Automorphism groups of some orbifold models of lattice VOAs

Ching Hung Lam

Academia Sinica

Based on joint works with Hiroki Shimakura and Koichi Betsumiya

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- We need to know the groups $\mathrm{Aut}\,(V_{L_{\mathfrak{g}}})$ and $\mathrm{Aut}\,(V_{\Lambda_g}^{\hat{g}})$.

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Let $f : (\mathrm{Irr}(V^1), q_1) \to (\mathrm{Irr}(V^2), -q_2)$ be an isometry such that

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Then $S = \{(M, f(M)) \mid M \in \mathrm{Irr}(V^1)\}$ is a maximal totally singular subspace of $(\mathrm{Irr}(V^1) \oplus \mathrm{Irr}(V^2), q_1 + q_2)$.

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By [Shimakura 2007], there is an exact sequence

$$1 \to S^* \to N_{\mathrm{Aut}\,(V)}(S^*) \to \mathrm{Stab}_{\mathrm{Aut}\,(V^1 \otimes V^2)}(S) \to 1,$$

where $\mathrm{Stab}_{\mathrm{Aut}\,(V^1 \otimes V^2)}(S) = \{g \in \mathrm{Aut}\,(V^1 \otimes V^2) \mid S \circ g = S\}$
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Note: $\mathrm{Aut}\,(V^1 \otimes V^2) = \mathrm{Aut}\,(V^1) \times \mathrm{Aut}\,(V^2)$ since $V^1 \not\cong V^2$.

Let

$$\mu_i : \mathrm{Aut}\,(V^i) \to O(\mathrm{Irr}(V^i), q_i), i = 1, 2,$$

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Lemma

$$\mathrm{Stab}_{\mathrm{Aut}\,(V^1 \otimes V^2)}(S)/(\ker \mu_1 \times \ker \mu_2) \cong (\mathrm{Im}\,\mu_1) \cap f^{-1}(\mathrm{Im}\,\mu_2)f.$$

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Proposition

Assume $\ker \mu_2 = id$. *Then we have*

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Lemma

We have $K(V) < \mathrm{Inn}\,(V)$ *and*

$$K(V) = \{\exp(-2\pi\sqrt{-1}x_{(0)}) \mid x \in \tilde{Q}^*/L_{\mathfrak{g}}\},$$

where $\tilde{Q} = \bigoplus_{i=1}^{s} \frac{1}{\sqrt{k_i}} Q^i$, Q^i *is the root lattice of* $\mathfrak{g}_i$ *and* $V_1 \cong \mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$.

For $\ker \mu_1$, let $X(L) = \{h \in O(L) \mid h = id \text{ on } \mathcal{D}(L) = L^*/L\}$ and

$$X(\hat{L}) = \{g \in O(\hat{L}) \mid \bar{g} \in X(L)\}.$$

Then we have

Lemma

$$\ker \mu_1 = \mathrm{Inn}\,(V_{L_\mathfrak{g}})X(\hat{L}_\mathfrak{g}) \quad \textit{and} \quad \mathrm{Im}\,\mu_1 \cong O(L_\mathfrak{g})/X(L_\mathfrak{g}).$$

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Moreover, there is an exact sequence of [FLM88, Proposition 5.4.1]

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When $L(2) = \{x \in L \mid \langle x, x\rangle = 2\} = \emptyset$, the normal subgroup $N(V_L) = \{\exp(\lambda\alpha(0)) \mid \alpha \in L,\ \lambda \in \mathbb{C}\}$ is abelian and we have

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In particular, we have an exact sequence

$$1 \to N(V_L) \to \mathrm{Aut}\,(V_L) \xrightarrow{\varphi} O(L) \to 1.$$

Theorem

Let L be an even positive definite lattice with $L(2) = \emptyset$. Let g be a fixed point free isometry of L and $\hat{g}$ a lift of g in $O(\hat{L})$. Then we have the following exact sequences.

$$1 \longrightarrow \mathrm{Hom}(L/(1-g)L, \mathbb{C}^*) \longrightarrow N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g} \rangle) \xrightarrow{\varphi} N_{O(L)}(\langle g \rangle) \longrightarrow 1;$$

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Definition

An automorphism $h \in \mathrm{Aut}\,(V_L^{\hat{g}})$ is said to be an extra automorphism if it is not in the image of f.

