Cosets of the large $N = 4$ superconformal algebra and the diagonal coset of $\mathfrak{sl}_2$

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Joint work with Thomas Creutzig and Boris Feigin

1. Two-parameter families of vertex algebras

Ex: Affine vertex algebra $V^k(D(2,1;\alpha))$ and its orbifolds, quotients, Hamiltonian reductions.

This includes the large $N=4$ superconformal vertex algebra $V^{k,\alpha}_{N=4}$.

It is the minimal $\mathcal{W}$-algebra of $D(2,1;\alpha)$ (Kac, Wakimoto, 2004).

Ex: Diagonal cosets.

Ex: Universal $\mathcal{W}_\infty$-algebras of types $\mathcal{W}(2,3,4,\dots)$ and $\mathcal{W}(2,4,6,\dots)$

Ex: More exotic universal algebras. One example has type $\mathcal{W}(2,3,4^2,5^2,6^4,7^4,8^7,\dots)$, and at least 3 parameters.

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2. Diagonal cosets

$\mathfrak{g}$ a simple, finite-dimensional Lie algebra over $\mathbb{C}$.

$V^k(\mathfrak{g})$ universal affine vertex algebra at level k.

Regard k as a **formal parameter**, so $V^k(\mathfrak{g})$ is defined over the ring $\mathbb{C}[k]$.

Given formal parameters k_1, k_2, we have diagonal embedding

$$V^{k_1+k_2}(\mathfrak{g}) \hookrightarrow V^{k_1}(\mathfrak{g}) \otimes V^{k_2}(\mathfrak{g}), \qquad a(z) \mapsto a(z) \otimes 1 + 1 \otimes a(z).$$

Diagonal coset

$$\mathcal{C}^{k_1,k_2}(\mathfrak{g}) = \mathrm{Com}(V^{k_1+k_2}(\mathfrak{g}),\ V^{k_1}(\mathfrak{g}) \otimes V^{k_2}(\mathfrak{g}))$$

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At special points, studied by many people, including Adamovic-Perse (2012), Jiang-Lin (2014).

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3. The case $\mathfrak{g} = \mathfrak{sl}_2$

Thm: As a two-parameter VOA, $C^{k_1,k_2} = C^{k_1,k_2}(\mathfrak{sl}_2)$ is of type $\mathcal{W}(2,4,6,6,8,8,9,10,10,12)$. Equivalently this holds for generic values of k_1, k_2.

First stated without proof by Blumenhagen, Eholzer, Honecker, Hornfeck, Hübel, 1995.

Step 1: For k_1 fixed, rescaling generators of $V^{k_2}(\mathfrak{sl}_2)$ by $\frac{1}{\sqrt{k_2}}$,

$$\lim_{k_2 \to \infty} C^{k_1,k_2} \cong V^{k_1}(\mathfrak{sl}_2)^{\mathrm{SL}_2}.$$

A strong generating set for $V^{k_1}(\mathfrak{sl}_2)^{\mathrm{SL}_2}$ will give rise to a strong generating set for C^{k_1,k_2} for generic k_2 (Creutzig, L., 2014).

Step 2: Rescaling the generators of $V^{k_1}(\mathfrak{sl}_2)$ by $\frac{1}{\sqrt{k_1}}$, we have

$$\lim_{k_1 \to \infty} V^{k_1}(\mathfrak{sl}_2)^{\mathrm{SL}_2} \cong \mathcal{H}(3)^{\mathrm{SL}_2},$$

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4. The case $\mathfrak{g} = \mathfrak{sl}_2$, cont'd

Note: Adjoint representation of SL_2 is the same as standard representation of SO_3.

So we can replace $\mathcal{H}(3)^{\mathrm{SL}_2}$ with $\mathcal{H}(3)^{\mathrm{SO}_3}$.

Strong generating set for $\mathcal{H}(3)^{\mathrm{SO}_3}$ give rise to strong generators for $V^{k_1}(\mathfrak{sl}_2)^{\mathrm{SL}_2}$ for generic values of k_1 (Creutzig, L., 2014).

Need to show that $\mathcal{H}(3)^{\mathrm{SO}_3}$ is of type $\mathcal{W}(2, 4, 6, 6, 8, 8, 9, 10, 10, 12)$.

This is a formal consequence of Weyl's first and second fundamental theorems of invariant theory for SO_3.

