A geometric view on Witt rings
Dubrovnik 2019

$\mathbb{k}$ will denote a commutative ring. $\mathcal{O}(X)$ are the functions on a space X and $\mathcal{O}^*(X)$ are the invertible functions.
This will be an elementary talk about distributions in the setting of algebraic geometry. Inside the projective line $\mathbb{P} = \mathbb{P}^1$ the formal neighborhood $d = \widehat{0}$ of 0 and the affine line $A_m \overset{\text{def}}{=} \mathbb{P} - 0$ interact on their intersection which is the punctured formal disc $d^* = d \cap A_m$. This roughly identifies functions on one with distributions on the other. We will make this precise in the multiplicative setting and notice how this mechanism appears in several examples.

Motivating example: **the Witt ring.**

Theorem. The formal power series $1 + T\Bbbk[[T]]$ with leading coefficient 1 has a natural structure of a ring. The addition operation is the multiplication $\cdot$ of formal power series. The multiplication operation $*$ is characterized by (for $a, b \in \Bbbk$)

$$(1 - aT) * (1 - bT) = \ 1 - abT.$$

Remarks. This is called the *Witt ring* over $\Bbbk$ (or *ring of big Witt vectors*). The usual proofs develop certain mastery of formulas. We will present it as a natural structure "without formulas".

1.1. Additive version The distributions are the dual of functions $\mathcal{D}_X = \mathcal{O}(X)^\vee$. There is a map $\delta : X \to \mathcal{D}_X$ where for $a \in X$, δ_a is the evaluation of functions at a.
In reasonable settings $\delta : X \to \mathcal{D}_X$ is the *linearization* of X, the universal linear object that X maps to. We also call it the *linear object freely generated by* X.
A small example in algebraic geometry. First, consider the affine line $\mathbb{A}_U$ over $\mathbb{k}$, with coordinate U, so functions are $\mathcal{O}(U) = \mathbb{k}[U]$. The functions on the formal disc $d = \widehat{0} \subseteq \mathbb{A}_U$ are the formal series $\mathbb{k}[[T]]$.
Lemma. The functions on one of the spaces $A_m = \mathbb{A}_{T^{-1}}$ and $d = \widehat{0}$ are distributions on the other:

$$\mathrm{Hom}_{\mathbb{k}}[\mathcal{O}(\widehat{0}), \mathbb{k}] \;=\; \mathcal{O}(A_m).$$

To make this precise we need to put some natural extra structures on these vector spaces of functions. $\mathcal{O}(\mathbb{A}_U) = \mathbb{k}[U]$ is an ind-system $\mathbb{k} + \cdots + \mathbb{k}U^n$ of finite dimensional vector subspaces of polynomials of degree $\leq n$. $\mathcal{O}(d) = k[[T]]$ is a pro-system of finite dimensional quotient vector spaces $\mathbb{k}[T]/T^n$.

Proof. The pairing of $f \in \mathbb{k}[[T]]$ and $g \in \mathbb{k}[T^{-1}]$ will be defined as

$$\langle f(T), g(T^{-1})\rangle \overset{\text{def}}{=} Res_0(fg\frac{dT}{T}).$$

Since $\langle T^i, T^{-j}\rangle = \delta_{ij}$ the pairing makes the above finite dimensional subs and quotients dual. So, the pairing makes the two systems of vector spaces dual. $\square$

Remark. This says that the vector space freely generated by the disc $\widehat{0}$ is $\mathcal{O}(\mathbb{P}^1 - 0)$. (More naturally, it is the 1-forms $\Omega^1(\mathbb{P}^1 - 0)$.)

1.2 Multiplicative distributions Now, for an ind-scheme X over k, instead of all functions $\mathcal{O}(X)$ let us consider just the invertible functions $\mathcal{O}^*(X)$.
In order for these to behave well we will again need some extra structure on these – we make them into objects of algebraic geometry. So, we replace the group $\mathcal{O}^*(X)$ with a commutative affine group indscheme $\underline{\mathcal{O}}^*(X)$. This means that instead of one group we consider the system of groups $\mathcal{O}^*(X_{\Bbbk'})$ for all maps of rings $k \to \Bbbk'$, where $X_{\Bbbk'}$ is the scheme over $\Bbbk'$, obtained by extension of scalars from $\Bbbk$ to $\Bbbk'$.
On commutative affine group ind-schemes we have a replacement for vector space duality, the *Cartier duality*

$$\mathbb{D}(A) \stackrel{\text{def}}{=} \underline{\text{Hom}}(A, G_m).$$

Here $\underline{\text{Hom}}$ means the inner Hom in affine group indschemes, i.e., again the system of all groups of homomorphisms $\underline{\text{Hom}}(A_{\Bbbk'}, G_{m\Bbbk'})$.
Examples (a) $\mathbb{D}G_m = \mathbb{Z}$.
(b) For a vector space V in characteristic zero the dual $\mathbb{D}V$ is the formal neighborhood $\widehat{0}_{V^*}$ of 0 in the dual vector space $V^\vee$.