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$$h_{A_n} : A \to P^{-1}AP \qquad \text{for } A \in sl(n+1, \mathbb{C}),$$

and

$$B^{-1}PB = \mathrm{diag}(\omega, \omega^2, ..., 1)$$

where

$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & \\ 0 & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{n+1}} \begin{pmatrix} \omega & \omega^2 & \cdots & \omega^n & 1 \\ \omega^2 & \omega^4 & \cdots & \omega^{2n} & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \omega^n & \omega^{2n} & \ddots & \omega^{n^2} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

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By a direct calculation, it follows that

$$\sigma_{A_n} h_{A_n} \sigma_{A_n}^{-1}(E_{ij}) = B^{-1}P^{-1}BE_{st}B^{-1}PB = \omega^{j-i}E_{ij}.$$

Let $\rho_{A_n} = \frac{1}{2}(n-1, n-2, \dots, -(n-2), -(n-1))$ be the Weyl vector.
Define $\eta_{A_n} = \exp(\frac{1}{n+1}(2\pi i \rho_{A_n}(0))$.

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Lemma

We have $\sigma_{A_n} h_{A_n} \sigma_{A_n}^{-1} = \eta_{A_n}$ and $\sigma_{A_n} \eta_{A_n} \sigma_{A_n}^{-1} = h_{A_n}^{-1}$ on $sl_{n+1}(\mathbb{C})$.

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Since they are inner automorphisms, we can extend them to V_L by using the same exponential expressions.

Let

$$R = A_{k_1} \oplus \cdots \oplus A_{k_j}$$

be an orthogonal sum of simple root lattices of type A.
Let L be an even overlattice of R and $\hat{\rho} = \sum_{i=1}^{j} \frac{1}{(k_i+1)} \rho_{A_{k_i}}$.
Set

$$X = L(\hat{\rho}) = \{\alpha \in L \mid \langle \alpha, \hat{\rho} \rangle \in \mathbb{Z}\}.$$

Then $L = \mathrm{Span}_{\mathbb{Z}} X \cup R$.
Set

$$h = h_{A_{k_1}} \otimes \cdots \otimes h_{A_{k_j}}, \eta = \eta_{A_{k_1}} \otimes \cdots \otimes \eta_{A_{k_j}}, \sigma = \sigma_{A_{k_1}} \otimes \cdots \otimes \sigma_{A_{k_j}}.$$

Since they are inner automorphisms, we can extend them to V_L by using the same exponential expressions.

Theorem

We have $\sigma(V_X^h) = V_X^h$ and σ induces an automorphism of V_X^h.

Next, we discuss several explicit examples (10 cases mentioned by Höhn).

Table: Standard lift of $g \in O(\Lambda)$

Class	Type	$rank(\Lambda^g)$	$\lvert\phi_g\rvert$	ρ_g	$O(\Lambda_g)$	$R(V_{\Lambda_g}^{\hat{g}})$
$2A$	$1^8 2^8$	16	2	1/2	$2.O_8^+(2)$	2^{10}
2C	2^{12}	12	**4**	3/4	$2^{11}.Sym_{12}$	$2^{10}4^2$
$3B$	$1^6 3^6$	12	3	2/3	$6.PSU_4(3).2^2$	3^8
$4C$	$1^4 2^2 4^4$	10	4	3/4	$[2^{13}].Sym_6$	$2^2 4^6$
$5B$	$1^4 5^4$	8	5	4/5	$(Frob_{20} \times O_4^+(5))/2$	5^6
$6E$	$1^2 2^2 3^2 6^2$	8	6	5/6	$D_{12}.(O_4^+(2) \times O_4^+(3))$	$2^6 3^6$
6G	$2^3 6^3$	6	**12**	11/12	$[2^{11}.3^4]$	$2^4.4^2.3^5$
$7B$	$1^3 7^3$	6	7	6/7	$7.3.2.L_2(7).2$	7^5
$8E$	$1^2 2^1 4^1 8^2$	6	8	7/8	$[2^{12}.3]$	$2.4.8^4$
10F	$2^2 10^2$	4	**20**	19/20	$5.2.[2^8]$	$2^2.4^2.5^4$