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5. The case $\mathfrak{g} = \mathfrak{sl}_2$, cont'd

Thm: (Weyl) For $n \geq 0$, let V_n be a copy of the standard representation $\mathbb{C}^3$ of SO_3, with orthonormal basis $\{a_n^1, a_n^2, a_n^3\}$.

Then $(\mathrm{Sym} \bigoplus_{n=0}^{\infty} V_n)^{SO_3}$ is generated by

$$q_{ij} = a_i^1 a_j^1 + a_i^2 a_j^2 + a_i^3 a_k^3, \qquad i, j \geq 0, \tag{1}$$

$$c_{klm} = \begin{vmatrix} a_k^1 & a_k^2 & a_k^2 \\ a_l^1 & a_l^2 & a_l^3 \\ a_m^1 & a_m^2 & a_m^3 \end{vmatrix}, \qquad 0 \leq k < l < m. \tag{2}$$

The ideal of relations among the variables q_{ij} and c_{klm} is generated by polynomials of the following two types:

$$q_{ij} c_{klm} - q_{kj} c_{ilm} + q_{lj} c_{kim} - q_{mj} c_{kli}, \tag{3}$$

$$c_{ijk} c_{lmn} - \begin{vmatrix} q_{il} & q_{im} & q_{in} \\ q_{jl} & q_{jm} & q_{jn} \\ q_{kl} & q_{km} & q_{kn} \end{vmatrix}. \tag{4}$$

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Step 3: We have linear isomorphisms

$$\mathcal{H}(3)^{\mathrm{SO}_3} \cong \mathrm{gr}(\mathcal{H}(3)^{\mathrm{SO}_3}) \cong \mathrm{gr}(\mathcal{H}(3))^{\mathrm{SO}_3} \cong (\mathrm{Sym} \bigoplus_{j \geq 0} V_j)^{\mathrm{SO}_3},$$

and isomorphisms of differential graded rings

$$\mathrm{gr}(\mathcal{H}(3)^{\mathrm{SO}_3}) \cong (\mathrm{Sym} \bigoplus_{j \geq 0} V_j)^{\mathrm{SO}_3}.$$

Generating set $\{q_{ij}, c_{klm}\}$ for $(\mathrm{Sym} \bigoplus_{j\geq 0} V_j)^{\mathrm{SO}_3}$ corresponds to a strong generating set $\{Q_{ij}, C_{klm}\}$ for $\mathcal{H}(3)^{\mathrm{SO}_3}$, where

$$Q_{i,j} = :\partial^i \alpha^1 \partial^j \alpha^1 + :\partial^i \alpha^2 \partial^j \alpha^2 : + :\partial^i \alpha^3 \partial^j \alpha^3 :,$$

$$C_{klm} = :\partial^k \alpha^1 \partial^l \alpha^2 \partial^m \alpha^3 : - :\partial^k \alpha^1 \partial^m \alpha^2 \partial^l \alpha^3 : - :\partial^l \alpha^1 \partial^k \alpha^2 \partial^m \alpha^3 :$$
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Note: Q_{ij} has weight $i + j + 2$ and C_{klm} has weight $k + l + m + 3$.

As a $\mathcal{H}(3)^{O_3}$-module, $\mathcal{H}(3)^{SO_3} \cong M_0 \oplus M_1$, where M_0, M_1 are irreducible $\mathcal{H}(3)^{O_3}$-modules (Dong, Li, Mason, 1998)

$M_0 \cong \mathcal{H}(3)^{O_3}$, which has lowest-weight vector 1.

M_1 has lowest-weight vector C_{012} and contains all cubics C_{klm}.

$\mathcal{H}(3)^{O_3}$ generated by $Q_{0,2}$, so $\mathcal{H}(3)^{SO_3}$ generated by $\{Q_{0,2}, C_{012}\}$.

One checks that the following set closes under OPE:

$$\{C_{01j} |\ j = 2, 4, 5, 6, 8\} \cup \{Q_{0,2k} |\ k = 0, 1, 2, 3, 4\}.$$

It follows that this set strongly generates $\mathcal{H}(3)^{SO_3}$.

Minimality follows from Weyl's second fundamental theorem.

7. The case $\mathfrak{g} = \mathfrak{sl}_2$, cont'd

Note: Q_{ij} has weight $i+j+2$ and C_{klm} has weight $k+l+m+3$.

