On irreducibility of Whittaker modules for cyclic orbifold vertex algebras and application to the Weyl algebra

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Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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3. Application to affine Lie algebra $\widehat{\mathfrak{sl}(2)}$
4. Application to Weyl vertex algebra orbifolds.

Summary of previous results

Whittaker modules for simple Lie algebras

- Anna Romanov talk

Whittaker VOA modules

- affine VOAs - Adamovic, Lu, Zhao
- Virasoro VOAs - Mazorchuk, Zhao, Lu, Ondrus, Wiesner
- Heisenberg VOAs - Yu, Hartwig, Tanabe

Whittaker non-VOA modules

- Imaginary Whittaker modules for Heisenberg and affine algebras - K. Christodoulopoulou (not restricted)

Vertex Operator Algebra

Vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)},$$

such that

- $\mathrm{wt} v = n$, for $v \in V_{(n)}$,
- $dim V_{(n)} < \infty$, for $n \in \mathbb{Z}$,
- $V_{(n)} = 0$, for n sufficiently small,

equipped with a linear map

$$V \to (End V)[[x, x^{-1}]]$$
$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1},$$

$Y(v, x)$ denoting the vertex operator associated with v,

and equipped also with two distinguished homogenous vectors $\mathbf{1} \in V_{(0)}$ (the vacuum) and $\omega \in V_{(2)}$.

Vertex Operator Algebra

The following conditions are assumed for $u, v \in V$:

- $u_n v = 0$ for n sufficiently large,
- $Y(\mathbf{1}, x) = 1$,
- $Y(v, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \to 0} Y(v, x)\mathbf{1} = v$,
- Jacobi identity holds:

$$x_0^{-1}\delta(\frac{x_1 - x_2}{x_0})Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta(\frac{x_2 - x_1}{-x_0})Y(v, x_2)Y(u, x_1)$$
$$= x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})Y(Y(u, x_0)v, x_2).$$

Also, the Virasoro algebra relations hold (acting on V):

$$[L(m), L(n)] = (m - n)L(n + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}(\mathrm{rk} V)\mathbf{1},$$

for $m, n \in \mathbb{Z}$, where $L(n) = \omega_{n+1}$ i.e., $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$

and rk $V \in \mathbb{C}$, $L(0)v = nv = (\mathrm{wt}\ v)v$ for $n \in \mathbb{Z}$ and $v \in V_{(n)}$,
$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x)$.

Weak module for a VOA

A **weak V–module** is a pair (M, Y_M) where M is a complex vector space, and $Y_M(\cdot, z)$ is a linear map

$$\begin{aligned} Y_M : \; V &\to \mathrm{End}(M)[[z, z^{-1}]], \\ a &\mapsto Y_M(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \end{aligned}$$

satisfying the following conditions on $a, b \in V$ and $v \in M$:

- $a_n v = 0$ for n sufficiently large.
- $Y_M(\mathbf{1}, z) = I_M$.
- The following Jacobi identity holds:
$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_M(a,z_1)Y_M(b,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y_M(b,z_2)Y_M(a,z_1)$
$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y_M(Y(a,z_0)b,z_2)$

Ordinary module for a VOA

Note: every weak V–module (M, Y_M) is a module for the Virasoro algebra generated by components of the field

$$Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$

Module (M, Y_M) is called **ordinary** if $L(0)$ acts diagonally on finite dimensional weight spaces + some extra conditions on grading.

Dong-Mason theorem

Let V be a VOA. An automorphism g of a VOA V is a mapping $V \to V$ such that $g(a_n b) = g(a)_n g(b)$, for all $a, b \in V$, $n \in \mathbb{Z}$ and such that $g(\omega) = \omega$.

Let M be a weak module for the VOA V. We define the composition $M \circ g$ to be the module whose vertex operator is given by $Y_{M \circ g}(v, z) = Y(gv, z)$, for all $v \in V$.

Theorem (Dong-Mason (1997))

Let (M, Y_M) be an irreducible **ordinary** *module for the vertex operator algebra V. Let g be an automorphism of V of prime order p, such that $M \circ g \not\cong M$. Then M is an irreducible module for the orbifold subalgebra V^g.*

Main goal: extend this result to **weak** modules for VOAs.

Main result

Let g be an automorphism of order n. Let W be a V–module. We define

$$\mathcal{M} = W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1},$$

where $W_i = W \circ g^i, i = 0, 1, \cdots, n-1$.

Lemma

Let W be an irreducible V–module and $\mathcal{M}$ be as above. Assume that $(w, \ldots, w)$ is cyclic in $\mathcal{M}$, for every $w \in W$, $w \neq 0$. Then W is an irreducible $V^{\langle g \rangle}$–module, where $V^{\langle g \rangle} = \{v \in V \mid gv = v\}$.

Theorem

Let W be an irreducible V–module, and g an automorphism of finite order n such that $W \circ g^i \not\cong W$ for $i = 1, \ldots, n-1$. Then W is an irreducible $V^{\langle g \rangle}$–module.

