Singularities of nilpotent Slodowy slices and collapsing levels for W-algebras

(joint work in progress with Tomoyuki Arakawa)

REPRESENTATION THEORY XVI – Dubrovnik

Anne Moreau

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Laboratoire Paul Painlevé, University of Lille

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$\rightsquigarrow$ *The local geometry of $\overline{\mathbb{O}}$ at $f \in \overline{\mathbb{O}}$ is therefore encoded in $\mathscr{S}_{\mathbb{O},f}$.*

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- More recently, Fu-Juteau-Levy-Sommers (2017) determined the generic singularities in the exceptional types.

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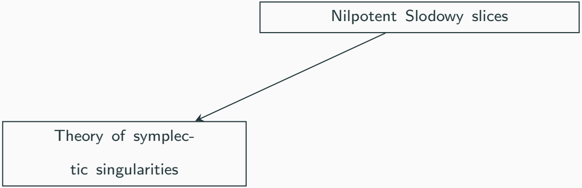

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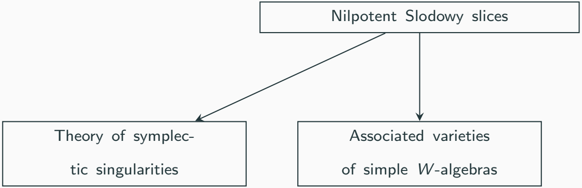

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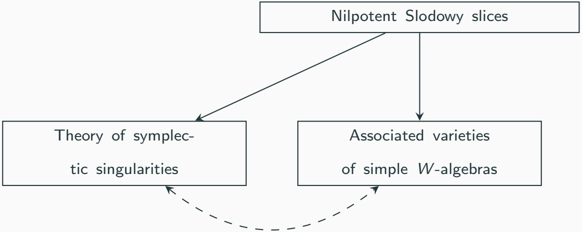

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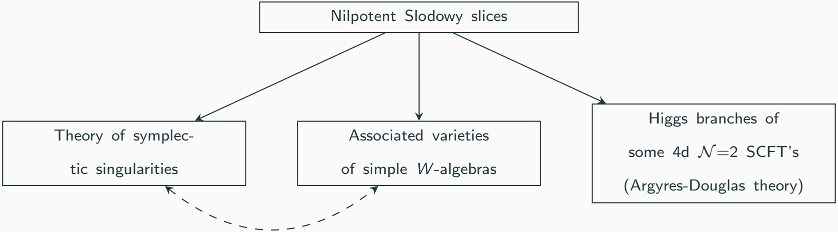

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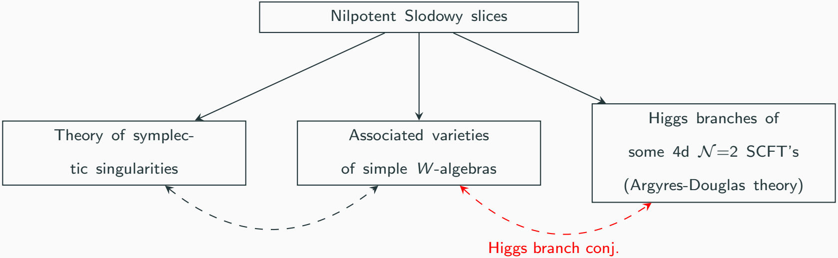

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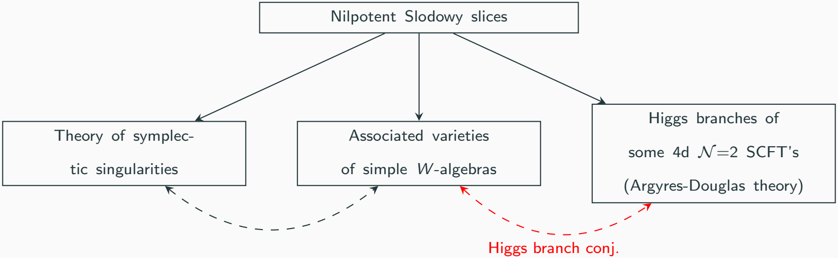