ϕ_g denotes the standard lift of g in $\mathrm{Aut}\,(V_\Lambda)$.

Holy construction for the Leech lattice

Let N be a Niemeier lattice with the root lattice $R = R_1 \oplus \cdots \oplus R_j$, where R_i's are A, D or E type .

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Define

$$N(\rho) = \{x \in N \mid \langle x, \rho \rangle \in \mathbb{Z}\},$$

and let $\alpha \in \rho + N$ such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$.

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$$N(\rho) = \{x \in N \mid \langle x, \rho\rangle \in \mathbb{Z}\},$$

and let $\alpha \in \rho + N$ such that $\langle \alpha, \alpha\rangle \in 2\mathbb{Z}$.

Then the lattice $\tilde{N}_\rho = \mathrm{Span}_{\mathbb{Z}} N(\rho) \cup \{\alpha\}$ is isomorphic to the Leech lattice [Conway-Sloane, Chapter 24].
In particular, the Leech lattice contains a sublattice isometric to $R(\rho) = \{x \in R \mid \langle x, \rho\rangle \in \mathbb{Z}\}$.

We verify that all coinvariant lattices mentioned in Table above can be realized as a lattice of the form $L(\hat{\rho})$.
The result is summarized in Table 2.

Table: Coinvariant lattices as $L(\hat{\rho})$

Class	Type	$rank(\Lambda_g)$	Niemeier	R	Glue
$2A$	$1^8 2^8$	8	A_1^{24}	A_1^8	(1^8)
2C	2^{12}	12	A_1^{24}	A_1^{12}	(1^{12})
$3B$	$1^6 3^6$	12	A_2^{12}	A_2^6	$(1^3, -1^3)$
$4C$	$1^4 2^2 4^4$	14	A_3^8	$A_3^4 A_1^2$	$(111-1\|11)$
$5B$	$1^4 5^4$	16	A_4^6	A_4^4	(1243)
$6E$	$1^2 2^2 3^2 6^2$	16	$A_5^4 D_4$	$A_5^2 A_2^2 A_1^2$	$(11\|11\|11)$
6G	$2^3 6^3$	18	$A_5^4 D_4$	$A_5^3 A_1^3$	$(551\|111)$
$7B$	$1^3 7^3$	18	A_6^4	A_6^3	(124)
$8E$	$1^2 2^1 4^1 8^2$	18	$A_7^2 D_5^2$	$A_7^2 A_3 A_1$	$(13\|1\|1)$
10F	$2^2 10^2$	20	$A_9^2 D_6$	$A_9^2 A_1^2$	$(32\|11)$

Note that $A_5 > A_2^2$, $D_4 > A_1^4$, $D_5 > A_3 A_1^2$ and $D_6 > A_1^6$ as sublattices.

Theorem

Let $g \in O(\Lambda)$. Suppose $C_{O(\Lambda_g)}(\langle g\rangle)/\langle g\rangle$ acts faithfully on Λ_g^*/Λ_g. Then the natural homomorphism

$$\mu_2 : \mathrm{Aut}\,(V_{\Lambda_g}^{\hat{g}}) \to O(R(V_{\Lambda_g}^{\hat{g}}), q)$$

is injective, i.e., $\ker \mu_2 = id$.

