Conformal embeddings in basic classical Lie superalgebras

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Definitions

Vertex operator algebra

A (super) Vertex Operator Algebra is a vertex algebra V equipped with a Virasoro vector ω_V, $(\omega_V)_0$ is diagonalizable with (half) integer eigenvalues and its spectrum is bounded below.

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Conformal embedding

A conformal embedding is a homomorphism of vertex operator algebras: it is an embedding $\phi : V \to W$ of vertex algebras such that $\phi(\omega_V) = \omega_W$.

Basic classical Lie superalgebras

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a basic classical simple Lie superalgebra if

- The even part $\mathfrak{g}_{\bar{0}}$ is reductive
- $\mathfrak{g}$ admits a non-degenerate invariant supersymmetric bilinear form $(\cdot|\cdot)$.

Basic example of conformal embedding

- $\mathfrak{g}$ basic classical Lie superalgebra and $k \in \mathbb{C}$.
- $h^\vee$ the dual Coxeter number of $\mathfrak{g}$ w.r.t. $(\cdot|\cdot)$.
- $V^k(\mathfrak{g})$ level k universal affine vertex algebra.
- $V_k(\mathfrak{g})$ level k simple affine vertex algebra.

If $k \neq -h^\vee$, both $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g})$ are vertex operator algebras with Virasoro vector given by Sugawara construction:

$$\omega_{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum : x^i x_i : .$$

($\{x_i\}$, $\{x^i\}$ dual bases of $\mathfrak{g}$ w.r.t the chosen invariant form).

Basic example continued

- $\mathfrak{g}^0$ quadratic Lie subalgebra of $\mathfrak{g}$ (i.e. $(\cdot|\cdot)_{|\mathfrak{g}^0\times\mathfrak{g}^0}$ is nondegenerate).
- Further assume that $\mathfrak{g}^0 = \mathfrak{g}^0_0 \oplus \cdots \oplus \mathfrak{g}^0_s$ with $\mathfrak{g}^0_0$ even abelian and $\mathfrak{g}^0_i$ basic classical simple ideals for $i > 0$.

Define $\widetilde{V}(\mathfrak{g}^0)$ the vertex subalgebra of $V_k(\mathfrak{g})$ generated by $x(-1)\mathbf{1}$, $x \in \mathfrak{g}^0$.

Basic conformal embedding

Since $\widetilde{V}(\mathfrak{g}^0)$ is a quotient of a universal affine vertex algebra, then it is a vertex operator algebras with Virasoro vector $\omega_{\mathfrak{g}^0}$ given by Sugawara construction.

The embedding $\widetilde{V}(\mathfrak{g}^0) \hookrightarrow V_k(\mathfrak{g})$ is a conformal embedding if

$$\omega_{\mathfrak{g}^0} = \omega_{\mathfrak{g}}.$$

Problems

Three general problems:

- Classification problem: find all conformal embeddings $\widetilde{V}(\mathfrak{g}^0) \hookrightarrow V_k(\mathfrak{g})$
- Simplicity problem: determine whether $\widetilde{V}(\mathfrak{g}^0)$ is simple
- Decomposition problem: describe $V_k(\mathfrak{g})$ as a $\widetilde{V}(\mathfrak{g}^0)$–module

Classification problem: AP-criterion

The main tool for detecting conformal embeddings is AP-criterion.

$\mathfrak{g}^0 = \mathfrak{g}^0_0 \oplus \cdots \oplus \mathfrak{g}^0_t$ quadratic subalgebra of $\mathfrak{g}$; let $\mathfrak{g}^1$ be its orthocomplement in $\mathfrak{g}$. Assume that $\mathfrak{g}^1$ is completely reducible as a $\mathfrak{g}^0$-module, and let

$$\mathfrak{g}^1 = \bigoplus_{i=1}^{t} V_{\mathfrak{g}^0}(\mu_i)$$

be its decomposition.

