C_2-cofiniteness of commutant subVOA

Masahiko Miyamoto

Institute of Mathematics, University of Tsukuba
Institute of Mathematics, Academia Sinica

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1. Motivation
2. Setting for commutant subVOA and the statement of our theorem.
3. Our strategy and V-internal operators
4. Matrix equations $AX = B$ and solutions $X = A^{-1}B$.
5. Functions and Rigidity
6. Borcherds-like identity
7. The case where V is generated by self-dual simple modules
8. The minimal counterexample and orbifold theory

Motivation

Powerful methods for construction (of infinite series) of SVOAs are
(1) Orbifold construction by finite automorphism group
(2) Commutant subVOA
(3) Homological construction, e.g. $\mathrm{Ker}\rho/\mathrm{Image}\rho$ for an endomorphism ρ of some SVOA satisfying $\rho^2 = 0$.

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Usually, when we expect subVOA to have finiteness property, like regularity we start with VOAs with finiteness properties. These are
(i) "C_2-cofinite" i.e. Zhu's Poisson algebra $R_2(V) = V/C_2(V)$ has finite-dim, where $C_m(W) = \mathrm{Span}_{\mathbb{C}}\{v_{-m}w \mid \mathrm{wt}(v) > 0, w \in W\}$ for $m \geq 1$ and $R_m(W) = W/C_m(W)$ for a V-mod W.

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(ii) "rationality"= all $\mathbb{N}$-gradable V-mods are direct sums of simple mods.

Many beautiful results (f.g. Verlinde formula) hold under two conditions. So, it is important to check them.

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On the other hand, Rationality is global, we need all $\mathbb{N}$-grad. mods.

Moreover, if we once get C_2-cof., then we can get global properties:
$\sharp$ of simple V-mods is finite,
Fusion products are well-defined,
modular invariance, etc.
These will help the proof for "Rationality".
So, let's start with the proof of C_2-cofiniteness.

SubVOA and Commutant subVOA 1

For the orbifold case (1), I have proved C_2-cofiniteness of orbifold models.

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If V is a simple VOA of CFT-type and $V' \cong V$ and $G \subseteq \mathrm{Aut}(V)$ is finite. If V is C_2-cofinite, then so is V^G.

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My talk is the second case (2), that is, a commutant subVOA.

Conj 1 (Fundamental)

V is C_2-cof. VOA, U is C_2-cof. subVOA, then $U^c := \{v \in V \mid \omega_0^U v = 0\}$ is also C_2-cofinite. More generally. If V is C_2-cof., U is subVOA and V is a finite direct sum of simple U-modules, then U is C_2-cofinite?

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We will give a partial answer to these conjectures. Most ideas from [M18].
Although, for a cyclic group auto, we use simple currents.
For a non-solvable group, we used a self-dual simple V^G-mod.

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$$V = \oplus_{i\in\Delta}(U^i \otimes W^i),$$

where U^i are simple U-mods, W^i are simple W-mods. and $i \neq j$, then $U^i \ncong U^j$, $W^i \ncong W^j$, by [Lin 2017] etc.

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Theorem 3

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I will explain our ideas to prove this theorem.

Our methods and V-internal fusion product

Since U is C_2-cof., $\mathrm{wt}(U^i) \in \mathbb{Q}$ and so $\mathrm{wt}(W^i)$, say in $\mathbb{Z}/R$.
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Answer: Just treat W^i and the inside of $V = \oplus_{i\in\Delta}(U^i \otimes W^i)$.

Our Strategy

As U-mods (W-mods), we treat only direct sums of U^i's, (of W^i's).
About intertwining op. we consider only intertwining ops. appeared in V.

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Notation 1

Since $V \cong V'$, $(U^i)' \cong U^j$ ($^\exists j$ denote by $\bar{i}$)
Set $W^\Delta = \oplus_{i\in\Delta} W^i$ (also $U^\Delta = \oplus_{i\in\Delta} U^i$)
Using $u \in U^\Delta$, we define W-hom: $\Omega_u : V \to W^\Delta$ by

$$\Omega_u(\textstyle\sum_{j\in\Delta} u^j \otimes w^j) = \sum_j \langle u, u^j\rangle w^j \in W^\Delta.$$

Similarly, we define U-homo $\Omega_w : V \to U^\Delta$ by

$$\Omega_w(\textstyle\sum_{j\in\Delta} u^j \otimes w^j) = \sum_j \langle w, w^j\rangle u^j \in U^\Delta.$$

V-internal operators

Definition 1

Using $u^1 \in U^i, u^2 \in U^j, u^3 \in U^{\bar{k}}$, we define

$$\mathcal{I}^{u^1,u^2,u^3}(w^1,z)w^2\rangle = \Omega_{u^3}(Y(u^1 \otimes w^1,z)(u^2 \otimes w^3))z^{\mathrm{wt}^U(u^1)+\mathrm{wt}^U(u^2)-\mathrm{wt}^U(u^3)}$$

for $w^1 \in W^i, w^2 \in W^j$.