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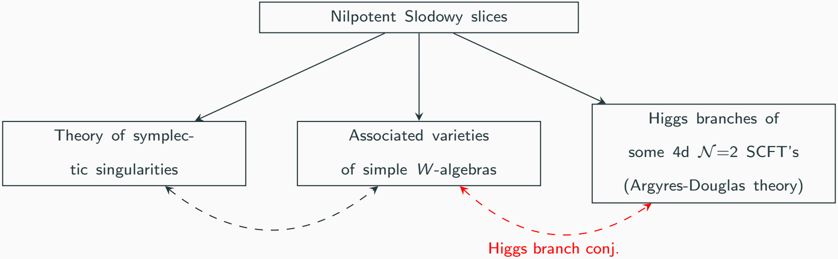

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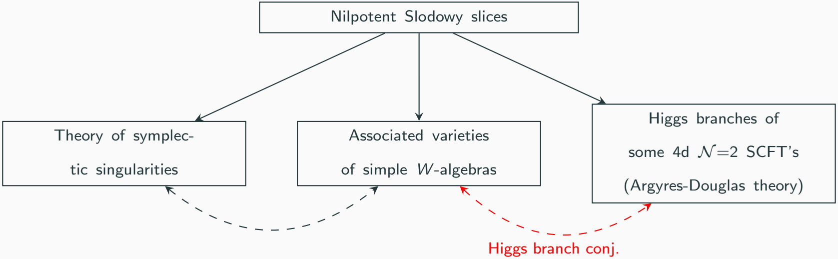

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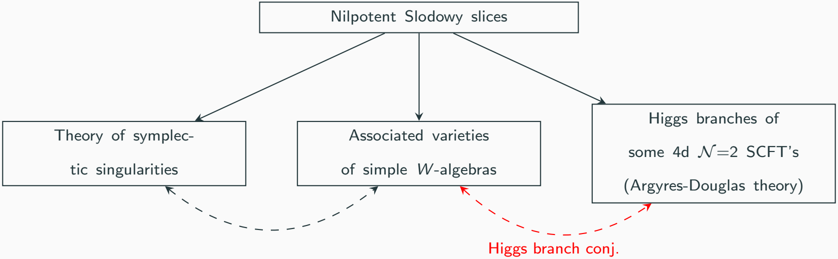

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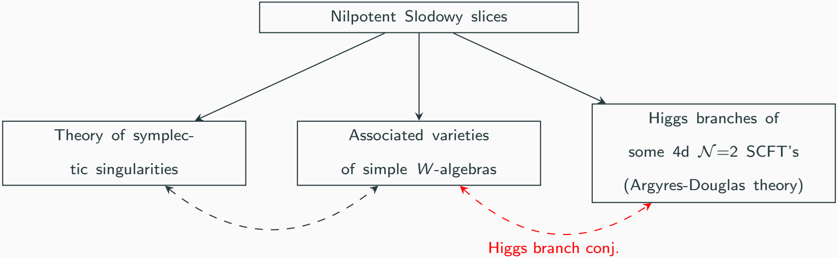

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Conjecturally (Kac-Wakimoto),

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provided that $H^0_{DS,f}(L_k(\mathfrak{g})) \neq 0$.

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For example, if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}$, then k is collapsing.

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If k is collapsing, then

$$X_{\mathcal{W}_k(\mathfrak{g},f)} \cong X_{L_{k^\natural}(\mathfrak{g}^\natural)},$$

and this is a very restrictive condition on (k, f) as we will see now...

2. Associated varieties of vertex algebras

Recall that a *vertex algebra* is a complex vector space V equipped with a distinguished vector $|0\rangle \in V$, Recall that a *vertex algebra* is a complex vector space V equipped with a distinguished vector $|0\rangle \in V$, an endomorphism $T \in \mathrm{End(V)}$

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The vertex algebra V is called *lisse* if $\dim X_V = \{0\}$.

- *The lisse condition implies for instance that V has only finitely many simple modules (Zhu 1996, Abe-Buhl-Dong 2004).*

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$\rightsquigarrow$ $X_{L_k(\mathfrak{g})}$ *is very difficult to compute in general.*

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In all the above cases, the associated variety of $L_k(\mathfrak{g})$ behaves like the associated variety of primitive ideals in the enveloping algebra.