Theorem (Adamovic-Perse)

$\widetilde{V}(\mathfrak{g}^0)$ *is conformally embedded in* $V_k(\mathfrak{g})$ *if and only if*

$$\sum_{j=0}^{t} \frac{(\mu_i^j, \mu_i^j + 2\rho_0^j)_j}{2(k_j + h_j^\vee)} = 1$$

Application of AP-criterion to the embedding $\widetilde{V}(\mathfrak{g}_{\bar{0}}) \hookrightarrow V_k(\mathfrak{g})$

If $\widetilde{V}_{(}\mathfrak{g}_{\bar{0}})$ embeds conformally in $V_k(\mathfrak{g})$ we call k a conformal level.

1. If $\mathfrak{g} = sl(m|n)$, $m > n$, the conformal levels are $k = 1$, $k = -1$ if $m \neq n+1$, $k = \frac{n-m}{2}$;
2. If $\mathfrak{g} = psl(m|m)$, the conformal levels are $k = 1, -1$;
3. If $\mathfrak{g}$ is of type $B(m,n)$, the conformal levels are $k = 1, \frac{3-2m+2n}{2}$;
4. If $\mathfrak{g}$ is of type $D(m,n)$, the conformal levels are $k = 1, 2-m+n$;
5. If $\mathfrak{g}$ is of type $C(n+1)$, the conformal levels are $k = -\frac{1}{2}, -\frac{1+n}{2}$;
6. If $\mathfrak{g}$ is of type $F(4)$, the conformal levels are $k = 1, -\frac{3}{2}$;
7. If $\mathfrak{g}$ is of type $G(3)$, the conformal levels are $k = 1, -\frac{4}{3}$;
8. If $\mathfrak{g}$ is of type $D(2,1,a)$, the conformal levels are $k = 1, -1-a, a$;

Other cases of conformal embedding

Consider the embeddings $\mathfrak{g}^0 \subset \mathfrak{g}$ with

- $gl(m|n) \subset sl(n+1|m)$,
- $sl(2) \times spo(2|3) \subset G(3)$.

Theorem

(1) Assume $n \neq m, m-1$. The conformal levels for the embedding $gl(n|m) \subset sl(n+1|m)$ are $k=1$ and $k=-\frac{n+1-m}{2}$.
(2) The conformal levels for the embedding $sl(2) \times spo(2|3) \subset G(3)$ are $k=1$ and $k=-4/3$.

Solving the simplicity and decomposition problems: the dot product

If U, W are subspaces in a vertex algebra then

$$U \cdot W = span(u(n)w \mid u \in U,\ w \in W,\ n \in \mathbb{Z}).$$

The dot product is associative

$$U \cdot (W \cdot Z) = (U \cdot W) \cdot Z.$$

and, if the subspaces are T-stable, commutative

$$U \cdot W = W \cdot U.$$

The dot product in a simple vertex algebra does not have zero divisors: if $U \cdot V = \{0\}$ then either $U = \{0\}$ or $W = \{0\}$.

Fusion rules argument

Suppose that $W \subset V$ is an embedding of vertex algebras. Let $\mathcal{M}$ be a collection of W–submodules of V that generates V as a vertex algebra. Then the structure of $span(\mathcal{M})$ under the dot product in the set of all W–submodules gives information about the simplicity and decomposition problem.

If the embedding is conformal then there are constraints that allow in many cases to recover the structure of $span(\mathcal{M})$ and solve the simplicity and sometimes also the decomposition problem.

Enhanced fusion rules argument

If V is semisimple as a W–module and M_1, M_2 are simple components, then projecting onto a simple component of $M_1 \cdot M_2$ defines an intertwining operator of type $\begin{bmatrix} M_3 \\ M_1 \ M_2 \end{bmatrix}$.

If the fusion coefficients $\dim \begin{bmatrix} M_3 \\ M_1 \ M_2 \end{bmatrix}$ are known then this gives further constraints for the computation of the structure of $span(\mathcal{M})$.