Remark. The system $\underline{\mathcal{O}}^*(X)$ is also a ring – for a certain tensor structure $B\otimes^* C \stackrel{\text{def}}{=} \mathbb{D}[\underline{\text{Hom}}(B, \underline{\text{Hom}}(C, G_m)]$ on commutative group ind-schemes. So, it is a part of some new multiplicative algebraic geometry. However, we will only be interested in the corresponding multiplicative notion of distributions.

$$A_X \stackrel{\text{def}}{=} \mathbb{D}[\underline{\mathcal{O}}^*(X)] = \underline{\text{Hom}}[\underline{\text{Map}}(X, G_m), G_m].$$

Remark. We will now only consider spaces X such that $B = \underline{\mathcal{O}}^*(X)$ satisfies $\mathbb{D}^2(B) = B$, Then A_X is the *affine commutative group ind-scheme freely generated by* X.

1.3 Multiplicative distributions $X \mapsto A_X$ as homology The Thom-Dold theorem in algebraic topology can be roughly interpreted as

homology $H_*(X, \mathbb{Z})$ of a topological space X is the abelian group object freely generated by the space X.

[The formulation is actually more complicated because at the time there was no adequate categorical setting.] The construction of multiplicative distributions A_X (when fully developed) will be a homology theory that is completely in Algebraic Geometry.

1.2. Multiplicative duality of functions on d and A_m Invertible functions $\mathcal{O}^*(X,a)$ on a pointed space $X\ni a$, are defined as as invertible functions $f:X\to G_m$ that vanish at a, i.e., $f(a)=1$. Then $\mathcal{O}^*(X)\cong G_m{\times}\mathcal{O}^*(X,a)$.

Example. (1) On the formal disc $\mathcal{O}^*(d,0)=\ 1+\Bbbk[[T]]$. The corresponding group scheme $\underline{\mathcal{O}}^*(d,0)$ is called the congruence subgroup $K=K_T$. So, $K(\Bbbk')=1+\Bbbk'[[T]]$.

(2) On an affine line $\mathbb{A}_U=\mathrm{Spec}(\Bbbk[U])$, $\mathcal{O}^*(\mathbb{A}_U,0)=1+U\mathcal{N}_k[U]$ where $\mathcal{N}_\Bbbk$ are the nilpotent elements in $\Bbbk$.
[For a polynomial $P\in\Bbbk[U]$, the inverse of $1+UP$ is again a polynomial iff P is nilpotent, i.e., iff all its coefficients are nilpotent.]
These form an indscheme $\underline{\mathcal{O}}^*(\mathbb{A}_U,0)$ which we can call the "small" congruence subgroup $K_U^s{\subseteq}K$.

Lemma. Multiplicative distributions on one of the pointed spaces $(d,0)=(\widehat{0},0)$ and $(A_m,\infty)=(\mathbb{P}^1-0,\infty)$ are invertible functions on the other.

Proof. On (A_m, ∞) we have a coordinate T^{-1} so an invertible function g is of the form $1 + a_1 T^{-1} + \cdots + a_n T^{-n}$. We rewrite it as $T^{-n}(T^n + a_1 T^{n-1} + \cdots + a_n)$ and the second factor is the equation of some finite subscheme D of the T-line $\mathbb{A}_T$. Since all a_i are nilpotent this scheme D lies in the formal disc $d \subseteq \mathbb{A}_T$.
Now the pairing of $f \in \mathcal{O}^*(d, 0)$ with this $g \in \underline{\mathcal{O}}^*(A_m, \infty)$ is the integral of f over the finite scheme D

$$\{f, g\} \stackrel{\text{def}}{=} \int_D f \in G_m.$$

This integral is usually called *norm*. If $x_1, ..., x_n$ are roots of D it just means $\prod f(x_i)$.
This pairing gives the natural isomorphism

$$\underline{\mathcal{O}}^*(A_m, \infty) \cong \underline{\mathrm{Hom}}(\underline{\mathcal{O}}^*(d, 0), G_m) = \underline{\mathcal{O}}^*(d, 0).$$