Automorphism groups

Class	Type	$rank(\Lambda^g)$	$\lvert\phi_g\rvert$	ρ_g	$O(\Lambda_g)$	$R(V_{\Lambda_g}^{\hat{g}})$	$\mathrm{Aut}\,(V_{\Lambda_g}^{\hat{g}})$
$2A$	$1^8 2^8$	16	2	1/2	$2.O_8^+(2)$	2^{10}	$O_{10}^+(2)$
2C	2^{12}	12	**4**	3/4	$2^{11}.Sym_{12}$	$2^{10}4^2$	$2^{12}.2^{10}.Sym_{12}.Sym_3$
$3B$	$1^6 3^6$	12	3	2/3	$6.PSU_4(3).2^2$	3^8	$\Omega_8^-(3).2$
$4C$	$1^4 2^2 4^4$	10	4	3/4	$[2^{13}].Sym_6$	$2^2 4^6$	index 2
$5B$	$1^4 5^4$	8	5	4/5	$(Frob_{20} \times O_4^+(5))/2$	5^6	$\Omega_6^+(5).2$
$6E$	$1^2 2^2 3^2 6^2$	8	6	5/6	$D_{12}.(O_4^+(2) \times O_4^+(3))$	$2^6 3^6$	index 2
6G	$2^3 6^3$	6	**12**	11/12	$[2^{11}.3^4]$	$2^4.4^2.3^5$	
$7B$	$1^3 7^3$	6	7	6/7	$7.3.2.L_2(7).2$	7^5	$\Omega_5(7).2$
$8E$	$1^2 2^1 4^1 8^2$	6	8	7/8	$[2^{12}.3]$	$2.4.8^4$	index 2
10F	$2^2 10^2$	4	**20**	19/20	$5.2.[2^8]$	$2^2.4^2.5^4$	

$2A$ element in $O(\Lambda)$

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Let L be an even lattice of rank 16 such that $\mathcal{D}(L) \cong \mathrm{Irr}(V_L) \cong \mathbb{Z}_2^{10}$ and $\langle\alpha|\alpha\rangle \in \mathbb{Z}$ for $\alpha \in L^*$.

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Set $N = \sqrt{2}L^*$. Then $\mathcal{D}(N) \cong \mathbb{Z}_2^6$. and N is a level 2 lattice. Such lattices has been classified.

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Set $N = \sqrt{2}L^*$. Then $\mathcal{D}(N) \cong \mathbb{Z}_2^6$. and N is a level 2 lattice. Such lattices has been classified.

Proposition ([SV01, Theorem 2])

Up to isometry, there exist exactly 17 *level* 2 *lattices of rank* 16 *with determinant* 2^6. *Moreover, they are uniquely determined by their root systems.*

The root systems and isometry groups of the lattices in the proposition above are summarized in Table below.