Application of f.r.a. to the case of $\mathfrak{g}^0$ fixed point set of an automorphism

Assume

- $\mathfrak{g}_0^0 = \{0\}$
- $\mathfrak{g}^0$ is the set of fixed points an automorphism σ of $\mathfrak{g}$ of order s and let $\mathfrak{g} = \oplus_{i \in \mathbb{Z}/s\mathbb{Z}} \mathfrak{g}^{(i)}$ be the corresponding eigenspace decomposition.
- Since $\mathfrak{g}^1$ is assumed to be completely reducible as $\mathfrak{g}^0$-module, we have

$$\mathfrak{g}^{(i)} = \sum_r V(\mu_r),$$

The map σ can be extended to a finite order automorphism of the simple vertex algebra $V_k(\mathfrak{g})$ which induces the eigenspace decomposition

$$V_k(\mathfrak{g}) = \oplus_{i \in \mathbb{Z}/s\mathbb{Z}} V_k(\mathfrak{g})^{(i)}.$$

Fusion rules argument

Theorem

Assume that, if ν is the weight of a $\mathfrak{g}^0$-primitive vector occurring in $V(\mu_i) \otimes V(\mu_j)$, then there is a $V^k(\mathfrak{g}^0)$-primitive vector in $V_k(\mathfrak{g})$ of weight ν if and only if $\nu = \mu_r$ for some r.
Then $\widetilde{V}(\mathfrak{g}^0)$ is simple and

$$V_k(\mathfrak{g}) = V_{\mathbf{k}}(\mathfrak{g}^0) \oplus (\oplus_{i=1}^{t} L_{\mathfrak{g}^0}(\mu_i)).$$

Proof: Set $M_i = \widetilde{V}(\mathfrak{g}^0) \cdot V(\mu_i)$. The hypothesis implies that $M_i \cdot M_j \subset M_r$ so $\sum M_i$ is a vertex algebra. This implies that $V_k(\mathfrak{g}) = \sum M_i$ so

$$M_i = V_k(\mathfrak{g})^{(i)}.$$

Numerical criterion

The fact that $\widetilde{V}(\mathfrak{g}^0)$ embeds conformally in $V_k(\mathfrak{g})$ leads to an easy sufficient condition for the hypothesis of the previous theorem to hold.

Let c_ν be the eigenvalue of $(\omega_{\mathfrak{g}^0})_0$ on the highest weight vector of $L_{\mathfrak{g}^0}(\nu)$. Since

$$\omega_{\mathfrak{g}} = \omega_{\mathfrak{g}^0},$$

the hypothesis of the previous theorem holds whenever for all primitive vectors of weight ν occurring in $V_{\mathfrak{g}^0}(\mu_i) \otimes V_{\mathfrak{g}^0}(\mu_j)$, one has that either $\nu = \mu_r$ for some r or $c_\nu \notin \mathbb{Z}_+$.

Examples

Applying the numerical criterion one obtains:

$$
\begin{aligned}
(1)\ V_{-4/3}(G(3)) &= V_1(sl(2)) \otimes V_{-4/3}(G_2) \oplus L_{sl(2)}(\omega_1) \otimes L_{G_2}(\omega_1),\\
(2)\ V_{-3/2}(F(4)) &= V_1(sl(2)) \otimes V_{-3/2}(so(7)) \oplus L_{sl(2)}(\omega_1) \otimes L_{so(7)}(\omega_3),\\
(3)\ V_1(B(m,n)) &= V_1(so(2m+1)) \otimes V_{-1/2}(sp(2n))\\
&\oplus L_{so(2m+1)}(\omega_1) \otimes L_{sp(2n)}(\omega_1), m \neq n,\\
(4)\ V_1(D(m,n)) &= V_1(so(2m)) \otimes V_{-1/2}(sp(2n))\\
&\oplus L_{so(2m)}(\omega_1) \otimes L_{sp(2n)}(\omega_1),\ \ m \neq n+1,
\end{aligned}
$$

The case of $\mathfrak{g}$ of type $D(n+1,n)$

In this case the numerical criterion fails, but the decomposition still holds:

$$V_1(D(n+1,n)) = V_1(so(2n+2)) \otimes V_{-1/2}(sp(2n)) \\ \oplus L_{so(2n+2)}(\omega_1) \otimes L_{sp(2n)}(\omega_1).$$

One has to work harder to show that primitive vectors of weight $(\omega_2, 2\omega_1)$ and $(2\omega_1, \omega_2)$ cannot occur.