$\square$

Remarks. (0) This means that the affine commutative group indschemes freely generated by the pointed disc and the pointed line are

$$A_{d,0} \;=\; \underline{\mathcal{O}}^*(\mathbb{P}^1 - 0, \infty) \quad \text{and} \quad A_{\mathbb{A}_m,\infty} \;=\; \underline{\mathcal{O}}^*(d, 0).$$

(1) The multiplicative world is simpler we do not need the 1-forms or a choice of a Haar measure for duality.
(2) In p-adic representation theory the above additive duality is a standard tool. However, the multiplicative duality is wrong since for $\mathbb{k}$ a field $\underline{\mathcal{O}}^*(\mathbb{A}_U, 0)$ is the trivial group, i.e., $\mathbb{k}[[U]]^*$ are just the constants $\mathbb{k}^*$. By passing to group indschemes we add the nilpotents and this makes the group sufficiently large for duality.

2. Witt ring

2.1. Restatement of Witt ring construction algebrai geometry. This uses the affine group indscheme K called the congruence subgroup. It is defined over $\mathbb{Z}$ and its points over a commutative ring $\Bbbk$ are $K(\Bbbk) \ = \ 1 + T\Bbbk[[T]]$.

Theorem. The congruence subgroup K has a natural structure of a ring in indschemes. The addition operation is the multiplication $\cdot$ of formal power series. The multiplication operation $*$ is the unique bilinear operation such that

$$(1-aT)*(1-bT) = \ 1-abT \quad \text{for} \quad a,b \in \Bbbk.$$

Remark. The ring structure on the indscheme K gives a ring structure on the set $\mathbb{K}(\Bbbk)$ of $\Bbbk$-points.)

Proof. We know that $K = \underline{\mathcal{O}}^*(d, 0)$ is the group indscheme freely generated by the pointed space $(\mathbb{P}^1 - 0, \infty)$. This makes it a group. Moreover, $(\mathbb{P}^1 - 0, \infty) \cong (\mathbb{A}, 0)$ has a natural structure of a monoid from the multiplication on the line $\mathbb{A}$ (0 is an ideal in $\mathbb{A}$, hence $(\mathbb{A}^1, 0)$ is still a monoid.)
Therefore, K is the algebro geometric monoid algebra of the commutative monoid $(\mathbb{A}, 0)$, hence it is a commutative ring in algebraic geometry.
Finally, this isomorphism $A_{\mathbb{A},0} \cong K$ restricts via $\mathbb{A} \to A_{\mathbb{A},0}$ to a map $\mathbb{A} \to K$ by $a \mapsto 1 - aT$ so the above relation is just the claim that $*$ comes from multiplication in $\mathbb{A}$. $\square$

2.2 Combinatorics. The Witt ring has a huge number of structures and applications (say, Borger's definition of the field with one element). For one thing it is the spectrum of the ring of symmetric functions in infinitely many variables $\Bbbk[x_1, ...]^{S_\infty}$ which is the home for classical combinatorics so it is an algebro geometric incarnation of combinatorics.

Proof. One system of coordinates $a_1, a_2, ..$ on the Witt ring K are the coefficients of the series $f = 1 + a_1(f)T + \cdots$. So, K is just an infinite dimensional affine space $\mathbb{A}^\infty$.

However, $K =$ is $\lim_{\leftarrow\, n\to\infty} K/K(n)$ for the n^{th} congruence subgroup $K(n) \stackrel{\text{def}}{=} 1 + T^n\Bbbk[[T]])$. Then $a_1, ..., a_n$ are the coordinates on $K/K(n)$ which is just the space of monic polynomials of the form $f = 1+a_1(f)T+\cdots+a_n(f)T^n = T^n(T^{-n}+a_1(f)T^{-(n-1)}+\cdots+a_n(f)]$. So, a_i's are the elementary symmetric functions in roots of the polynomial $T^{-n} + a_1(f)T^{-(n-1)} + \cdots + a_n(f)$. Then $\mathcal{O}(K/K(n)) = \Bbbk[x_1, ..., x_n]^{S_n}$ and $\mathcal{O}(K)$ is $\Bbbk[x_1, ...]^{S_\infty}$. □