Similarly, using $w^1 \in W^i, w^2 \in W^j, w^3 \in W^{\bar{k}}$, we define

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Lemma 4

$\mathcal{I}^{w^1,w^2,w^3} \in I\binom{U^k}{U^i\ U^j}$ *and* $\mathcal{J}^{u^1,u^2,u^3} \in I\binom{W^k}{W^i\ W^j}$.

Definition 2

Set $I^V_U\binom{U^k}{U^i, U^j}$, $I^V_W\binom{W^k}{W^i, W^j}$ the subspaces of $I\binom{U^k}{U^i, U^j}$, $I\binom{W^k}{W^i, W^j}$ spanned by the above intertwining operators.
We call elements in these subspaces "V-internal operators".

V-internal operators

Since U is C_2-cofinite, $\dim I_U^V\binom{U^k}{U^i,U^j} \le \dim I\binom{U^k}{U^i,U^j} < \infty$.

Choose a basis $\{\mathcal{I}_{i,j,k}^s \mid s \in B_{i,j,k}\}$ of $I_U^V\binom{U^k}{U^i,U^j}$.

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Theorem 5

There are $\mathcal{J}^s_{i,j,k} \in I_W^V\binom{W^k}{W^i,W^j}$ *such that*

$$Y = \sum_{i,j,k}\sum_{s\in B_{i,j,k}} \mathcal{I}^s_{i,j,k} \otimes \mathcal{J}^s_{i,j,k}.$$

Furthermore, $\{\mathcal{J}^s_{i,j,k} \mid s \in B_{i,j,k}\}$ *is a basis of* $I_W^V\binom{W^k}{W^i,W^j}$.

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Namely, if $v^1 = u^1 \otimes w^1 \in U^i \otimes W^i$, $v^2 = u^2 \otimes w^2 \in U^j \otimes W^j$ and $\pi_k : V \to U^k \otimes W^k$ a projection, then
$\pi_k(Y(v^1,z)v^2) = \sum_{s \in B_{i,j,k}} \mathcal{I}^s_{i,j,k}(u^1,z)u^2 \otimes \mathcal{J}^s_{i,j,k}(w^1,z)w^2$

$W = V/U$, self-knowledge comes from knowing other men.

Our proof is: we get information on $I_W^V\binom{W^k}{W^i,W^j}$ from $I_U^V\binom{U^k}{U^i,U^j}$ and Y. Simply, write $Y = \sum_{a\in B}\mathcal{I}^a\otimes\mathcal{J}^a$. Then we have

$$\langle u^4\otimes w^4, Y(Y(u^1\otimes w^1, x-y)(u^2\otimes w^2), y)(u^3\otimes w^3)\rangle$$
$$= \sum_{a,b\in B}\langle u^4, \mathcal{I}^a(\mathcal{I}^b(u^1, x-y)u^2, y)u^3\rangle\langle w^4, \mathcal{J}^a(\mathcal{J}^b(w^1, x-y)w^2, y)w^3\rangle.$$

We simply write $Y(Y()) = \sum_{i\in D}\mathcal{I}^{a_i}(\mathcal{I}^{b_i})\otimes\mathcal{J}^{a_i}(\mathcal{J}^{b_i})$

Theorem 6

$$\langle w^4, \mathcal{J}^a(\mathcal{J}^b(w^1, x-y)w^2, y)w^3\rangle.$$

are absolutely convergent on $0 < |x-y| < |y|$ and analytically extended to the multi-valued analytic functns whose poles are at most $x, y, x-y$. We have the same statement for V-internal operators of U-modules. We also have the similar statement for

$$\langle u^4, \mathcal{I}^s(u^1, x)\mathcal{I}^t(u^2, y)u^3\rangle \text{ and } \langle w^4, \mathcal{J}^s(w^1, x)\mathcal{J}^t(w^2, y)w^3\rangle.$$

A proof.