$osp(m|2n)$

Consider the superspace $\mathbb{C}^{m|2n}$ equipped with the standard supersymmetric form $\langle \cdot, \cdot \rangle$. Let $F_{m|2n}$ be the universal vertex algebra generated by $\mathbb{C}^{m|2n}$ with λ–bracket

$$[v_\lambda w] = \langle w, v \rangle.$$

Let $\{e_i\}$ be the standard basis of $\mathbb{C}^{m|2n}$ and let $\{e^i\}$ be its dual basis with respect to $\langle \cdot, \cdot \rangle$ (i. e. $\langle e_i, e^j \rangle = \delta_{ij}$).
There is a non-trivial homomorphism $\Phi : V^1(osp(m|2n)) \to F_{m|2n}$ uniquely determined by

$$X \mapsto 1/2 \sum_i : X(e_i) e^i :, \quad X \in osp(m|2n).$$

Set $\tilde{V}(osp(m|2n))$ the image of Φ. Embed $\mathbb{C}^{m|2n}$ in $F_{m|2n}$ via $v \mapsto v(-1)\mathbf{1}$ and set

$$M = \tilde{V}(osp(m|2n)) \cdot \mathbb{C}^{m|2n}.$$

Free field realization of $V_1(osp(m|2n))$

The map Φ induces a conformal embedding $V_1(osp(m|2n)) \hookrightarrow F_{m|2n}$ and one has the decomposition

$$F_{m|2n} = V_1(osp(m|2n)) \oplus L_{osp(m|2n)}(\mathbb{C}^{m|2n}).$$

(free field realization of $V_1(osp(m|2n))$ due to Kac-Wakimoto)

Proof: a generalization of AP-criterion reduces the check that the embedding is conformal to the check that, if $\lambda_{m|2n}$ is the highest weight of $\mathbb{C}^{m|2n}$, then

$$\frac{(\lambda_{m|2n}, \lambda_{m|2n} + 2\rho)}{2(1 + h^\vee)} = \frac{1}{2}$$

Fusion rules argument for free field realization

The map $-Id$ on $\mathbb{C}^{m|2n}$ induces an involution of $F_{m|2n}$. Write

$$F_{m|2n} = F^+ \oplus F^-$$

for the eigenspace decomposition. Clearly $\tilde{V}(osp(m|2n)) \subset F^+$ and $M \subset F^-$. If

$$M \cdot M \subset \tilde{V}(osp(m|2n))$$

then $\tilde{V}(osp(m|2n)) + M$ is a vertex algebra and, since it generates $F_{m|2n}$, $\tilde{V}(osp(m|2n)) + M = F_{m|2n}$ so

$$\tilde{V}(osp(m|2n)) = F^+, \quad M = F^-.$$

$M \cdot M \subset \tilde{V}(osp(m|2n))$

To check that $M \cdot M \subset \tilde{V}(osp(m|2n))$ we compute the composition factors of $\mathbb{C}^{m|2n} \otimes \mathbb{C}^{m|2n}$.
In δ, ϵ notation for roots and weights these are:

n,m	highest weights of composition factors
$n \geq 2,\ m \geq 2$	$2\delta_1,\ \delta_1 + \delta_2,\ 0$
$n = 1,\ m > 2$	$2\delta_1,\ \delta_1 + \epsilon_1,\ 0$
$n = 1,\ m = 2$	$2\delta_1,\ \delta_1 + \epsilon_1,\ \delta_1 - \epsilon_1,\ 0$
$n = 1,\ m = 1$	$2\delta_1,\ \delta_1,\ 0$

One checks that, for λ in the list above, $\frac{(\lambda,\lambda+2\rho)}{2(k+h^\vee)} \in \mathbb{Z}_+$ iff $\lambda = 0$.