As a $\mathcal{H}(3)^{\mathrm{O}_3}$-module, $\mathcal{H}(3)^{\mathrm{SO}_3} \cong M_0 \oplus M_1$, where M_0, M_1 are irreducible $\mathcal{H}(3)^{\mathrm{O}_3}$-modules (Dong, Li, Mason, 1998)

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8. Large $N = 4$ superconformal algebra $V^{k,\alpha}_{N=4}$

Weight 1: $\{e, f, h, e', f', h'\}$ generate $V^{\ell}(\mathfrak{sl}_2) \otimes V^{\ell'}(\mathfrak{sl}_2)$ where $\ell = -\frac{\alpha+1}{\alpha}k - 1$ and $\ell' = -(\alpha+1)k - 1$, where $\alpha \neq 0,\ -1$.

Weight 2: Virasoro field L of central charge $c = -6k - 3$.

Weight $\frac{3}{2}$: Odd fields $G^{\pm\pm}$ which transform as $\mathbb{C}^2 \otimes \mathbb{C}^2$ under $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and satisfy complicated OPE relations. For example,

$$\begin{aligned}
G^{++}(z)G^{--}(w) &\sim -2\left(k(k+1) + \frac{\alpha}{(\alpha+1)^2}\right)(z-w)^{-3} \\
&+ \left(\frac{\alpha + k + \alpha k}{(1+a)^2}h' + \frac{\alpha(1+k+\alpha k)}{(1+\alpha)^2}h\right)(w)(z-w)^{-2} \\
&+ \left(kL + \frac{\alpha}{4(1+\alpha)^2} :h'h': + \frac{\alpha}{4(1+\alpha)^2} :hh: - \frac{\alpha}{2(1+\alpha)^2} :hh': \right. \\
&+ \frac{\alpha}{(1+\alpha)^2} :e'f': + \frac{\alpha}{(1+\alpha)^2} :ef: + \frac{\alpha k}{2(1+\alpha)}\partial h \\
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9. Affine coset of $V^{k,\alpha}_{N=4}$

Let $D^{k,\alpha} = \mathrm{Com}(V^{\ell}(\mathfrak{sl}_2) \otimes V^{\ell'}(\mathfrak{sl}_2), V^{k,\alpha}_{N=4})$.

Thm: For generic values of k and α, $D^{k,\alpha}$ is of type $\mathcal{W}(2,4,6,6,8,8,9,10,10,12)$.

Step 1: Rescale x, y, h, x', y', h', L by $\frac{1}{\sqrt{k}}$ and rescale $G^{\pm\pm}$ by $\frac{1}{k}$.

Then $V^{k,\alpha}$ admits a well defined limit $k \to \infty$ limit

$$\lim_{k\to\infty} V^{k,\alpha} \cong \mathcal{H}(6) \otimes \mathcal{T} \otimes \mathcal{G}_{\mathrm{odd}}(4).$$

$\mathcal{H}(6) = \lim_{k\to\infty} V^{\ell}(\mathfrak{sl}_2) \otimes V^{\ell'}(\mathfrak{sl}_2)$ a rank 6 Heisenberg algebra.

$\mathcal{T}$ has even generator L satisfying $L(z)L(w) \sim (z-w)^{-4}$.

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Step 2: By a general result of Arakawa, Creutzig, L., Kawasetsu (2017),

$$\lim_{k\to\infty} D^{k,\alpha} \cong \mathcal{T} \otimes \left(\mathcal{G}_{\text{odd}}(4)\right)^{SL_2\times SL_2}.$$

Action of $SL_2 \times SL_2$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is the same as the action of SO_4 on its standard module $\mathbb{C}^4$.

We can replace $\left(\mathcal{G}_{\text{odd}}(4)\right)^{SL_2\times SL_2}$ with $\left(\mathcal{G}_{\text{odd}}(4)\right)^{\mathrm{SO}_4}$.

Generator of $\mathcal{T}$ has weight 2 and corresponds to the Virasoro field.

Suffices to prove that $(\mathcal{G}_{\text{odd}}(4))^{\mathrm{SO}_4}$ is of type $\mathcal{W}(4,6,6,8,8,9,10,10,12)$.

Step 3: This is a formal consequence of Weyl's first and second fundamental theorems of invariant theory of SO_4.