2.3 Transfers Recall that A_X is a version of homology. The ordinary homology is characterized by having transfers for finite maps so we expect them for multiplicative distributions.
Any finite map $\chi : X \to Y$ defines the *transfer* map of groups

$$\chi^{tr} : A_Y \to A_X, \ \chi^{tr}(y) \overset{\text{def}}{=} \sum_{x \in \phi^{-1} y} x. \quad \square$$

The endomorphisms of the monoid $(\mathbb{A}, \cdot)$ form a semiring $\chi : (\mathbb{N}, +, \cdot) \cong \mathrm{End}(\mathbb{A}, \cdot)$ where $\chi_n(x) = x^n$. These are fine maps so we have transfers χ_n^{tr} and

$$\chi_n^{tr}(1 - aT) = \prod_{\alpha^n = a} (1 - \alpha T) \ = \ 1 - aT^n.$$

Notice that for $a, b \in \mathbb{k}$,

$$a\chi_n^{tr}(b) \ = \ \chi_n^{tr}(a^n b) \quad \text{and} \quad \chi_n^{tr}(a) \cdot \chi_m^{tr}(b) \ = \ \left(\chi_{[n,m]}(a^{\frac{[n,m]}{n}} b^{\frac{[n,m]}{m}})\right)^{(n,m)}.$$

Lemma. Any multiplicatively closed $S \subseteq \mathbb{N}$ defines an ideal $K_S \subseteq K$ which is the subgroup generated by all images of χ_n^{tr} for $n \in S$.
Example. (1) If $S = \{n, n+1, ...\}$ then $K_S(\mathbb{k}) = 1 + T^n\mathbb{k}[[T]]$ is an ideal in W.
(2) If S is all numbers not divisible by a fixed prime p then K/K_S is the "ring of p-typical Witt vectors".
Corollary. For any $a_n \in \mathbb{k}$, $\sum_n \ \chi_n^{tr}(a_n)$ converges in $A_{\mathbb{A},0}$. We call a_n's the system of *Witt* coordinates $\mathbb{A}^{\mathbb{N}} \cong A_{\mathbb{A},0}$.
The multiplicative formulation in K says that the Witt coordinates of an element α of $K(\mathbb{k}) = 1 + t\mathbb{k}[[T]]$ are given by the unique factorization

$$\alpha \ = \ \prod_n (1 - a_n T^n) \ \text{ for } \quad a_n \in \mathbb{k}.$$

2.4 Factorization formula for Witt multiplication This coordinate system allows us to write the product $*$ explicitly

$$\prod_{n>0} 1 - aT^n \cdot \prod_{m>0} 1 - b_m T^m = \prod_{m,n>0} \left(1 - a_n^{[n,m]/n} b_m^{[n,m]/m} T^{[n,m]}\right)^{(n,m)}.$$

Proof. This is just the formula for the product of transfers written multiplicatively. $\square$

3. Geometric Class Field Theory

3.1. Uniform formulation local and clobal cases. This is closest to Contou-Carrere.
For a smooth curve C over a ring $\Bbbk$ the Geometric Class Field Theory says that

$(*)$ The commutative group indsceme A_C freely generated by C is the moduli $Bun^c_{G_m}(C)$ of line bundles on C with compact support.

This follows from the *Abel-Jacobi* map

$$AJ: \ C \ \to \ Bun^c_{G_m}(C), \qquad AJ_a \stackrel{\text{def}}{=} \ \mathcal{O}_C(-a) = \ \mathcal{I}_a.$$

Corollary. This gives a Cartier duality formulation of geometric CFT:

$$\underline{Map}(C, G_m) \ \cong \ \mathbb{D}[Bun^c_{G_m}(C)].$$

Proof. The RHS is $\underline{\mathrm{Hom}}[A_C, \mathbb{G}_m)$ and this the same as $\underline{Map}(C, G_m)$.
Remarks. (0) A compactly supported line bundle on C means a line bundle on a compactification $\overline{C}$ endowed with a trivialization on the formal neighborhood of the boundary $\overline{C} - C$.

(1) For complete C this subtlety disappears but now $Bun^c_{G_m}(C)$ is a stack. We will only be interested in local curves.

3.2. Example $C = d$**.**
Here $Bun^c_{G_m}(d)$ is the space of line bundles on $\mathbb{P}^1$ with a trivialization on $\mathbb{P}^1 - 0$.
This is the same as line bundles on d with a trivialization on $d \cap (\mathbb{P}^1 - 0) = d^*$. This is called the loop Grassmannian $\mathcal{G}(G_m)$ and there is a simple formula $G_m((z))/G_m[[z]]$. The duality statement is

$$\mathbb{D}[\underline{\mathcal{O}}^*(d)] \cong \mathcal{G}(G_m)$$

which is the combination of $\mathbb{D}(\mathbb{Z}) = G_m$ and $\mathbb{D}(K^s_{T^{-1}}) = K$. The second claim is what we have proved earlier. The Abel-Jacobi point of view on this duality is really the same as what we have been doing.