Order 2 case

Let θ be an automorphism of order two of V. Let

$$V^+ = \{v \in V \mid \theta(v) = v\}, \quad V^- = \{v \in V \mid \theta(v) = -v\}.$$

Then V^+ is a vertex subalgebra of V and V^- is a V^+-module.

Lemma

Let L_i, $i = 1, \dots, t$, be non-isomorphic irreducible weak V–modules and $\mathcal{L} = \oplus_{i=1}^{t} L_i$. Then for each $w_i \neq 0$, $w_i \in L_i$, vector $(w_1, w_2, \dots, w_t)$ is cyclic in $\mathcal{L}$.

Theorem

Let W be an irreducible weak V– module such that $W \circ \theta \ncong W$. Then W is an irreducible V^+–module.

Order 2 case: proof

Consider $V-$ module $\mathcal{M} = W \oplus W_\theta$. Let us define

$$\Delta^{\pm} : W \to \mathcal{M}, \quad w \mapsto (w, \pm w).$$

Then $\Delta^{\pm}$ are $V^{+}-$ homomorphisms and we have

$$\mathcal{M} = \Delta^{+}(W) \bigoplus \Delta^{-}(W).$$

Also,

$$V^{+}.\Delta^{+}(W) = \Delta^{+}(W)$$

and

$$V^{-}.\Delta^{+}(W) = \Delta^{-}(W)$$

Assume W is not an irreducible V^{+}- module and $0 \neq S \subsetneqq W$. In particular, $0 \neq \Delta^{+}(S) \subsetneqq \Delta^{+}(W)$.

On the other hand, $V.\Delta^{+}(S) = \mathcal{M}$. Since

$$V^{\pm}.\Delta^{+}(S) \subset \Delta^{\pm}(W),$$

we must have $V^{+}.\Delta^{+}(S) = \Delta^{+}(W)$. Contradiction.

Whittaker modules for Lie algebras: general case

Let $\mathfrak{g}$ be a (possibly infinite dimensional) Lie algebra with triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+.$$

Let $\lambda \in \mathfrak{n}_+^*$ be a Lie homomorphism.

The universal Whittaker module with Whittaker function λ is defined as

$$\widetilde{W}(\lambda) = U(\mathfrak{g})/J(\lambda),$$

where $J(\lambda) = U(\mathfrak{g}).\{x - \lambda(x)1 \mid x \in \mathfrak{n}_+\}$. (= the left ideal generated by $\{x - \lambda(x)1 \mid x \in \mathfrak{n}_+\}$).

If the simple quotient of $\widetilde{W(\lambda)}$ is unique, we denote it by $W(\lambda)$.

<u>Problem:</u> Determine the structure of $W(\lambda)$.

Affine Lie algebras

Let $\mathfrak{g}$ be a finite - dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ be the associated triangular decomposition.

The (untwisted) affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$.

For example, if $\mathfrak{g} = sl_{n+1}$, then we say that $\widehat{\mathfrak{g}}$ is of type $A_n^{(1)}$, where K is the canonical central element and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}K.$$

We say that M is a $\widehat{\mathfrak{g}}$**–module of level k** if the central element K acts on M as a multiplication with k.

We have the following triangular decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- + \widehat{\mathfrak{h}} + \widehat{\mathfrak{n}}_+,$$

$$\widehat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}], \ \widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K.$$

Affine Lie algebra $A_1^{(1)}$

Let $\mathfrak{g} = sl_2$ with the standard basis e, f, h. The corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ is of type $A_1^{(1)}$.

Then $\widehat{\mathfrak{n}}_+$ is generated as a Lie algebra by

$$e_0 = e \otimes t^0; e_1 = f \otimes t^1.$$

Lie algebra homomorphism $\lambda : \widehat{\mathfrak{n}}_+ \to \mathbb{C}$ is uniquely determined by

$$(\lambda_1, \lambda_2) = (\lambda(e_0), \lambda(e_1)).$$

Let $\widetilde{W}(k, \lambda_1, \lambda_2)$ and $W(k, \lambda_1, \lambda_2)$ denote the universal and simple Whittaker modules of level k and type (λ_1, λ_2). If $\lambda_1 \cdot \lambda_2 \neq 0$, then the Whittaker module $W(k, \lambda_1, \lambda_2)$ is called **non-degenerate**.

The universal affine vertex algebra $V^k(\mathfrak{g})$

Let us set $x(n) = x \otimes t^n$, for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, and identify $\mathfrak{g}$ as the subalgebra $\mathfrak{g} \otimes t^0$.

Let $\mathbb{C}1_k$ be the 1-dimensional $\hat{\mathfrak{g}}_{\geq 0} + \mathbb{C}K$–module on which $\hat{\mathfrak{g}}_{\geq 0} = \mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially and K acts as multiplication by the complex number k.

