Some Zhu reduction formula and applications

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Recap: The original Zhu story

Zhu's work: The study of n-point functions.

Definition (n-point functions)

Let V be a VOA with Virasoro vector ω of central charge $\mathbf{c}$. For $v_1, \ldots, v_n \in V$ and a weak V-module M, the n-point function is

$$\begin{aligned} &Z_M\left((v_1, x_1), \ldots, (v_n, x_n); \;\; \right) \\ &\quad := \mathrm{tr}_M Y\left(e^{x_1 L(0)} v_1, e^{x_1}\right) \cdots Y\left(e^{x_n L(0)} v_n, e^{x_n}\right) q^{L(0) - \frac{\mathbf{c}}{24}}, \end{aligned}$$

where $q := e^{2\pi i \tau}$ with $\tau \in \mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$.

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- Established their modularity.

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Zhu's core results concerning n-point functions:

- Established their modularity.
- Established their convergence.

Recap: The original Zhu story

Theorem

Suppose V is a rational and C_2-cofinite and $V = M_0, M_1, \ldots, M_k$ be its inequivalent irreducible modules. Moreover let $v_s \in V_{[\mathrm{wt}\, v_s]}$ for $1 \le s \le n$. Then

1. *each $Z_M\left((v_1, x_1); \tau\right)$ converges on $\mathbb{H}$, and*
2. *for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have there exists scalars $\alpha_{ij} \in \mathbb{C}$ such that*

$$Z_{M_i}\left((v_1, x_1), \ldots, (v_n, x_n); \frac{a\tau + b}{c\tau + d}\right)$$
$$= (c\tau + d)^{\sum \mathrm{wt}\, v_j} \sum_{j=1}^{k} \alpha_{ij} Z_{M_j}\left((v_1, x_1), \ldots, (v_n, x_n); \tau\right).$$

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Core idea of proof

Zhu expressed n-point functions as linear combinations of $(n-1)$-point functions.

- Reduced the study of n-point functions to the study of 1-point functions.
- Allowed the creation of ODEs whose solution space consisted of 1-point functions.

Recap: Original Zhu reduction formula, Part I

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We have

$$Z_M\left((a,y),(v_1,x_1),\ldots,(v_n,x_n);\tau\right)$$
$$=\mathrm{tr}_M\, v(\mathrm{wt}\, a-1)Y\left(e^{x_1L(0)}v_1,e^{x_1}\right)\cdots Y\left(e^{x_nL(0)}v_n,e^{x_n}\right)q^{L(0)-\frac{c}{24}}$$
$$+\sum_{j=1}^{n}\sum_{m\geq 0}P_{m+1}\left(\frac{y-x_j}{2\pi i},\tau\right)Z_M\left((v_1,x_1),\ldots,(a[m]v_j,x_j),\ldots,(v_n,x_n);\tau\right).$$

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Where ($q_w=e^{2\pi i w}$)

$$P_{m+1}(w,\tau):=\frac{(-1)^{m+1}}{m!}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{n^m q_w^n}{1-q^n}-\delta_{m,0}\frac{1}{2}$$
$$=\frac{(-1)^m}{m!}\left(\frac{1}{2\pi i}\frac{d}{dw}\right)^n\left(P_1(w,\tau)\right).$$

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Let $a, v_1, \ldots v_n \in V$. For $N \geq 1$ we have

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& Z_M\left((a[-N]v_1, x_1), \ldots, (v_n, x_n); \tau\right) \\
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& + \sum_{m \geq 0} (-1)^{m+1} \binom{m+N-1}{m} G_{m+N}(\tau) Z_M\left((a[m]v_1, x_1), \ldots, (v_n, x_n); \tau\right) \\
& \quad + \sum_{j=2}^{n} \sum_{m \geq 0} (-1)^{N+1} \binom{m+N-1}{m} P_{m+N}\left(\frac{x_1 - x_j}{2\pi i}, \tau\right) \\
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Here,

$$P_1(w, \tau) = \frac{1}{2\pi i w} - \sum_{k \geq 1} G_k(\tau) (2\pi i w)^{k-1}.$$

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Rested heavily on the coefficient functions: For $k \geq 1$,

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\tau + n)^{2k}}.$$

- Modular forms for $k \geq 2$.
- Quasi-modular form when $k = 1$.
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Notes:

1. $G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{c(c\tau+d)}{2\pi i}$.

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2. The 'modular derivative' defined as $\vartheta_k := \frac{1}{2\pi i}\frac{d}{d\tau} + kG_2(\tau)$ is the unique holomorphic differential operator that maps modular forms to modular forms.