Application: realization of $osp(2m|2n)$ at level -2

Consider $\mathbb{C}^{0|2} \otimes \mathbb{C}^{2m|2n} \simeq \mathbb{C}^{4n|4m}$. This isomorphism induces an embedding

$$sl(2) \times osp(2m|2n) \hookrightarrow osp(4n|4m)$$

hence homomorphisms

$$V^{-2}(osp(2m|2n)) \to \widetilde{V}(sl(2) \times osp(2m|2n)) \subset V_1(osp(4n|4m)) \hookrightarrow F_{4n|4m}.$$

Let $\tilde{V}(osp(2m|2n)$ be the image of the first homomorphism.

Restricting to the even part we have a homomorphism

$$V^{-2}(so(2m)) \otimes V_1(sp(2n)) \to \tilde{V}(osp(2m|2n))$$

Let $\tilde{V}(so(2m)) \otimes V_1(sp(2n))$ be its image.

Special case: $osp(2n+8|2n)$

The algebras $\tilde{V}(so(2n+8))$ and $\tilde{V}(osp(2m|2n))$ are not simple.

The embedding

$$\tilde{V}(so(2n+8)) \otimes V_1(sp(2n)) \hookrightarrow \tilde{V}(osp(2n+8|2n))$$

is not conformal and the coset Virasoro has central charge 0

We have a homomorphism

$$\tilde{V}(so(2n+8)) \otimes V_1(sp(2n)) \to V_{-2}(osp(2n+8|2n))$$

Let $\overline{V}(so(2n+8)) \otimes V_1(sp(2n)) \hookrightarrow V_{-2}(osp(2n+8|2n))$ be its image.

The embedding $\overline{V}(so(2n+8)) \otimes V_1(sp(2n)) \hookrightarrow V_{-2}(osp(2n+8|2n))$

- The embedding $\overline{V}(so(2n+8)) \otimes V_1(sp(2n)) \hookrightarrow V_{-2}(osp(2n+8|2n))$ is conformal.
- $\overline{V}(so(2n+8)) \otimes V_1(sp(2n))$ is simple
- The action of $\overline{V}(so(2n+8)) \otimes V_1(sp(2n)) = V_{-2}(so(2n+8)) \otimes V_1(sp(2n))$ on $V_{-2}(osp(2n+8|2n))$ is semisimple

Proof

- The embedding is conformal by AP-criterion.
- $\overline{V}(so(2n+8)) \otimes V_1(sp(2n))$ is simple by a nice application of the fusion rules argument: set U to be the submodule of $V_{-2}(osp(2n+8|2n)$ generated by $\mathfrak{g}_{\bar{1}}$.
  It is known that
  $$V_{-2}(so(2n+8)) = \tilde{V}(so(2n+8))/\langle v\rangle$$
  with v an explicit singular vector. Let V be the submodule generated by v.
  One shows that there is r such that $U^r \cdot V = 0$. Since $U \neq 0$, $V = 0$ in $V_{-2}(osp(2n+8|2n)$.
- The action of $V_{-2}(so(2n+8)) \otimes V_1(sp(2n))$ is semisimple because Kazhdan-Lusztig category for $V_{-2}(so(2n+8))$ is semisimple at level -2.

Decomposition of $V_{-2}(osp(2n+8|2n)$

As $V_{-2}(so(2n+8)) \otimes V_1(sp(2n))$-module

$$V_{-2}(osp(2n+8|2n)) = \bigoplus_{i=0}^{n} V_{-2}(i\omega_1) \otimes V_1(\omega_i).$$

Proof: One constructs singular vectors w_i of the correct weight. Set W_i to be the submodule generated by w_i.
We have the following dot product inside of $V_{-2}(osp(2n+8|2n))$:

$$W_1 \cdot W_i = W_{i-1} \oplus W_{i+1} \quad (1 \le i \le n-1)$$
$$W_1 \cdot W_n = W_{n-1}.$$

It follows that $\sum W_i$ is a sub-vertex algebra. Since W_1 generates $V_{-2}(osp(2n+8|2n)$ we have $V_{-2}(osp(2n+8|2n) = \sum W_i$

Other examples of simplicity of $\tilde{V}(\mathfrak{g}^0)$

The fusion rules argument can be used in solving the simplicity problem also in the following example.