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11. Isomorphism $C^{k_1,k_2} \cong D^{k,\alpha}$

Thm: We have an isomorphism of two-parameter vertex algebras

$$C^{k_1,k_2} \cong D^{k,\alpha}.$$

Parameters are related by

$$k_1 = -\frac{1+k+\alpha k}{(1+\alpha)k}, \qquad k_2 = -\frac{\alpha+k+\alpha k}{(1+\alpha)k}.$$

Note: symmetry $k_1 \leftrightarrow k_2$ corresponds to symmetry $\alpha \leftrightarrow \frac{1}{\alpha}$.

Idea of proof: Both algebras are generated by the weight 4 primary field, which is unique up to scaling.

It follows from VOA axioms that the full OPE algebra is determined by a small set of structure constants.

These can be found directly by computer.

11. Isomorphism $C^{k_1,k_2} \cong D^{k,\alpha}$

Thm: We have an isomorphism of two-parameter vertex algebras

$$C^{k_1,k_2} \cong D^{k,\alpha}.$$

Parameters are related by

$$k_1 = -\frac{1+k+\alpha k}{(1+\alpha)k}, \qquad k_2 = -\frac{\alpha+k+\alpha k}{(1+\alpha)k}.$$

Note: symmetry $k_1 \leftrightarrow k_2$ corresponds to symmetry $\alpha \leftrightarrow \frac{1}{\alpha}$.

Idea of proof: Both algebras are generated by the weight 4 primary field, which is unique up to scaling.

It follows from VOA axioms that the full OPE algebra is determined by a small set of structure constants.

These can be found directly by computer.

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12. Simple one-parameter quotients of C^{k_1,k_2}

C^{k_1,k_2} is **simple** as a VOA over $\mathbb{C}[k_1,k_2]$: for every proper graded ideal $\mathcal{I} \subseteq C^{k_1,k_2}$, $\mathcal{I}[0] \neq \{0\}$.

Equivalently, C^{k_1,k_2} is simple for generic k_1, k_2.

There exist curves in the parameter space $\mathbb{C}^2$ given by polynomials $p(k_1,k_2) = 0$, where C^{k_1,k_2} degenerates.

Ex: $p(k_1,k_2) = k_2 - 1$. Then $C^{k_1,1}$ has singular vector in weight 4.

Simple quotient $C_1^{k_1}$ coincides with

$$\mathrm{Com}(V^{k_1+1}(\mathfrak{sl}_2),\ V^{k_1}(\mathfrak{sl}_2)\otimes L_1(\mathfrak{sl}_2)).$$

This is well-known to be just the Virasoro algebra.

Ex: $p(k_1,k_2) = k_2 - n$, where $n \geq 1$ is a positive integer.

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13. Simple one-parameter quotients, cont'd

Thm: In the case $n = 2$, $C_2^{k_1}$ is of type $\mathcal{W}(2,4,6)$.

$C_2^{k_1}$ is isomorphic as a simple, one-parameter vertex algebra to the $\mathbb{Z}_2$-orbifold of the $N = 1$ superconformal vertex algebras.

Previously stated without proof in Blumenhagen et al (1995).

Thm: In the case $n = -\frac{1}{2}$, $C_{-1/2}^{k_1}$ is of type $\mathcal{W}(2,4,6)$, but not generically isomorphic to $C_2^{k_1}$.

Thm: In the case $n = -\frac{4}{3}$, $C_{-4/3}^{k_1}$ is of type $\mathcal{W}(2,6,8,10,12)$.

Rem: $(\mathcal{W}_3)^{\mathbb{Z}_2}$ is another one-parameter VOA of type $\mathcal{W}(2,6,8,10,12)$, but not generically isomorphic to $C_{-4/3}^{k_1}$.

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14. Simple zero-parameter quotients

Let k be **admissible**: $k = -2 + \frac{p}{q}$ where $(p, q) = 1$ and $p \geq 2$.

Thm:

1. The diagonal homomorphism $V^{k+2}(\mathfrak{sl}_2) \to L_k(\mathfrak{sl}_2) \otimes L_2(\mathfrak{sl}_2)$ descends to a map

$$L_{k+2}(\mathfrak{sl}_2) \hookrightarrow L_k(\mathfrak{sl}_2) \otimes L_2(\mathfrak{sl}_2).$$

2. The simple quotient $C_{k,2}$ of C^k_2 coincides with the coset

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3. $C_{k,2}$ is lisse and rational.

Statements (1) and (2) hold if 2 is replaced with an arbitrary positive integer n.