Recap: Further iterations

To tackle modularity of trace functions:

1. Dong-Li-Mason: As mentioned before, extended to orbifold case
2. Miyamoto: C_2-cofinite modularity
3. Miyamoto/Yamauchi: application to a single intertwining operator
4. Huang: Developed new theory and created a type of reduction formula for n-many intertwining operators of a new type
5. Dong-Zhao: modularity of $\mathbb{Z}$-graded VOSAs
6. Van Ekeren: modularity of $\mathbb{Q}$-graded VOSAs and twisted modules
7. Miyamoto: Theta functions
8. Mason-Tuite-Zuesky: $\mathbb{R}$-graded VOSAs
9. Etc.

Recap: Other uses/occurrences

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2. Gaberdiel-Keller: To study differential operators and CFTs
3. Mason-K: Study of two-variable (Jacobi) 1-point functions (see also Heluani-Van Ekeren)

Note: It is this last two cases that we discuss further here.

"Jacobi" n-point functions

Definition

Let V be a VOA with Virasoro vector ω of central charge $\mathbf{c}$. Consider $J \in V_1$ such that $J(0)$ acts semisimply on V. For $v_1, \ldots, v_n \in V$ and a weak V-module M, the Jacobi n-point function is

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Core notes:

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Core notes:

1. Much of Zhu's theory carries over (now Jacobi or quasi-Jacobi forms)
2. Modularity exists, but is developed differently than in Zhu (since no ODE Frobenius-Fuch's theory)

Reduction formula, Part I

Reduction formula I

Let $a, v_1, \ldots v_n \in V$ with $J(0)a = \alpha a$ $(\alpha z \notin \mathbb{Z}\tau + \mathbb{Z})$. We have

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& Z_M^J\left((a, y), (v_1, x_1), \ldots, (v_n, x_n); z, \tau\right) \\
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Here, (again $q_x = e^{2\pi i x}$)

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Note: $\widetilde{P}_{m+1}(w,z,\tau)$ has simple poles at $z = \lambda\tau + \mu$ for $\lambda, \mu \in \mathbb{Z}$.

Reduction formula, Part II

Reduction formula II

Let $a, v_1, \dots v_n \in V$ with $J(0)a = \alpha a$ ($\alpha z \notin \mathbb{Z}\tau + \mathbb{Z}$). For $N \geq 1$ we have

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&\qquad + \sum_{j=2}^{n} \sum_{m\geq 0} (-1)^{N+1} \binom{m+N-1}{m} \widetilde{P}_{m+N}\left(\frac{x_1 - x_j}{2\pi i}, \alpha z, \tau\right) \\
&\qquad\qquad \times Z_M^J\left((v_1, x_1), \ldots, (a[m]v_j, x_j), \ldots, (v_n, x_n); z, \tau\right).
\end{aligned}
$$

Here, (with poles at $z \in \mathbb{Z}\tau + \mathbb{Z}$)

$$
\widetilde{G}_k(z, \tau) = -\delta_{k,1}\frac{q_z}{q_z - 1} - \frac{B_k}{k!} + \frac{1}{(k-1)!}\sum_{n\geq 1}\left(\frac{n^{k-1}q_z q^n}{1 - q_z q^n} + (-1)^k \frac{n^{k-1}q_z^{-1}q^n}{1 - q_z^{-1}q^n}\right)
$$

Why look at the reduction formula?

Previous uses:

1. Extend modularity of two-variable 1-point functions to *n*-point functions (Mason-K)
2. Gain some control of convergence (Heluani-Van Ekeren: poles on $\mathbb{Z}\tau + \mathbb{Z}$ for certain structures)
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3. To study differential operators of $N = 2$ theories (Gaberdiel-Keller)

Our motivation:

1. Study the possible poles more closely
2. Study more elaborate differential operators of Jacobi forms
3. Use as a tool to write sums and products of quasi-Jacobi forms to create Jacobi forms
4. Possibly gain insight in the structure of the VOA?

Results/Progress I: Introduction of poles

Since $\widetilde{P}_m(w, \alpha z, \tau)$ and $G_k(\alpha z, \tau)$ have poles for $z \in \frac{1}{\alpha}\mathbb{Z}\tau + \frac{1}{\alpha}\mathbb{Z}$, it appears many poles can be introduced.

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Recall that the previous reduction formulas were given for $\alpha z \notin \mathbb{Z}\tau + \mathbb{Z}$.

