfrak{sl}(4), f) \cong L_{-\frac{4}{3}}(\mathfrak{sl}(2))$$

and

$$\mathcal{W}_{-\frac{3}{2}}(\mathfrak{sl}(4), f) \cong L_1(\mathfrak{sl}(2))$$

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A level that collapses to a Virasoro VOA

Proposition

We have

$$\mathcal{W}_{-2}(\mathfrak{sl}(4), f) \cong L(1, 0)$$

Consider the rank 2 $\beta\gamma$ system, $S(2)$, generated by even, weight 1/2 fields $\beta_1, \beta_2, \gamma_1, \gamma_2$ subject to the non-trivial OPE

$$\beta_i\gamma_j \sim \frac{\delta_{i,j}}{z-w}.$$

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Consider the field

$$h = :\beta_1\gamma_1: + :\beta_2\gamma_2:,$$

which generates a rank 1 Heisenberg subalgebra of $S(2)$, which we denote by $\mathcal{H}$.

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

Consider the rank 2 $\beta\gamma$ system, $S(2)$, generated by even, weight 1/2 fields $\beta_1, \beta_2, \gamma_1, \gamma_2$ subject to the non-trivial OPE

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Consider the field

$$h = :\beta_1\gamma_1: + :\beta_2\gamma_2:,$$

which generates a rank 1 Heisenberg subalgebra of $S(2)$, which we denote by $\mathcal{H}$.Finally consider the coset

$$C(2) = \mathrm{Comm}(\mathcal{H}, S(2)).$$

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

Theorem (Creutzig-Kanade-Linshaw-Ridout)

$C(2)$ is simple and of type $\mathcal{W}(1,1,1,2,2,2)$. Moreover, explicit primary generators are given by

$$
\begin{aligned}
x_{1,2} &= -{}^\circ_\circ\beta_1\gamma_2{}^\circ_\circ \\
x_{2,1} &= -{}^\circ_\circ\beta_2\gamma_1{}^\circ_\circ \\
h_1 &= -{}^\circ_\circ\beta_1\gamma_1{}^\circ_\circ + {}^\circ_\circ\beta_2\gamma_2{}^\circ_\circ \\
P &= {}^\circ_\circ\beta_1\partial\gamma_2{}^\circ_\circ - {}^\circ_\circ(\partial\beta_1)\gamma_2{}^\circ_\circ + \frac{1}{3}{}^\circ_\circ\beta_1\beta_1\gamma_1\gamma_2{}^\circ_\circ + \frac{2}{3}{}^\circ_\circ\beta_1\beta_2\gamma_2\gamma_2{}^\circ_\circ \\
Q &= {}^\circ_\circ\beta_2\partial\gamma_1{}^\circ_\circ - {}^\circ_\circ(\partial\beta_2)\gamma_1{}^\circ_\circ + \frac{1}{3}{}^\circ_\circ\beta_1\beta_2\gamma_1\gamma_1{}^\circ_\circ + \frac{2}{3}{}^\circ_\circ\beta_2\beta_2\gamma_1\gamma_2{}^\circ_\circ \\
R &= {}^\circ_\circ\beta_1\beta_1\gamma_1\gamma_1{}^\circ_\circ - {}^\circ_\circ\beta_2\beta_2\gamma_2\gamma_2{}^\circ_\circ + 2{}^\circ_\circ\beta_1\partial\gamma_1{}^\circ_\circ - 2{}^\circ_\circ\beta_2\partial\gamma_2{}^\circ_\circ - 2{}^\circ_\circ(\partial\beta_1)\gamma_1{}^\circ_\circ +
\end{aligned}
$$

where $x_{1,2}, x_{2,1}, h_1$ generate a subalgebra isomorphic to $L_{-1}(\mathfrak{sl}_2)$.

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

Following from the identification

$$\begin{aligned}
P &\mapsto 2E - \frac{1}{6} {}^{\circ}_{\circ} J^0 J^+ {}^{\circ}_{\circ} + \frac{1}{6} \partial J^+ \\
Q &\mapsto 2F - \frac{1}{6} {}^{\circ}_{\circ} J^0 J^- {}^{\circ}_{\circ} - \frac{1}{6} \partial J^- \\
R &\mapsto -4H + \frac{4}{3} L - \frac{4}{3} {}^{\circ}_{\circ} J^+ J^- {}^{\circ}_{\circ} - \frac{1}{3} {}^{\circ}_{\circ} J^0 J^0 {}^{\circ}_{\circ} + \frac{2}{3} \partial J^0,
\end{aligned}$$

Theorem (Adamovic-Milas-P. also Creutzig-Kanade-Linshaw-Ridout)

We have

$$\mathcal{W}_{-\frac{5}{2}}(\mathfrak{sl}(4), f) \cong C(2).$$

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

Theorem (Adamovic-Milas-P.)

We have
$$\mathcal{W}_{-\frac{7}{3}}(\mathfrak{sl}(4), f) \cong (L_{N=3}(-2,0) \otimes \mathcal{F})^{\mathbb{Z}_2}.$$

Theorem (Adamovic-Milas-P.)

We have

$$\mathcal{W}_{-\frac{7}{3}}(\mathfrak{sl}(4), f) \cong (L_{N=3}(-2,0) \otimes \mathcal{F})^{\mathbb{Z}_2}.$$

Proof.

(sketch)
It is straightforward to check that appropriate identification is

$$\begin{aligned} j^0 &\mapsto J^0, & j^{\pm} &\mapsto J^{\pm}, \\ \tilde{L} &\mapsto L, & w^0_{0,0} &\mapsto -\frac{i}{2\sqrt{6}} H, \\ w^+_{0,0} &\mapsto \frac{i}{2\sqrt{6}} E, & w^-_{0,0} &\mapsto -\frac{i}{2\sqrt{6}} F, \end{aligned}$$

$\square$

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

At level $k = -\frac{1}{2}$ the central charge of $\mathcal{W}^k(\mathfrak{sl}(4), f)$ is

$$-\frac{29}{7} = \frac{9}{5} - \frac{208}{35}.$$

$\mathcal{W}^k(\mathfrak{sl}(4), f_{short})$

At level $k = -\frac{1}{2}$ the central charge of $\mathcal{W}^k(\mathfrak{sl}(4), f)$ is

$$-\frac{29}{7} = \frac{9}{5} - \frac{208}{35}.$$

This provides some motivation for $\mathcal{W}_{-\frac{1}{2}}(\mathfrak{sl}(4), f)$ to be an extension of $L_3(\mathfrak{sl}(2)) \otimes L(c_{5,14}, 0)$.

Thank You!