Level 2 lattices N of rank 16 with $\mathcal{D}(N) \cong \mathbb{Z}_2^6$

Root system $R(N)$	N/Q	$O(N)/W(R(N))$	Isometry group $O(N)$
A_1^{16}	2^5	$\mathrm{AGL}_4(2)$	$W(A_1) \wr \mathrm{AGL}_4(2)$
$A_3^4(\sqrt{2}A_1)^4$	$2^3 4^1$	$W(D_4)$	$(W(A_3)^4 \times W(A_1)^4).W(D_4)$
$D_4^2 C_2^4$	2^3	$2 \times \mathrm{Sym}_4$	$(W(D_4)^2 \times W(C_2)^4).(2 \times \mathrm{Sym}_4)$
$A_5^2(\sqrt{2}A_2)^2 C_2$	$3^1 6^1$	Dih_8	$(W(A_5)^2 \times W(A_2)^2 \times W(C_2)).Dih_8$
$A_7(\sqrt{2}A_3)C_3^2$	$2^1 4^1$	$\mathbb{Z}_2^2$	$(W(A_7) \times W(A_3) \times W(C_3)^2).\mathbb{Z}_2^2$
$D_5^2(\sqrt{2}A_3)^2$	4^2	Dih_8	$(W(D_5)^2 \times W(A_3)^2).Dih_8$
C_4^4	2^1	Sym_4	$W(C_4) \wr \mathrm{Sym}_4$
$D_6 C_4(\sqrt{2}B_3)^2$	2^2	$\mathbb{Z}_2$	$(W(D_6) \times W(C_4) \times W(B_3)^2).\mathbb{Z}_2$
$A_9(\sqrt{2}A_4)(\sqrt{2}B_3)$	10^1	$\mathbb{Z}_2$	$(W(A_9) \times W(A_4) \times W(B_3)).\mathbb{Z}_2$
$E_6(\sqrt{2}A_5)C_5$	6^1	$\mathbb{Z}_2$	$(W(E_6) \times W(A_5) \times W(C_5)).\mathbb{Z}_2$
$C_6^2(\sqrt{2}B_4)$	2^1	$\mathbb{Z}_2$	$W(C_6) \wr 2 \times W(B_4)$
$D_8(\sqrt{2}B_4)^2$	2^2	$\mathbb{Z}_2$	$W(D_8) \times W(B_4) \wr \mathbb{Z}_2$
$D_9(\sqrt{2}A_7)$	8^1	$\mathbb{Z}_2$	$(W(D_9) \times W(A_7)).\mathbb{Z}_2$
$C_8 F_4^2$	1	$\mathbb{Z}_2$	$W(C_8) \times W(F_4) \wr \mathbb{Z}_2$
$E_7(\sqrt{2}B_5)F_4$	2^1	1	$W(E_7) \times W(B_5) \times W(F_4)$
$C_{10}(\sqrt{2}B_6)$	2^1	1	$W(C_{10}) \times W(B_6)$
$E_8(\sqrt{2}B_8)$	2^1	1	$W(B_8) \times W(E_8)$

Level 2 lattices N of rank 16 with $\mathcal{D}(N) \cong \mathbb{Z}_2^6$

Root system $R(N)$	N/Q	$O(N)/W(R(N))$	Isometry group $O(N)$
A_1^{16}	2^5	$\mathrm{AGL}_4(2)$	$W(A_1) \wr \mathrm{AGL}_4(2)$
$A_3^4(\sqrt{2}A_1)^4$	$2^3 4^1$	$W(D_4)$	$(W(A_3)^4 \times W(A_1)^4).W(D_4)$
$D_4^2 C_2^4$	2^3	$2 \times \mathrm{Sym}_4$	$(W(D_4)^2 \times W(C_2)^4).(2 \times \mathrm{Sym}_4)$
$A_5^2(\sqrt{2}A_2)^2 C_2$	$3^1 6^1$	Dih_8	$(W(A_5)^2 \times W(A_2)^2 \times W(C_2)).Dih_8$
$A_7(\sqrt{2}A_3)C_3^2$	$2^1 4^1$	$\mathbb{Z}_2^2$	$(W(A_7) \times W(A_3) \times W(C_3)^2).\mathbb{Z}_2^2$
$D_5^2(\sqrt{2}A_3)^2$	4^2	Dih_8	$(W(D_5)^2 \times W(A_3)^2).Dih_8$
C_4^4	2^1	Sym_4	$W(C_4) \wr \mathrm{Sym}_4$
$D_6 C_4(\sqrt{2}B_3)^2$	2^2	$\mathbb{Z}_2$	$(W(D_6) \times W(C_4) \times W(B_3)^2).\mathbb{Z}_2$
$A_9(\sqrt{2}A_4)(\sqrt{2}B_3)$	10^1	$\mathbb{Z}_2$	$(W(A_9) \times W(A_4) \times W(B_3)).\mathbb{Z}_2$
$E_6(\sqrt{2}A_5)C_5$	6^1	$\mathbb{Z}_2$	$(W(E_6) \times W(A_5) \times W(C_5)).\mathbb{Z}_2$
$C_6^2(\sqrt{2}B_4)$	2^1	$\mathbb{Z}_2$	$W(C_6) \wr 2 \times W(B_4)$
$D_8(\sqrt{2}B_4)^2$	2^2	$\mathbb{Z}_2$	$W(D_8) \times W(B_4) \wr \mathbb{Z}_2$
$D_9(\sqrt{2}A_7)$	8^1	$\mathbb{Z}_2$	$(W(D_9) \times W(A_7)).\mathbb{Z}_2$
$C_8 F_4^2$	1	$\mathbb{Z}_2$	$W(C_8) \times W(F_4) \wr \mathbb{Z}_2$
$E_7(\sqrt{2}B_5)F_4$	2^1	1	$W(E_7) \times W(B_5) \times W(F_4)$
$C_{10}(\sqrt{2}B_6)$	2^1	1	$W(C_{10}) \times W(B_6)$
$E_8(\sqrt{2}B_8)$	2^1	1	$W(B_8) \times W(E_8)$