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Ultimately, comes down to the facts that

- $\widetilde{P}_{m+1}(w, z, \tau)$ has simple poles at $z = \lambda\tau + \mu$ for $\lambda, \mu \in \mathbb{Z}$ with residue $\dfrac{\lambda^m q_w^{-\lambda}}{m!2\pi i}$ and no other poles; and
- This pole cancels with a zero of the trace function in the reduction formula.

Results/Progress I: Introduction of poles

Thus, for $\alpha z = \lambda\tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}$

- the functions $\widetilde{P}_{m+1}$ and $\widetilde{G}_k$ above can be replace with functions of the form

$$P_{m+1,\lambda}(w,\tau) := \lim_{z\to\lambda\tau+\mu}\left(\widetilde{P}_{m+1}(w,z,\tau) - \frac{1}{(z-\lambda\tau-\mu)}\frac{\lambda^m q_w^{-\lambda}}{m!2\pi i}\right)$$
$$= \frac{(-1)^{m+1}}{m!}\sum_{n\in\mathbb{Z}\setminus\{-\lambda\}}\frac{n^m q_w^n}{1-q^{n+\lambda}}$$

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- and the $\widetilde{G}_k$ can be replaced with

$$P_{1,\lambda}(\tau) =: \frac{1}{2\pi i w} - \sum_{k\geq 1} G_{k,\lambda}(2\pi i w)^{k-1},$$

where it can be shown

$$G_{k,\lambda}(\tau) = \sum_{j=0}^{k}\frac{\lambda^j}{j!}G_{k-j}(\tau).$$

Results/Progress I: Introduction of poles

Theorem (Bringmann-K-Tuite)

A Jacobi n-point function for a VOA V does not have poles in $\mathbb{C} \times \mathbb{H}$ if the $(n-1)$-point functions do not contain poles for any choice of $n-1$ vectors in V.

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In other words, the reduction formulas do not introduce poles.

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In other words, the reduction formulas do not introduce poles.

Note: These formula can also be deduced using a shifted Virasoro vector.

Results/Progress II: Differential operators

Definition (Jacobi forms)

A (holomorphic) **Jacobi form of weight** k **and index** m ($k, m \in \mathbb{Z}$) on $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is a (holomorphic) function $\phi\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ such that for all $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), [\lambda, \mu]\right) \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ we have

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right) = (c\tau + d)^k \, e^{2\pi i m \left(\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} - \left(\lambda^2\tau + 2\lambda z\right)\right)} \phi(\tau, z)$$

for all $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, and ϕ also has a Fourier expansion

$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{r^2 \leq 4mn} c(n, r) q^n \zeta^r.$$

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$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{r^2 \leq 4mn} c(n, r) q^n \zeta^r.$$

It is a **weak Jacobi form** if $r^2 \leq 4mn$ is replaced with $r \in \mathbb{Z}$.

Results/Progress II: Differential operators

Definition (Quasi-Jacobi forms)

- A function ϕ is a **quasi-Jacobi form** of weight k, index 0, and depth (s,t) if there are meromorphic functions S_j^ϕ and T_j^ϕ dependent only on ϕ such that

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k\sum_{j=0}^{s}S_j^\phi(\tau,z)\left(\frac{ca}{c\tau+d}\right)^j$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$ and

$$\phi\left(\tau,z+\lambda\tau+\mu\right)=\sum_{j=0}^{t}T_j^\phi(\tau,z)\lambda^j$$

for all $[\lambda,\mu]\in\mathbb{Z}^2$.

Results/Progress II: Differential operators

- Using the above reduction formulas and the fact certain 1-point functions in VOAs are weak Jacobi forms, we find differential operators which must preserve the Jacobi form transformation laws.

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- Gaberdiel-Keller did such an analysis for $N = 2$ VOSAs before and realized the heat operator (for degree 2)

$$\mathcal{H}_{k,m} = \vartheta_k - D_z^2 - \frac{1}{2}G_2(\tau),$$

where $D_x = \frac{1}{2\pi i}\frac{d}{dx}$.

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  where $D_x = \frac{1}{2\pi i}\frac{d}{dx}$.
- Are there differential operators that contain the quasi-Jacobi forms $\widetilde{G}_1(z,\tau)$ and $\widetilde{G}_2(z,\tau)$? (and others?)

An example

Consider the VOA $V := L_{\widehat{\mathfrak{sl}}_2}(m,0)$ associated to the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ of level $m \in \mathbb{N}$

- where $h, x, y \in \mathfrak{sl}_2$ are the typical basis elements of the Lie algebra, and
- we have $h(0)x = [h,x] = 2x$ and $\langle x, y \rangle = m$.