3.3. Example $C = d^*$ Here $Bun^c_{G_m}(d^*) = G_m((z))$. Here, the boundary of d^* in the compactification $\mathbb{P}^1$ consists of $\mathbb{P}^1 - 0$ (as before) and of formal neighborhood d of 0. The duality statement is

$$\mathbb{D}[G_m((z))] \cong G_m((z)).$$

The corresponding pairing $\{,\}: \ G_m((z)) \times G_m((z)) \to G_m$ is the *Contou Carrere* refinement of the tame symbol in NT.

Remark. We can write $f \in G_m((z))$ as $f = f_0 \cdot T^{\mathrm{ord}(f)} \cdot f_+ \cdot f_-$ in terms of the factorization

$$G_m((z)) \ \cong \ G_m \times T^{\mathbb{Z}} \times K_T \times K^s_{T^{-1}}.$$

Then the formula for the symbol is

$$\{f, g\} \ = \ (-1)^{\mathrm{ord}(f)\cdot\mathrm{ord}(g)} \cdot \frac{g_0^{\mathrm{ord}(f)}}{f_0^{\mathrm{ord}(g)}} \cdot \{f_+, g_-\} \cdot \{g_+, f_-\}^{-1},$$

where the last two terms are the pairing of K_T and $K^s_{T^{-1}}$ from above. [The first two factors are simple algebraically but deep geometrically.]

3.4. Comparison of the symbol and the Witt multiplication

Lemma. [Beilinson-Bloch-Esnault] For $f \in K_T$ and $g \in K^s_{T^{-1}}$

$$\{f(T), g(T^{-1})\} \;=\; \big(f(T) * g(T)\big)|_{T=1}$$

and

$$[f(T) * g(T)](c) \;=\; \{f(T), g(cT^{-1})\}.$$

Proof. This is now a formal consequence of both constructions using the pairing of K_T and $K^s_{T^{-1}}$. $\square$

4. Vertex algebras/operators

The geometric theory of vertex algebras has been constructed by Beilinson-Drinfeld as *chiral algebras*. I will only mention some elementary relations to the above symbol pairing, i.e., the duality of K_T and $K^s_{T^{-1}}$.

4.1. Heisenberg central extensions A split torus T is of the form $L\otimes G_m$ for the lattice $L = X_*(T)$ of cocharacters of T. Then the loop group $T((z))$ is $L\otimes G_m((z))$. A quadratic form $\kappa : L\times L \to \mathbb{Z}$ on the lattice combines with the Contou-Carrere symbol to give a pairing

$$\{,\}^\kappa : \ T((z))\times T((z)) \ \to \ G_m, \quad \{\lambda\otimes f, \mu\otimes g\}^\kappa \ \stackrel{\text{def}}{=} \ \{f,g\}^{\kappa(\lambda,\mu)}.$$

A *κ-Heisenberg extension* is a central extension

$$0 \to G_m \ \to \ T_\kappa \ \to \ T(((z)) \to 0$$

such that the corresponding commutator pairing $T((z))\times T((z)) \ \to \ G_m$ is $\{,\}^\kappa$. This T_κ is unique up to isomorphism (which is not unique).

These correspond to a vertex lattice algebras V so that V is Morita equivalent to the kernel of the pairing.

4.2 Vertex operators The notion of *vertex operators* is essentially equivalent to vertex algebras. Vertex operators appear in various places in math/physics. They are usually written by formulas as generating series controlling the numerical invariants of some interesting phenomena.

4.3 Differential geometry of vertex operators [Skirm]. It is in terms of mapping spaces $Map(\mathbb{S}, \mathbb{T})$ of circles $\mathbb{S}, \mathbb{T}$. Here, $\mathbb{S}$ is thought of as a geometric space and $\mathbb{T}$ as a group. So, $Map(\mathbb{S}, \mathbb{T})$ is a called the loop group $\mathcal{L}\mathbb{T}$ of the group $\mathbb{T}$.

Now, a *kink* or *blip* at $s \in \mathbb{S}$ is a map $\phi_s : \mathbb{S} \to \mathbb{T}$ which has constant value $1 \in \mathbb{T}$ outside s and at s it quickly runs once around the circle $\mathbb{T}$ in the positive direction.