We expect (3) to hold as well, but we are unable to prove it.

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15. Simple zero-parameter quotients, cont'd

Proof of (3): Let $F(4)$ be the algebra of 4 free fermions.

Regarding $F(4)$ as $F(2) \otimes F(2)$, it is a simple current extension of $L_1(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2)$.

Regarding $F(4)$ as $F(3) \otimes F(1)$, it is a simple current extension of $L_2(\mathfrak{sl}_2) \otimes F(1)$.

Then $\mathrm{Com}(L_{k+2}(\mathfrak{sl}_2),\ L_k(\mathfrak{sl}_2) \otimes F(4))$ is both a simple current extension of $C_{k,1} \otimes C_{k+1,1}$, and a simple current extension of $C_{k,2} \otimes F(1)$.

Rationality of $C_{k,2}$ follows from rationality of $C_{k,1} \otimes C_{k+1,1}$.

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Then $\mathrm{Com}(L_{k+2}(\mathfrak{sl}_2),\ L_k(\mathfrak{sl}_2) \otimes F(4))$ is both a simple current extension of $\mathcal{C}_{k,1} \otimes \mathcal{C}_{k+1,1}$, and a simple current extension of $\mathcal{C}_{k,2} \otimes F(1)$.

Rationality of $\mathcal{C}_{k,2}$ follows from rationality of $\mathcal{C}_{k,1} \otimes \mathcal{C}_{k+1,1}$.

16. $C_{k,2}$ and principle $\mathcal{W}$-algebras of type C

Thm: We have the following isomorphisms $C_{k,2} \cong \mathcal{W}_\ell(\mathfrak{sp}_{2n}, f_{\text{prin}})$ for $n \geq 2$.

1. $k = -\dfrac{4n}{1+2n}, \qquad \ell = -(n+1) + \dfrac{1+2n}{4(1+n)},$
2. $k = \dfrac{3-2n}{n}, \qquad \ell = -(n+1) + \dfrac{3+2n}{4n},$
3. $k = 4n-6, \qquad \ell = -(n+1) + \dfrac{2n-1}{4(n-1)}.$

Rem: In cases (1) and (2), the levels ℓ are nondegenerate admissible, so the rationality of $\mathcal{W}_\ell(\mathfrak{sp}_{2n}, f_{\text{prin}})$ is already known (Arakawa, Annals of Math. 2015).

In case (3), the level ℓ is **degenerate admissible**.

Since $C_{k,2}$ is rational and lisse, we obtain new examples of rational and lisse principal $\mathcal{W}$-algebras.

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Rem: In cases (1) and (2), the levels ℓ are nondegenerate admissible, so the rationality of $\mathcal{W}_\ell(\mathfrak{sp}_{2n}, f_{\text{prin}})$ is already known (Arakawa, Annals of Math. 2015).

In case (3), the level ℓ is **degenerate admissible**.

Since $C_{k,2}$ is rational and lisse, we obtain new examples of rational and lisse principal $\mathcal{W}$-algebras.

16. $C_{k,2}$ and principle $\mathcal{W}$-algebras of type C

Thm: We have the following isomorphisms $C_{k,2} \cong \mathcal{W}_\ell(\mathfrak{sp}_{2n}, f_{\mathrm{prin}})$ for $n \geq 2$.

1. $k = -\dfrac{4n}{1+2n}, \qquad \ell = -(n+1) + \dfrac{1+2n}{4(1+n)},$
2. $k = \dfrac{3-2n}{n}, \qquad \ell = -(n+1) + \dfrac{3+2n}{4n},$
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17. Universal even spin $\mathcal{W}_\infty$-algebra

The following was conjectured by physicists Candu, Gaberdiel, Kelm, Vollenweider (2013).

There exists a universal 2-parameter VOA $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ of type $\mathcal{W}(2,4,\dots)$ with following properties:

- Generated by Virasoro field L and weight 4 primary field W^4.
- Freely generated of type $\mathcal{W}(2,4,6,\dots)$.
- All VOAs of type $\mathcal{W}(2,4,\dots,2N)$ for some N satisfying some mild hypotheses, arise as quotients.
- This includes principal $\mathcal{W}$-algebras of types B and C, as well as $\mathbb{Z}_2$-orbifold of type D principal $\mathcal{W}$-algebras.