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Consider the endomorphism

$$S := L[-2] - \frac{1}{3}\left(h[-1]^2 - \frac{1}{2}x[-1]y[-1]\right).$$

An example

Then the reduction formulas show

$$S = L[-2] - \frac{1}{3}h[-1]^2 + \frac{1}{6}x[-1]y[-1],$$

satisfies (for a V-module W)

$$Z_W\left(S\mathbf{1}; z, \tau\right) = \mathcal{S}\left(Z_W(\mathbf{1}; z, \tau)\right),$$

where

$$\mathcal{S} := \vartheta_k - \frac{1}{3}\left[\left(D_z^2 + 2G_2(\tau)\right) + \frac{1}{2}\left(\widetilde{G}_1(2z,\tau)D_z - \widetilde{G}_2(2z,\tau)\right)\right].$$

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That is,

$$\frac{1}{\eta(\tau)}\begin{pmatrix}\mathcal{S}\left(\theta_3(z, 2\tau)\right)\\ \mathcal{S}\left(\theta_2(z, 2\tau)\right)\end{pmatrix}$$

is a vector-valued Jacobi form.

Another example

The same analysis for the holomorphic VOA V_{E_8} gives

$$\mathcal{S}\left(Z_{V_{E_8}}(\mathbf{1}; z, \tau)\right) = Z_{V_{E_8}}\left(S\mathbf{1}; z, \tau\right).$$

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From which one can deduce

$$\mathcal{S}\left(E_{4,1}(z, \tau)\right) = -\frac{7}{24} E_{6,1}(z, \tau),$$

where $E_{k,m}$ are the Jacobi-Eisenstein series of weight k and index m.

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Such expressions are similar to one of the three Ramanujan equations studies for modular derivatives and modular forms.

Results/Progress II: Differential operators

- A more general analysis finds

$$\begin{aligned}\mathcal{M} = \mathcal{M}_{k,\alpha} = \mathcal{M}_{(A,B),k,\alpha,m} \qquad & (A, B, \alpha \in \mathbb{Z}) \\ := \vartheta_k + \frac{1}{A - 4mB}\Bigg[& B\left(D_z^2 + 2mG_2(\tau)\right). \\ & + \frac{A}{\alpha}\left(\widetilde{G}_1(\alpha z, \tau) D_z - \frac{2m}{\alpha}\widetilde{G}_2(\alpha z, \tau)\right)\Bigg],\end{aligned}$$

preserves the transformation properties of Jacobi forms (and also the convergence for appropriate A, B, k, α).

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  preserves the transformation properties of Jacobi forms (and also the convergence for appropriate A, B, k, α).
- Higher degree differential operators can also be found.

Results/Progress II: Differential operators

Above we realized a deviation of the 'Serre' derivative (which was studied by Oberdieck)

$$\vartheta_k + \frac{1}{\alpha}\widetilde{G}_1(\alpha z, \tau) D_z - \frac{2m}{\alpha^2}\widetilde{G}_2(\alpha z, \tau).$$

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- Answer: Depends.

To see how this looks, we let $N_1, N_2 \in \mathbb{N}$ be uniquely defined for a multiplier χ by

$$\chi \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = e^{2\pi i \frac{a_1}{N_1}} \quad \text{and} \quad \chi(0,1) = e^{2\pi i \frac{a_2}{N_2}},$$

where $a_j \in \mathbb{N}$ satisfy $\gcd(a_j, N_j) = 1$.

Results/Progress II: Differential operators

Lemma

Let $\alpha \in \mathbb{Z}$.

1. *The operator $\mathcal{M}_{k,\alpha}$ maps forms transforming like Jacobi forms of weight k with multiplier χ to forms of weight $k+2$ with multiplier χ.*
2. *Assume that $\frac{2}{\alpha N_2} \in \mathbb{Z}$, $\chi\left(\frac{2}{\alpha}, 0\right) = e^{2\pi i \frac{a}{N}}$ for $a, N \in \mathbb{N}$ with $\gcd(a, N) = 1$ and N odd, and $\chi\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = (-1)^k$. Then we have*

$$\mathcal{M}_{k,\alpha} \colon J_{k,m,\chi} \to J_{k+2,m,\chi}.$$

Results/Progress II: Differential operators

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I.e., NO poles for $\alpha = \pm 1, \pm 2$. But for other α, poles could be introduced (generically).