Consider $\tilde{V}(sl(2)\times G_2)\subset V_1(G(3))$. We have $\tilde{V}(sl(2)\times G_2)=\tilde{V}_{-3/4}(sl(2))\otimes V_1(G_2)$. If $\tilde{V}_{-3/4}(sl(2))\neq V_{-3/4}(sl(2))$ then there is a singular vector in $\tilde{V}_{-3/4}(sl(2))$ of $sl(2)$-weight $8\omega_1$.

Let $v_{n,m}$ be the set of $\mathfrak{g}_{\bar{0}}$ singular vector in $V_1(G(3))$ of $\mathfrak{g}_0$ weight $(n\omega_1, m\omega_2)$, where $n\in\mathbb{Z}_{\geq 0}$ and $m\in\{0,1\}$. Let $V_{n,m}=\widetilde{V}_1(\mathfrak{g}_{\bar{0}})\cdot v_{n,m}$.

Example continued

The fusion rules for $V_1(G_2)$ and Clebsch-Gordan decomposition imply that

$$V_{n,0} \cdot V_{1,1} \subset V_{n+1,1} + V_{n-1,1}.$$
$$V_{n,1} \cdot V_{1,1} \subset V_{n+1,1} + V_{n-1,1} + V_{n+1,0} + V_{n-1,0}.$$

By conformal embedding we can drop the summands that give noninteger conformal weight. Then it follows that

$$\begin{aligned} V_{8,0} \cdot V_{1,1} &\subset V_{7,1}, \\ V_{7,1} \cdot V_{1,1} &\subset V_{6,1} + V_{8,0}, \\ V_{6,1} \cdot V_{1,1} &\subset V_{5,0} + V_{7,1}, \\ V_{5,0} \cdot V_{1,1} &\subset V_{6,1}, \end{aligned}$$

and this implies that $V_{8,0}$ generates a proper ideal in $V_1(\mathfrak{g})$. A contradiction.

Other cases

Similar arguments give the simplicity of

- $\tilde{V}_1(so(7) \times sl(2))$ in $V_1(F(4))$
- $\tilde{V}_{-3/4}(sl(2) \times sl(2))$ in $V_{-3/4}(spo(2|3))$

Once the simplicity of $\tilde{V}_k(\mathfrak{g}^0)$ is established, then its action on $V_k(\mathfrak{g})$ is semisimple and one can hope to use fusion rules to compute the decomposition. Unfortunately fusion rules alone do not suffice in this cases.

Example: $V_{-3/4}(sl(2)) \otimes V_3(sl(2))$ in $V_{-3/4}(spo(2|3))$

Let $v_{n,m}$ be the space of $\widehat{\mathfrak{g}}_{\bar{0}}$–singular vectors in $V_{-3/4}(spo(2|3))$ of $sl(2) \times sl(2)$–weight $(n\omega_1, m\omega_1)$. Let $V_{n,m} = \widetilde{V}_k(\mathfrak{g}_{\bar{0}}) \cdot v_{n,m}$. We know that $V_{1,2} \neq \{0\}$. There are two possibilities

$$V_{1,2} \cdot V_{1,2} \subset V_{0,0}.$$

or

$$\begin{aligned}
V_{1,2} \cdot V_{1,2} &\subset V_{0,0} + V_{2,2} \\
V_{1,2} \cdot V_{2,2} &\subset V_{3,0} + V_{1,2}, \\
V_{1,2} \cdot V_{3,0} &\subset V_{2,2} \\
V_{2,2} \cdot V_{2,2} &\subset V_{0,0} + V_{2,2} \\
V_{2,2} \cdot V_{3,0} &\subset V_{3,0} + V_{1,2} \\
V_{3,0} \cdot V_{3,0} &\subset V_{0,0}
\end{aligned}$$

Example continued

To check the correct structure of $span(V_{1,2})$ under dot product it is enough to check if $V_{3,0} \neq \{0\}$. Let $V^{-3/4}(\mathfrak{g})_{3,0,3}$ be the space of vectors in $V^{-3/4}(\mathfrak{g})$ of weight $(3\omega_1, 0)$ and conformal weight 3. The maximal ideal of $V^{-3/4}(\mathfrak{g})$ intersects $V^{-3/4}(\mathfrak{g})_{3,0,3}$ in a one dimensional subspace that we compute explicitly and we can then find a singular vector in $V^{-3/4}(\mathfrak{g})_{3,0,3}$ and check that it is not in the maximal ideal.