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18. Idea of proof

For fields a, b, c in any VOA, and $r, s \geq 0$, we have identity

$$a_{(r)}(b_{(s)}c) = (-1)^{|a||b|}b_{(s)}(a_{(r)}c) + \sum_{i=0}^{r}\binom{r}{i}(a_{(i)}b)_{(r+s-i)}c.$$

These are called **Jacobi relations** of type (a, b, c).

Imposing relations of type (W^{2i}, W^{2j}, W^{2k}) for $2i + 2j + 2k \leq 2n + 2$ uniquely determines OPEs $W^{2a}(z)W^{2b}(w)$ for $a + b \leq 2n$.

We obtain a **nonlinear Lie conformal algebra** over ring $\mathbb{C}[c, \lambda]$.

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19. 1-parameter quotients of $\mathcal{W}^{\mathbf{ev}}(c,\lambda)$

Each weight space of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ is a free module over $\mathbb{C}[c,\lambda]$.

Let $I \subseteq \mathbb{C}[c,\lambda]$ be a prime ideal and let $I \cdot \mathcal{W}^{\mathrm{ev}}(c,\lambda)$ be the VOA ideal generated by I.

The quotient

$$\mathcal{W}^{\mathrm{ev},I}(c,\lambda) = \mathcal{W}^{\mathrm{ev}}(c,\lambda)/(I \cdot \mathcal{W}^{\mathrm{ev}}(c,\lambda))$$

is a VOA over $R = \mathbb{C}[c,\lambda]/I$.

Weight spaces are free R-modules, same rank as before.

$\mathcal{W}^{\mathrm{ev},I}(c,\lambda)$ is simple for a generic ideal I. But for certain discrete families of ideals I, $\mathcal{W}^{\mathrm{ev},I}(c,\lambda)$ is not simple.

Let $\mathcal{W}^{\mathrm{ev}}_I(c,\lambda)$ be simple graded quotient of $\mathcal{W}^{\mathrm{ev},I}(c,\lambda)$.

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20. Truncation curve $V(I_{2n})$ for $\mathcal{W}^k(\mathfrak{sp}_{2n}, f_{\mathbf{prin}})$

Let $I_{2n} = (p_{2n}(c,\lambda))$, where

$$p_{2n}(c,\lambda) = f(c,n) + \lambda g(c,n) + \lambda^2 h(c,n), \text{ and}$$

$$\begin{aligned}
f(c,n) = &-204c^2 - 192c^3 + 171c^4 + 952cn - 4612c^2n + 2348c^3n \\
&- 38c^4n + 1568n^2 - 7708cn^2 + 1788c^2n^2 + 2401c^3n^2 - 74c^4n^2 \\
&+ 560n^3 - 18936cn^3 + 22280c^2n^3 - 2112c^3n^3 + 8c^4n^3 \\
&- 16304n^4 + 18640cn^4 + 3420c^2n^4 - 364c^3n^4 + 8c^4n^4 \\
&- 17408n^5 + 27680cn^5 - 10576c^2n^5 + 304c^3n^5 - 3264n^6 \\
&- 3072cn^6 + 2736c^2n^6,
\end{aligned}$$

$$\begin{aligned}
g(c,n) = &-14(-1+c)(-1+2c)(22+5c)(-2+n)(-1+n) \\
&(3c + 10n + 2cn + 12n^2)(5c + 28n + 2cn + 40n^2),
\end{aligned}$$

$$\begin{aligned}
h(c,n) = &\,49(-1+c)^2(22+5c)^2(21c^2 + 70cn - 14c^2n + 200n^2 \\
&- 135cn^2 - 26c^2n^2 + 380n^3 - 176cn^3 + 8c^2n^3 + 436n^4 \\
&+ 132cn^4 + 8c^2n^4 + 448n^5 + 112cn^5 + 336n^6).
\end{aligned}$$

21. One-parameter VOAs of type $\mathcal{W}(2,4,6)$

Thm: There are exactly three distinct one-parameter VOAs of type $\mathcal{W}(2,4,6)$ that arise as quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$.