Results/Progress II: Differential operators

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    - For example, suppose V is a strongly regular VOA with an $\widehat{\mathfrak{sl}}$-subalgebra and that $J(1)J = 2$. Then $\dim V_{1,\pm 2} = 1$ and $V_{n,\pm 2n} = \{0\}$ for all $n \geq 2$.

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- Such differential operators can give information about functions that satisfy the Jacobi form transformation properties.
  - Take $\mathcal{T}_{k,\alpha} = \vartheta_k - \frac{1}{4m} D_z^2 - \frac{1}{\alpha} \widetilde{G}_1(\alpha z, \tau) + \frac{2m}{\alpha} \widetilde{G}_2(\alpha z, \tau)$, and
  - let $c(n,r)$ be the coefficients in $\phi = \sum_{n \geq 0, r^2 \leq 4nm} c(n,r) q^n \zeta^r$.
  - Example: If ϕ is a Jacobi form of weight k and index m and $\mathcal{T}_{k,\alpha}(\phi)$ has no poles $(|\alpha| \geq 1)$, then for $h_m := \lfloor 2\sqrt{m} \rfloor$ such that $h_m \neq \frac{2m}{\alpha}$ we have $c(1, h_m) = 0$.

The end

Thank you!

The end

Thank you!

Fermionic models

Reductions can also provide interesting sums and products of quasi-Jacobi and Jacobi forms without differential operators.
Assume

- $V = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} V_k$ is an appropriate VOSA (of CFT-type, etc).
- There are $2R$ 'free fermion' vectors $\psi_r^{\pm} \in V_{\frac{1}{2}}$ for $r = 1, \ldots, R$ with vertex operators $Y(\psi_r^{\pm}, z) = \sum_{n \in \mathbb{Z}} \psi_r^{\pm}(n) z^{-n-1}$ such that $\psi_r^{+}(0)\psi_s^{-} = \delta_{r,s}\mathbf{1}$ and $\psi_r^{\pm}(0)\psi_s^{\pm} = 0$.

Then

$$J = \sum_{r=1}^{R} \psi_r^{+}(-1)\psi_r^{-}$$

satisfies

$$J(0)\psi_r^{\pm} = \pm\psi_r^{\pm} \qquad \text{and} \qquad J(1)J = R\mathbf{1},$$

i.e., $\langle J, J \rangle = R$.

Fermionic models

We also note that by defining $Y_\sigma(v,z) = Y(\Delta(\sigma,z)v,z)$ where

- $\sigma = e^{\pi i J(0)}$ (the fermion number automorphism), and
- $\Delta(\sigma,z) := z^{\frac{1}{2}J(0)} \exp\left(-\frac{1}{2}\sum_{n\geq 1}\frac{J(n)}{n}(-z)^{-n}\right)$,

we have (V, Y_σ) is the σ-twisted V-module (by Li).

Thus we can consider

$$Z_V^J(z,\tau) := \mathrm{str}_V\, \zeta^{J(0)} q^{L(0)-\frac{\mathbf{c}}{24}}$$

so that using that

$$J_\sigma(0) = J(0) + \frac{1}{2} \qquad \text{and} \qquad L_\sigma(0) = L(0) + \frac{1}{2}J(0) + \frac{R}{8}$$

we have

$$Z_{V_\sigma}^J(z,\tau) := \mathrm{tr}_V\, e^{i\pi J_\sigma(0)} \zeta^{J_\sigma(0)} q^{L_\sigma(0)-\frac{\mathbf{c}}{24}} = i\,\mathrm{str}_V\, \zeta^{J(0)+\frac{1}{2}} q^{L(0)+\frac{1}{2}J(0)-\frac{(\mathbf{c}-3R)}{24}}.$$

Fermion model: Finding well-known functions

Taking specific endomorphism Φ^R and Ψ^R one can find

$$Z^J_{V^R_\sigma}\left(\Phi^R; z, \tau\right) = F_R(z,\tau)\left(\frac{\theta_1(z,\tau)}{\eta(\tau)}\right)^R,$$

$$Z^J_{V^R_\sigma}\left(\Psi^R; z, \tau\right) = K_R(z,\tau)\left(\frac{\theta_1(z,\tau)}{\eta(\tau)}\right)^R,$$

where

$$F_k(z,\tau) := \frac{(-1)^{k+1}}{k}\left(P_k(z,\tau) - G_k(\tau)\right)$$

$$K_n(z,\tau) := \sum_{m=0}^{n} \frac{1}{m!}\widetilde{G}_{n-m}(z,\tau)\widetilde{G}_1(z,\tau)^m.$$

The end (seriously!)

Thank you!