Note: The group $\mathrm{AGL}_4(2)$ can be regarded as a subgroup of Sym_{16} via the action on the first order Reed-Muller code $\mathrm{RM}(1,4)$ of length 16, which is the glue code of the lattice with respect to A_1^{16}.

$K(V)$ and $\mathrm{Out}(V)$ for the case $g \in 2A$

No. in [Sc93]	$R(\sqrt{2}L^*)$	V_1	dim V_1	$K(V)$	$\mathrm{Out}(V)$
5	A_1^{16}	$A_{1,2}^{16}$	48	$\mathbb{Z}_2^5$	$\mathrm{AGL}_4(2)$
16	$A_3^4(\sqrt{2}A_1)^4$	$A_{3,2}^4 A_{1,1}^4$	72	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$W(D_4)$
22	$A_5^2(\sqrt{2}A_2)^2 C_2$	$A_{5,2}^2 C_{2,1} A_{2,1}^2$	96	$\mathbb{Z}_3 \times \mathbb{Z}_6$	Dih_8
25	$D_4^2 C_2^2$	$D_{4,2}^2 C_{2,1}^4$	96	$\mathbb{Z}_2^3$	$\mathbb{Z}_2 \times \mathrm{Sym}_4$
31	$D_5^2(\sqrt{2}A_3)^2$	$D_{5,2}^2 A_{3,1}^2$	120	$\mathbb{Z}_4^2$	Dih_8
33	$A_7(\sqrt{2}A_3) C_3^2$	$A_{7,2} C_{3,1}^2 A_{3,1}$	120	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2$
38	C_4^4	$C_{4,1}^4$	144	$\mathbb{Z}_2$	Sym_4
39	$D_6 C_4(\sqrt{2}B_3)^2$	$D_{6,2} C_{4,1} B_{3,1}^2$	144	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
40	$A_9(\sqrt{2}A_4)(\sqrt{2}B_3)$	$A_{9,2} A_{4,1} B_{3,1}$	144	$\mathbb{Z}_{10}$	$\mathbb{Z}_2$
44	$E_6 A(\sqrt{2}A_5) C_5$	$E_{6,2} C_{5,1} A_{5,1}$	168	$\mathbb{Z}_6$	$\mathbb{Z}_2$
47	$D_8(\sqrt{2}B_4)^2$	$D_{8,2} B_{4,1}^2$	192	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
48	$C_6^2(\sqrt{2}B_4)$	$C_{6,1}^2 B_{4,1}$	192	$\mathbb{Z}_2$	$\mathbb{Z}_2$
50	$D_9(\sqrt{2}A_7)$	$D_{9,2} A_{7,1}$	216	$\mathbb{Z}_8$	$\mathbb{Z}_2$
52	$C_8 F_4^2$	$C_{8,1} F_{4,1}^2$	240	1	$\mathbb{Z}_2$
53	$D_7(\sqrt{2}B_5) F_4$	$E_{7,2} B_{5,1} F_{4,1}$	240	$\mathbb{Z}_2$	1
56	$C_{10}(\sqrt{2}B_6)$	$C_{10,1} B_{6,1}$	288	$\mathbb{Z}_2$	1
62	$(\sqrt{2}B_8) E_8$	$B_{8,1} E_{8,2}$	384	$\mathbb{Z}_2$	1