Define the following generalized Verma module

$$V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0} + \mathbb{C}K)} \mathbb{C}.1_k$$

$V^k(\mathfrak{g})$ has the structure of a vertex algebra which is uniquely determined by the fields

$$x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1} \quad (x \in \mathfrak{g}).$$

$V^k(\mathfrak{g})$ has the role of the universal affine vertex algebra of level k.

If the simple quotient is unique, we denote it by $V_k(\mathfrak{g})$.

Whittaker modules for $A_1^{(1)}$

The following result gives the irreducibility of non-degenerate Whittaker modules:

Theorem (Adamovic-Lu-Zhao, Advances in Math. (2016))

Assume that $k \neq -2$ and $\lambda_1 \cdot \lambda_2 \neq 0$. Then the universal Whittaker module $\widetilde{W}(k, \lambda_1, \lambda_2)$ is an irreducible $\widehat{\mathfrak{sl}(2)}$–module.

Problem: Find a higher rank generalization of Adamovic-Lu-Zhao theorem.

Application to the affine Lie algebra $\widehat{\mathfrak{sl}(2)}$

Let $\widetilde{W}(k, \lambda_1, \lambda_2)$ be the universal Whittaker module for the affine Lie algebra $\widehat{\mathfrak{sl}(2)}$, where k is not the critical level, and λ_1, λ_2 are Whittaker functions such that $\lambda_1 \cdot \lambda_2 \neq 0$. Then $\widetilde{W}(k, \lambda_1, \lambda_2)$ is irreducible as a $V^k(\mathfrak{sl}(2))^{\langle\theta\rangle}$–module, where θ is an automorphism of order 2, uniquely determined by the folllowing relations:

$$e(n) \mapsto -e(n)$$

$$f(n) \mapsto -f(n)$$

$$h(n) \mapsto h(n)$$

Weyl vertex algebra

Weyl algebra $\widehat{\mathcal{A}}$ is an associative algebra with generators $a(n), a^*(n)$, $n \in \mathbb{Z}$, and relations (for $n, m \in \mathbb{Z}$)

$$[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0.$$

Let M be a simple *Weyl module* generated by the cyclic vector $\mathbf{1}$ such that

$$a(n)\mathbf{1} = a^*(n+1)\mathbf{1} = 0. \quad (n \geq 0).$$

As a vector space,

$$M \;\cong\; \mathbb{C}[a(-n), a^*(-m) \mid n > 0,\; m \geq 0].$$

There is a unique vertex algebra $(M, Y, \mathbf{1})$ where the vertex operator is given by $Y : M \to \mathrm{End}(M)[[z, z^{-1}]]$, such that

$$Y(a(-1)\mathbf{1}, z) = a(z), \qquad Y(a^*(0)\mathbf{1}, z) = a^*(z),$$
$$a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}, \qquad a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n) z^{-n}.$$

Whittaker module for the Weyl algebra

Whittaker module for $\widehat{\mathcal{A}}$ is the quotient

$$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \widehat{\mathcal{A}}/I,$$

where $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and I is the left ideal

$$I = \left\langle a(0)-\lambda_0, \ldots, a(n)-\lambda_n, a^*(1)-\mu_1, \ldots, a^*(n)-\mu_n, a(n+1), a^*(n+1), \ldots \right\rangle.$$

Proposition

We have:

(*i*) *$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible $\widehat{\mathcal{A}}$–module.*

(*ii*) *$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible weak module for the Weyl VOA M.*

Irreducible module for orbifold subalgebras

Let $\zeta_n = e^{2\pi i/n}$ be the primitive n-th root of unity. Let g_n be the automorphism of vertex algebra M, uniquely determined by the following automorphism of Weyl algebra $\widehat{\mathcal{A}}$:

$$a(n) \mapsto \zeta_n a(n), \quad a^*(n) \mapsto \zeta_n^{-1} a^*(n) \quad (n \in \mathbb{Z}).$$

Then g_n is the automorphism of M of order n.

Theorem

Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible module for the orbifold subalgebra $M^{\mathbb{Z}_n} = M^{\langle g_n \rangle}$ for each $n \geq 1$.

Example of irreducible modules: $n = 2$

Corollary

Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$. Then $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible module for the affine Lie algebra $\widehat{sl(2)}$ at the level $k = -\frac{1}{2}$.

Proof.
For $n = 2$, $M^{\mathbb{Z}_2}$ is isomorphic to the affine VOA $V_{-\frac{1}{2}}(sl(2))$. Therefore, module $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is irreducible for $V_{-\frac{1}{2}}(sl(2))$. $\square$

Example
For $k = -\frac{1}{2}$, $\widetilde{W}(k, \lambda_1, \lambda_2)$, $\lambda_1 \cdot \lambda_2 \neq 0$ is an irreducible module only for universal affine VOA $V^k(sl(2))$.

However, $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a module for the simple VOA $V_k(sl(2))$.

Thank you!