The poles of LHS are at most $x, y, x-y$. View $\langle \cdot, \mathcal{I}^{a_i}(\mathcal{I}^{b_i}(\cdot, x-y)\cdot, y)\cdot \rangle_{i=1,\dots,S}$ as a function on $(\oplus_{i\in\Delta} U^i)^{\oplus 4}$ with values in functions of x and y. Since $\{\langle \cdot, \mathcal{I}^{a_i}(\mathcal{I}^{b_i}(\cdot, x-y)\cdot, y)\cdot \rangle \mid i = 1, \dots, S\}$ is linearly independent, we may choose qualtets $(u_1^1, u_1^2, u_1^3, u_1^4), \dots, (u_S^1, u_S^2, u_S^3, u_S^4)$ such that $S \times S$-matrix $A := (\langle u_j^4, \mathcal{I}^{a_i}(\mathcal{I}^{b_i}(u_j^1, x-y)u_j^2, y)u_j^3 \rangle)$ has nonzero determinant.

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Equations on vectors over function fields

$$\begin{aligned}(\langle v_i^4, Y(Y(v_i^1, x-y)v_i^2, y)v_i^3\rangle_{i=1,\dots,S}\\ = A(\langle w^4, \mathcal{J}^{s_i}(\mathcal{J}^{t_i}(w^1, x-y)w^2, y)w^3\rangle_{j=1,\dots,s}\end{aligned}$$

as column vectors, where $v_i^a = u_i^a \otimes w^a$ for $a = 1,2,3,4$. So we have:

$$\langle w^4, \mathcal{J}^{s_i}(\mathcal{J}^{t_i}(w^1, x-y)w^2, y)w^3\rangle = A^{-1}\langle v_i^4, Y(Y(v_i^1, x-y)v_i^2, y)v_i^3\rangle$$

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If we assume that $\langle w^4, \mathcal{J}^a(\mathcal{J}^b(w^1, x-y)w^2, y)w^3\rangle$ has a pole at $x-ry$, then we can choose the above determinant is not zero at $x-ry$, then solving the equation, we have a contradiction.

Fusion product

Because we have $I_W^V\binom{W^k}{W^i,W^j}$, we can naturally define

$$W^i \boxtimes_W^V W^j := \oplus_k (W^k)^{\dim I_W^V\binom{W^k}{W^i,W^j}}$$

and surjective intertwining op. $\mathcal{F} \in I_W^V\binom{W^i\boxtimes_W^V W^j}{W^i,W^j}$ to define fusion product.

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Similarly, we can define a fusion product of three modules.
Not only W, but we can also define the fusion products of U^i's.

Fusion product

Because we have $I_W^V\binom{W^k}{W^i, W^j}$, we can naturally define

$$W^i \boxtimes_W^V W^j := \oplus_k (W^k)^{\dim I_W^V\binom{W^k}{W^i, W^j}}$$

and surjective intertwining op. $\mathcal{F} \in I_W^V\binom{W^i \boxtimes_W^V W^j}{W^i, W^j}$ to define fusion product.

Similarly, we can define a fusion product of three modules.
Not only W, but we can also define the fusion products of U^i's.

Remark 1

There is a natural def. of fusion products for V-internal ops, as a projective limit of increasing series of surjective (linear combs. of) V-internal ops. Our definition of the V-internal fusion products depends on the choice of a basis $\{\mathcal{I}_{i,j,k}^s \mid s \in B_{i,j,k}\}$ of $I_U^V\binom{U^k}{U^i\ U^j}$, but the isomorphism class of V-internal fusion product does not depend on the choice of bases and coincides with the natural one.

Rigidity, the relation between U^i and W^i are dual.

Moving to $y = 0$ according to a suitable path, we can expand $\langle w^4, \mathcal{J}^a\mathcal{J}^b(w^1, x-y)w^2, y)w^3\rangle$ in the form

$$\sum_j \langle w^4, \tilde{\mathcal{J}}^{a_j}(w^1, x)(\tilde{\mathcal{J}}^{b_j}(w^2, y)w^3\rangle.$$

Using the previous arguments, we can prove that $\tilde{\mathcal{J}}^{a_j}\tilde{\mathcal{J}}^{b_j}$ are replaced by linear combinations of products of V-internal operators.

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Similarly, we have
$\langle u^4, \mathcal{I}^{a_i}(\mathcal{I}^{b_i}(u^1, x-y)u^2, y)u^3\rangle = \sum \kappa_{i,j}\langle u^4, \mathcal{I}^{s_j}(u^1, x)\mathcal{I}^{t_j}(u^2, y)w^3\rangle$.

Rigidity, the relation between U^i and W^i are dual.

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Theorem 7

${}^t(\kappa_{i,j})(\tau_{i,j}) = I$. So, U^j satisifies right (left) rigidity if and only if W^j satisfies left (right) rigidity.