$3B$ element in $O(\Lambda)$

Assume that g belongs to the conjugacy class $3B$ of $O(\Lambda)$;
its cycle shape is $1^6 3^6$.

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The coinvariant lattice Λ_g is isometric to the Coxeter-Todd lattice K_{12} of rank 12.
Then $\mathrm{Irr}(V^2) \cong \mathbb{Z}_3^8$ and $\mathrm{Aut}\,(V^2) \cong \Omega^-(8,3){:}2$, which is an index 2 subgroup of the full orthogonal group

$$O(\mathrm{Irr}(V^2), q_2) = GO^-(8,3) \cong 2 \times \Omega^-(8,3){:}2.$$

$3B$ element in $O(\Lambda)$

Assume that g belongs to the conjugacy class $3B$ of $O(\Lambda)$;
its cycle shape is $1^6 3^6$.
The coinvariant lattice Λ_g is isometric to the Coxeter-Todd lattice K_{12} of rank 12.
Then $\mathrm{Irr}(V^2) \cong \mathbb{Z}_3^8$ and $\mathrm{Aut}\,(V^2) \cong \Omega^-(8,3){:}2$, which is an index 2 subgroup of the full orthogonal group

$$O(\mathrm{Irr}(V^2), q_2) = GO^-(8,3) \cong 2 \times \Omega^-(8,3){:}2.$$

Hence $\mu_2 : \mathrm{Aut}\,(V^2) \to O(\mathrm{Irr}(V^2), q_2)$ is injective.

Let L be an even lattice of rank 12 such that $\mathcal{D}(L) \cong \mathrm{Irr}(V_L) \cong \mathbb{Z}_3^8$ and $\langle \alpha | \alpha \rangle \in (2/3)\mathbb{Z}$ for all $\alpha \in L^*$.

Let L be an even lattice of rank 12 such that $\mathcal{D}(L) \cong \mathrm{Irr}(V_L) \cong \mathbb{Z}_3^8$ and $\langle \alpha | \alpha \rangle \in (2/3)\mathbb{Z}$ for all $\alpha \in L^*$.

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Set $N = \sqrt{3}L^*$. Then $\mathcal{D}(N) \cong \mathbb{Z}_3^4$ and N is a level 3 lattice.

Such lattices are also classified.

Proposition ([SV01, Theorem 3])

Up to isometry, there exist exactly 6 level 3 lattices of rank 12 with determinant 3^4. Moreover, they are uniquely determined by their root systems.

Table: Level 3 lattices N of rank 12 with $\mathcal{D}(N) \cong \mathbb{Z}_3^4$

Root system $R(N)$	N/Q	$O(N)/W(R(N))$	Isometry group $O(N)$
A_2^6	3^1	$\mathbb{Z}_2 \times \mathrm{Sym}_6$	$(W(A_2) \wr \mathrm{Sym}_6).\mathbb{Z}_2$
$A_5 D_4(\sqrt{3}A_1)^3$	2^3	Dih_{12}	$(W(A_5) \times W(D_4) \times W(A_1)^3).Dih_{12}$
$A_8(\sqrt{3}A_2)^2$	3^2	$\mathbb{Z}_2^2$	$(W(A_8) \times W(A_2)^2).\mathbb{Z}_2^2$
$D_7(\sqrt{3}A_3)G_2$	4^1	$\mathbb{Z}_2$	$(W(D_7) \times W(A_3) \times W(G_2)).\mathbb{Z}_2$
$E_6 G_2^3$	1	$\mathbb{Z}_2 \times \mathrm{Sym}_3$	$(W(E_6) \times W(G_2) \wr \mathrm{Sym}_3).\mathbb{Z}_2$
$E_7(\sqrt{3}A_5)$	6^1	$\mathbb{Z}_2$	$(W(E_7) \times W(A_5)).\mathbb{Z}_2$