1. $\mathcal{W}^k(\mathfrak{sp}_6, f_{\mathrm{prin}})$ corresponds to the ideal I_6.
2. C_2^k corresponds to the ideal $J_2 = (q_2(c,\lambda))$ where

$$q_2(c,\lambda) = 7\lambda(-1+c)(-17+2c)(22+5c)+82-47c-10c^2.$$

3. $C_{-1/2}^k$ corresponds to the ideal $J_{-1/2} = (q_{-1/2}(c,\lambda))$ where

$$q_{-1/2}(c,\lambda) = 7\lambda(-41+c)(-1+c)(22+5c)-14+309c+5c^2.$$

The proof of our isomorphisms $C_{k,2} \cong \mathcal{W}_\ell(\mathfrak{sp}_{2n}, f_{\mathrm{prin}})$ involves finding **intersection points** on the curves $V(J_2)$ and $V(I_{2n})$.

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22. **A word about coincidences**

Let I, J be ideals in $\mathbb{C}[c,\lambda]$, and $\mathcal{W}^{\mathrm{ev}}_I(c,\lambda)$, $\mathcal{W}^{\mathrm{ev}}_J(c,\lambda)$ the corresponding simple, one-parameter quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$.

Aside from degenerate cases, pointwise coincidences between the simple quotients correspond to intersection points in $V(I)\cap V(J)$.

Often, $\mathcal{W}^{\mathrm{ev}}_I(c,\lambda)$ and $\mathcal{W}^{\mathrm{ev}}_I(c,\lambda)$ are isomorphic to vertex algebras $\mathcal{A}^k$ and $\mathcal{B}^\ell$ via rational parametrizations

$$k \mapsto ((c(k),\lambda(k)), \qquad \ell \mapsto (c(\ell),\lambda(\ell))$$

of the curves $V(I)$ and $V(J)$, respectively. In our examples, $\mathcal{A}^k = \mathcal{C}^k_2$ and $\mathcal{B}^\ell = \mathcal{W}_\ell(\mathfrak{sp}_{2n}, f_{\mathrm{prin}})$.

Subtlety 1: Specialization of C^k_2 at a number $k=k_0$ can be a *proper subset* of the coset $\mathrm{Com}(V_{k_0+2}(\mathfrak{sl}_2),\ V^{k_0}(\mathfrak{sl}_2), L_2(\mathfrak{sl}_2))$.

Subtlety 2: If k_0 is a pole of $c(k)$ or $\lambda(k)$, even if $C_{k_0,2}$ is defined, it is not obtained as a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ at this point.

Neither of these problems occur in our examples.

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Subtlety 1: Specialization of C^k_2 at a number $k=k_0$ can be a *proper subset* of the coset $\mathrm{Com}(V_{k_0+2}(\mathfrak{sl}_2),\ V^{k_0}(\mathfrak{sl}_2), L_2(\mathfrak{sl}_2))$.

Subtlety 2: If k_0 is a pole of $c(k)$ or $\lambda(k)$, even if $C_{k_0,2}$ is defined, it is not obtained as a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ at this point.

Neither of these problems occur in our examples.

23. Some open problems

Consider the diagonal coset

$$C^{k_1,k_2}(\mathfrak{sl}_n) = \mathrm{Com}(V^{k_1+k_2}(\mathfrak{sl}_n),\ V^{k_1}(\mathfrak{sl}_n)\otimes V^{k_2}(\mathfrak{sl}_n)).$$

This has a **stabilization property** as $n \to \infty$.

Both the graded character up to weight k, and the strong generating type up to weight k, are independent of N for $N > k$.

In the stable limit, the algebra is of type $\mathcal{W}(2,3,4^2,5^2,6^4,7^4,8^7,\dots)$.

Idea: Generating type of $C^{k_1,k_2}(\mathfrak{sl}_n)$ is the same as $V^k(\mathfrak{sl}_n)^{SL_n}$.

Need first and second fundamental theorems of invariant theory for the adjoint representation of SL_n (Procesi, 1976).

Description of generators (FFT) is independent of n, although relations depend on n.

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24. Some open problems

Thm: There exists a 3-parameter vertex algebra which is freely generated of type $\mathcal{W}(2,3,4^2,5^2,6^4,7^4,8^7,\dots)$.

For each $n \geq 3$, the 2-parameter coset $C^{k_1,k_2}(\mathfrak{sl}_n)$ arises as a quotient of this algebra.

It is not clear if this is the **universal algebra** of this kind.

Question: For $n \geq 3$, is there a vertex superalgebra $V^{k,\alpha}(\mathfrak{sl}_n)$ containing two copies of affine $\mathfrak{sl}_n$ in weight 1, which is an analogue of the large $N=4$ algebra $V^{k,\alpha}_{N=4}$?

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