We carry out a similar argument for $V_1(F(4))$ using an explicit computation of the structure constants of $F(4)$.

Outcome

We obtain the following decomposition

$$V_{-3/4}(spo(2|3)) = (V_{-3/4}(sl(2)) \oplus L_{sl(2)}(3\omega_1)) \otimes V_3(sl(2)) \bigoplus (L_{sl(2)}(\omega_1) \oplus L_{sl(2)}(2\omega_1)) \otimes L_{sl(2)}(2\omega_1).$$

$$V_1(F(4)) = \quad V_1(so(7)) \otimes V_{-\frac{2}{3}}(sl(2)) \bigoplus L_{so(7)}(\omega_3) \otimes L_{sl(2)}(\omega_1) \bigoplus L_{so(7)}(\omega_1) \otimes L_{sl(2)}(2\omega_1).$$

Remarks

An easy argument of Creutzig shows how to obtain the decomposition of $V_1(G(3))$ from the decomposition for $V_{-3/4}(spo(2|3))$. Namely

$$\begin{aligned} V_1(G(3)) = \quad & \left(V_{-\frac{3}{4}}(sl(2)) \oplus L_{sl(2)}(3\omega_1)\right) \otimes V_1(G_2) \\ & \bigoplus \left(L_{sl(2)}(\omega_1) \oplus L_{sl(2)}(2\omega_1)\right) \otimes L_{G_2}(\omega_1) \end{aligned}$$

The $F(4)$ and $spo(2|3)$ decompositions appear in T. Creutzig's RIMS lecture notes. There the crucial fact that $V_{3,0} \neq \{0\}$ is obtained using vertex tensor category theory and Huang-Kirillov-Lepowski extension theory.

Application of f.r.a. to the case of $\mathfrak{g}_0^0 = \mathbb{C}\varpi$

Assume that $\mathfrak{g}_0^0 = \mathbb{C}\varpi$ and that $\mathfrak{g}^1$ decomposes as

$$\mathfrak{g}^1 = V_{\mathfrak{g}^0}(\mu) \oplus V_{\mathfrak{g}^0}(\mu)^*.$$

By a suitable choice of ϖ we can assume that ϖ acts as the identity on $V_{\mathfrak{g}_0}(\mu)$ and as minus the identity on its dual.

If $q \in \mathbb{Z}$, let $V_k(\mathfrak{g})^{(q)}$ be the eigenspace for the action of $\varpi(0)$ on $V_k(\mathfrak{g})$ relative to the eigenvalue q. Let $\{0, \nu_1, \cdots, \nu_m\}$ be the set of weights of $\mathfrak{g}^0$–primitive vectors occurring in $V_{\mathfrak{g}^0}(\mu) \otimes V_{\mathfrak{g}^0}(\mu)^*$.

Fusion rules argument

Theorem

Assume that $V_k(\mathfrak{g})^{(0)}$ does not contain primitive vectors of weight ν_r, where $r = 1, \ldots, m$, then

$$\widetilde{V}(\mathfrak{g}^0) \cong V_{\mathbf{k}}(\mathfrak{g}^0) = V_k(\mathfrak{g})^{(0)}$$

and $V_k(\mathfrak{g})^{(q)}$ is a simple $V_{\mathbf{k}}(\mathfrak{g}^0)$–module so that $V_k(\mathfrak{g})$ is completely reducible.
Moreover, if $M^+ = \widetilde{V}(\mathfrak{g}^0) \cdot V_{\mathfrak{g}^0}(\mu)$, $M^- = \widetilde{V}(\mathfrak{g}^0) \cdot V_{\mathfrak{g}^0}(\mu)^*$,

$$V_k(\mathfrak{g})^{(q)} = \underbrace{M^+ \cdot M^+ \cdot \ldots \cdot M^+}_{q \text{ times}} \qquad \text{if } q > 0,$$

$$V_k(\mathfrak{g})^{(q)} = \underbrace{M^- \cdot M^- \cdot \ldots \cdot M^-}_{|q| \text{ times}} \qquad \text{if } q < 0.$$