[This is actually a distributional map, a limit of approximations ϕ_ε that do the run on an interval of size ε around s.]

The vertex operator Ψ_s at s lies in the central extension $\widehat{\mathbb{T}}$ of the loop group $\mathcal{L}\mathbb{T}$ which is defined by the symbol pairing.
(This is the Heisenberg extension $\mathbb{T}_\kappa$ where κ is the multiplication on the lattice $L = \mathbb{Z}$). Ψ_s is the normal ordering lift : ϕ_s : of the blip $\phi_s \in \underline{Map}(\mathbb{S}, \mathbb{T})$ to the central extension
Precisely, Ψ_s is the normal ordering lift : ϕ_s : of the kink $\phi_s \in \underline{Map}(\mathbb{S}, \mathbb{T})$ to the central extension. So, the map $\widehat{\mathbb{T}} \rightarrow \mathcal{L}\mathbb{T}$ takes : Ψ_s : to the kink ϕ_s.

4.4.Calculation in a presence of a cohomology class

This happens frequently in physics (normal ordering, *disorder operators*, ...). The idea is that the relevant calculation ihappens on a space $\mathcal{X}$ above X which is a geometric realization of the class c. The method is to choose a trivialization of c over some open large U. This reduces the calculation on the restriction $\mathcal{X}|_U$ to a calculation on U, plus some "*rules*" that deal with non-naturality of the trivialization and with its singularity on the boundary $X - U$.

4.5 Normal ordering lift Here the relevant cohomology class appears as the extension $\mathbb{T}_\kappa$ of the loop group $\mathcal{L}\mathbb{T}$. The extension splits canonically on the subgroups $\mathcal{L}^\pm$ of positive and negative loops (in polynomial loops these subgroups correspond to the above $K_T,\ K^s_{T^{-1}}$). So, we can regard $\mathcal{L}^\pm\mathbb{T}$ as subgroups of $\widehat{\mathbb{T}}$.
The multiplication gives an isomorphism $\mathcal{L}^+\times\mathcal{L}^- \cong \mathcal{L}_0\mathbb{T}$ onto the connected component of $\mathcal{L}\mathbb{T}$. So, one can write $f \in \mathcal{L}_0\mathbb{T}$ uniquely as $f = f_+f_-$ with $f_\pm \in \mathcal{L}^\pm$ and define the normal ordering lift of f as the product

$$:f: \ \stackrel{\text{def}}{=}\ f_+\cdot_{\widehat{\mathbb{T}}} f_-$$

in the extension.
The "normal ordering" refers to the necessity to make a choice "$+$ before $-$". This lack of naturality is accounted by the rule that $f_+\cdot_{\widehat{\mathbb{T}}}f_-$ and $f_-\cdot_{\widehat{\mathbb{T}}}f_+$ differ by the commutator which is the symbol $\{f_+, f_-\}$.
Remark. An analytic way to pass to a central extension is that to normalization the blip ϕ_s to $\Psi_s =\ \phi_s/\{(\phi_s)_+, (\phi_s)_-\}$ where the correction factor is again the symbol pairing

4.6 Quasimaps One can see that when one translates this picture of a kink into algebraic geometry one gets a *quasimap* from $\mathbb{P}^1$ to $\mathbb{P}^1$

$$\frac{z(z-1)}{z-1}.$$

These quasimaps are Drinfeld's compactification of maps $Map(\mathbb{P}^1, \mathbb{P}^1)$ to $Map(\mathbb{P}^1, \overline{\mathbb{P}^1})$ where $\overline{\mathbb{P}^1}$ is a stack compactification $\mathbb{C}^2/G_m$ of $\mathbb{P}^1 = (\mathbb{C}^2 - 0)/G_m$.

The above formula is of course symbolic – one can not cancel a factor. The meaning is the limit in quasimaps of maps $z(z-1)/(z-\lambda)$.

Quasimaps are enormously useful in geometric representation theory but I have not connected this with vertex operators.

5. My motivation

A lattice vertex algebra arises from a torus T, a quadratic form κ and some infinitesimal geometry.

The Kac-Segal paper lifted this to an observation that affine Lie algebras arise effectively (i.e., not just combinatorially) in the same way.

I expect to globalize this to

1. reductive groups G and their generalizations the Kac-Moody groups corresponding to quivers;
2. to the cohomology moduli of such groups such as $Bun_G(C)$ and $\mathcal{G}(G)$.