Table: $K(V)$ and $\mathrm{Out}(V)$ for the case $g \in 3B$

No. in [Sc93]	$R(N)$	V_1	dim V_1	$K(V)$	$\mathrm{Out}(V)$
6	A_2^6	$A_{2,3}^6$	48	$\mathbb{Z}_3$	Sym_6
17	$A_5 D_4(\sqrt{3}A_1)^3$	$A_{5,3}D_{4,3}A_{1,1}^3$	72	$\mathbb{Z}_2^3$	Sym_3
27	$A_8(\sqrt{3}A_2)^2$	$A_{8,3}A_{2,1}^2$	96	$\mathbb{Z}_3^2$	$\mathbb{Z}_2$
32	$E_6 G_2^3$	$E_{6,3}{G_{2,1}}^3$	120	1	Sym_3
34	$D_7(\sqrt{3}A_3)G_2$	$D_{7,3}A_{3,1}G_{2,1}$	120	$\mathbb{Z}_4$	1
45	$E_7(\sqrt{3}A_5)$	$E_{7,3}A_{5,1}$	168	$\mathbb{Z}_6$	1

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Then $N = \sqrt{5}L^*$ is even and $\mathcal{D}(N) \cong \mathbb{Z}_5^2$.
There are two such lattices and their root system are A_4^2 and $D_6(\sqrt{5}A_1^2)$.

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Table: Level 5 lattices N of rank 8 with $\mathcal{D}(N) \cong \mathbb{Z}_5^2$

Root system $R(N)$	N/Q	$O(N)/W(R(N))$	Isometry group $O(N)$
A_4^2	1	Dih_8	$(2 \times W(A_4)) \wr \mathrm{Sym}_2$
$D_6(\sqrt{5}A_1^2)$	2^2	Sym_2	$(W(D_6) \times W(A_1)^2).2$

Table: $K(V)$ and $\mathrm{Out}(V)$ for the case $g \in 5B$

No. in [Sc93]	$R(N)$	V_1	$\dim V_1$	$K(V)$	$\mathrm{Out}(V)$
9	A_4^5	$A_{4,5}^2$	48	1	$\mathbb{Z}_2^2$
20	$D_6(\sqrt{5}A_1^2)$	$D_{6,5}A_{1,1}^2$	72	$\mathbb{Z}_2^2$	1

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The coinvariant lattice Λ_g has rank 18 and $\mathrm{Irr}(V^2) \cong \mathbb{Z}_7^5$.
Moreover, $\mathrm{Aut}\,(V^2)$ is isomorphic to an index 2 subgroup of $O(\mathrm{Irr}(V^2), q_2) \cong GO_5(7)$.

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Table: Level 7 lattices N of rank 6 with $\mathcal{D}(N) \cong \mathbb{Z}_7$

Root system $R(N)$	N/Q	$O(N)/W(R(N))$	Isometry group $O(N)$
A_6	1	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times W(A_6)$

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The group structures of $K(V)$ and $\mathrm{Out}(V)$ are as follows.

Table: $K(V)$ and $\mathrm{Out}(V)$ for the case $g \in 7B$

No. in [Sc93]	$R(N)$	V_1	$\dim V_1$	$K(V)$	$\mathrm{Out}(V)$
11	A_6	$A_{6,7}$	48	1	1

Thank You

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