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3. Main results

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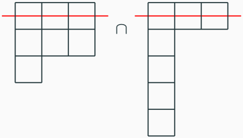

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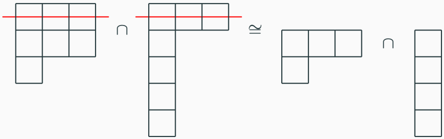

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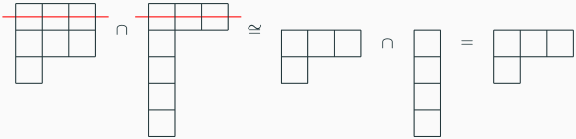

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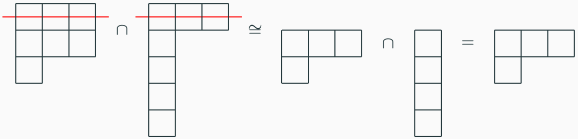

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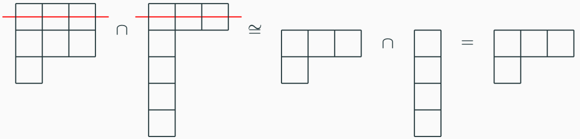

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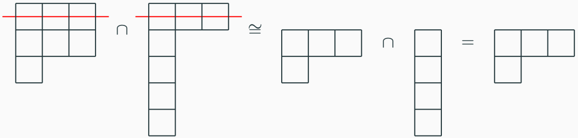

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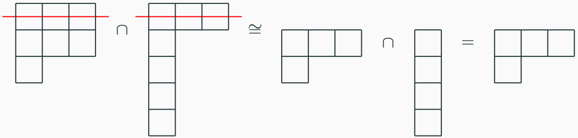

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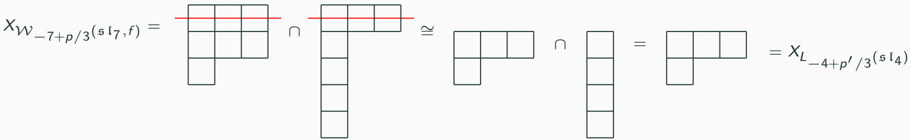

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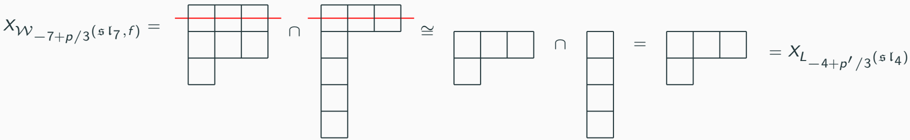

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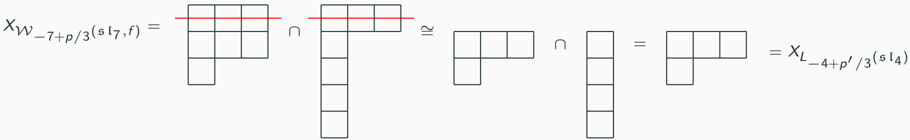

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4. Applications and motivations coming from physics

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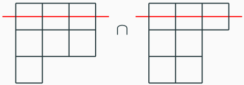

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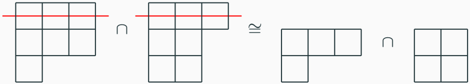

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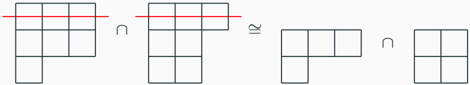

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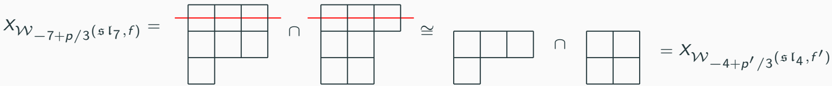

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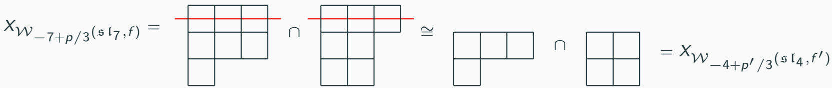

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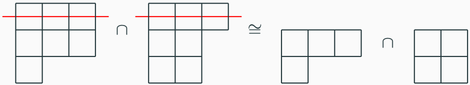

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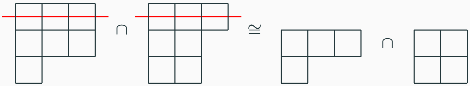

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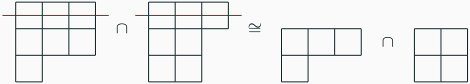

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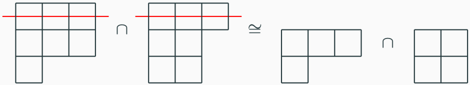

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Thank you !