Numerical criterion

Let c_ν be the eigenvalue of $(\omega_{\mathfrak{g}^0})_0$ on the highest weight vector of $L_{\mathfrak{g}^0}(\nu)$. Since

$$\omega_{\mathfrak{g}} = \omega_{\mathfrak{g}^0},$$

the hypothesis of the previous theorem holds whenever $c_{\nu_i} \notin \mathbb{Z}_+$ for all i.

Example: $\mathfrak{g}$ of type $C(n+1)$

Here the numerical criterion suffices:

$$\begin{aligned} V_1(C(n+1)) = {} & M_c(1) \otimes V_{-1/2}(sp(2n)) \\ & \oplus \sum_{q\in\mathbb{Z}\setminus\{0\}} M_c(1,2q) \otimes V_{-1/2}(sp(2n)) \\ & \oplus \sum_{q\in\mathbb{Z}} M_c(1,2q+1) \otimes L_{sp(2n)}(\omega_1) \end{aligned}$$

First one uses the fusion rules argument to show that the action of $\tilde{V}(\mathfrak{g}_{\bar{0}})$ is semisimple, then one uses the enhanced fusion rules argument and the fact that the rules are known to compute the decomposition.

Another example: $\mathfrak{g} = sl(m|n)$

The numerical criterion suffices except when $m = n - 2$ and the decomposition is

$$V_1(sl(m|n)) = M_c(1) \otimes V_1(sl(m)) \otimes V_{-1}(sl(n))$$
$$\oplus \sum_{q \in \mathbb{Z} \setminus \{0\}} M_c(1, \sqrt{\tfrac{n-m}{nm}} qm) \otimes V_1(sl(m)) \otimes U^{(n)}_{-qm}$$
$$\oplus \sum_{j=1}^{m-1} \sum_{q \in \mathbb{Z}} M_c(1, \sqrt{\tfrac{n-m}{nm}} (qm + j)) \otimes L_{sl(m)}(\omega_j) \otimes U^{(n)}_{-qm-j}$$
$$m \neq n, n-2, m \geq 2, n \geq 3.$$

where $U_r^{(n)} = L_{sl(n)}(r\omega_1)$ if $r > 0$ and $U_r^{(n)} = L_{sl(n)}(-r\omega_{n-1})$ if $r < 0$.

$m = n - 2$

For $m = n - 2$ the numerical criterion fails but the decomposition still holds.
Indeed, if one proves that the action of $\tilde{V}(\mathfrak{g}_{\bar{0}})$ is semisimple, the one can use the enhanced fusion rules argument and compute the decomposition.
Semisimplicity follows from the free field realization of $V_1(sl(m|n))$ of Kac-Wakimoto.

The case $\mathfrak{g} = psl(m|m)$

The free field realization implies that the action of $\tilde{V}(\mathfrak{g}_{\bar{0}})$ on $V_1(psl(m|m))$ is semisimple so one can use fusion rules and obtain the decomposition for $m \geq 3$:

$$V_1(psl(m|m)) = \sum_{j=0}^{m-1} \sum_{q \in \mathbb{Z}} L_{sl(m)}(\omega_j) \otimes U^{(m)}_{-qm-j}.$$

For $m = 2$ it has been shown by T. Creutzig and D. Gaiotto that

$$V_1(psl(2|2)) = \bigoplus_{i=0}^{\infty} \Big((2i+1) V_1(sl(2)) \otimes U^{(2)}_{2i} \Big) \oplus \bigoplus_{i=0}^{\infty} \Big((2i+2) L_{sl(2)}(\omega_1) \otimes U^{(2)}_{2i+1} \Big).$$