Borcherds'-like identity

We assume that $P := W^j$ is self-dual and W^j generates V.
We want to show that $V^i = U^i \otimes W^i$ is C_2-cofinite for some j.
Finiteness $V^j/C_2(V^i)$ is not information for small weights.
So, Ignore small weights.

Theorem 8 (Borcherds' like identity)

Let $\pi : V \to U \otimes W$ a projection. For $\theta \in (C_2(V^j))^{\perp}$ and $v^i \in U^j \otimes W^j$ with $\mathrm{wt}(\theta) > \mathrm{wt}(v^1) + \mathrm{wt}(v^2) + \mathrm{wt}(v^3) + 1$, we have

$$\langle \theta, \pi(v^1_n v^2)_m v^3 \rangle = \lambda \langle \theta, \sum_{i=0}^{\infty} \binom{n}{i} (-1)^i \{ v^1_{n-i} \pi(v^2_{m+i} w) - (-1)^n v^2_{n+m-i} \pi(v^1_i v^3) \} \rangle$$

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In particular, by taking $m \leq -2$, we have

Corollary 1

$$\langle \theta, \sum_{i=0}^{\infty} \binom{n}{i} (-1)^i v^1_{n-i} \pi(v^2_{i-2} v^3) \rangle = \langle \theta, \sum_{i=0}^{\infty} \binom{n}{i} (-1)^{i+n} v^2_{n-2-i} \pi(v^1_i v^3) \rangle$$

The case V is generated by a self-dual simple module P

So we will consider the set $\mathrm{Map}(\mathbb{N},\mathbb{C})$ of all maps from $\mathbb{Z}$ to $\mathbb{C}$ satisfying $f(n)=0$ for $n\in\mathbb{Z}_{<0}$. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be the spaces of coefficients $f(x)$ of $a_{(-x+M-1)}b$ at $\alpha_{(-x-1)}\mathbf{1}$ modulo K for $a\in T, b\in P$ and $a\in P, b\in T$, that is,

$$\begin{aligned}\mathcal{F}_0 = \mathrm{Span}_{\mathbb{C}}\Big\{ f\in\mathrm{Map}(\mathbb{N},\mathbb{C}) \mid {}^{\exists}a\in T, {}^{\exists}b\in P \text{ s.t.}\\ \langle\theta, a_{(-x+M-1)}b\rangle = f(x) \text{ for } x\in\mathbb{N}\Big\},\\ \mathcal{F}_1 = \mathrm{Span}_{\mathbb{C}}\Big\{ f\in\mathrm{Map}(\mathbb{N},\mathbb{C}) \mid {}^{\exists}a\in P, {}^{\exists}b\in T \text{ s.t.}\\ \langle\theta, a_{(-x+M-1)}b\rangle = f(x) \text{ for } x\in\mathbb{N}\Big\}.\end{aligned}$$

As we did in the proof for cyclic automorphism group, we define

$$\begin{aligned}Sf(n) &= (-1)^n\textstyle\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(n-k) && \text{for } n\in\mathbb{N},\\ Tf(n) &= (-1)^n f(n) && \text{for } n\in\mathbb{N}.\end{aligned}$$

and we can get a contradiction.

Final Step

Consider all counterexamples $(V, H, U \otimes W)$, where V and U are C_2-cofinite, $U \otimes W \subseteq H$ are not C_2-cofinite and V and H are direct sums of distinct simple $U \otimes W$-modules. In particular, $V = \oplus H^i$ with simple H-modules H^i. Choose $(V, H, U \otimes W)$ with minimal number of simple sub H-modules. By the theorem, we may assume $H^i \ncong H^{\bar{i}}$ for $H^i \ncong H$.

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Perm. tensor product $V^{\otimes 2}$ and orbifold model

Consider orbifold $(V \otimes V)^\sigma$ with $\sigma = (1,2)$.
Then we can check that $((V \otimes V)^\sigma, (H \otimes H)^\sigma, (U \otimes U)^\sigma \otimes (W \otimes W)^\sigma)$ satisfies the assumption of Theorem.

Then $H^i \otimes H^{\bar{i}} + H^{\bar{i}} \otimes H^i$ is self-dual simple $(H \otimes H)^\sigma$-module. Let $K :=$ subVOA generated by self-dual $(H \otimes H)^\sigma$-submods, then since (V, H) is minimal counterexample, i.e. $\geq ((V \otimes V)^\sigma, K, U \otimes W)$, we can get that finite $G \subseteq \mathrm{Aut}((V \otimes V)^\sigma)$ s.t. $K = ((V \otimes V)^\sigma)^G$.

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This completes the proof of the main theorem.

Thank you for listening !!