Smith algebra and classification of irreducible modules for certain W-algebras

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Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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PROVEDBA VRHUNSKIH ISTRAŽIVANJA U SKLOPU
ZNANSTVENOG CENTRA IZVRSNOSTI
ZA KVANTNE I KOMPLEKSNE SUSTAVE
TE REPREZENTACIJE LIEJEVIH ALGEBRI

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Overview

1. Bershadsky-Polyakov vertex algebra $\mathcal{W}^k$
2. Structure of the Zhu algebra $A(\mathcal{W}^k)$
3. Smith-type algebra
4. Classification of irreducible ordinary $\mathcal{W}_k$-modules for $k = -\frac{5}{3}$
5. Irreducible ordinary $\mathcal{W}_k$-modules for integer levels k

Minimal affine $\mathcal{W}$–algebras

Minimal affine $\mathcal{W}$–algebra $\mathcal{W}^k(\mathfrak{g}, f_\theta)$, where f_θ is a minimal nilpotent element, is the vertex algebra obtained by quantum Drinfeld-Sokolov reduction from the affine vertex algebra $V^k(\mathfrak{g})$.

Vertex algebra $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ is strongly generated by vectors

- $G^{\{u\}}$, $u \in \mathfrak{g}_{-\frac{1}{2}}$, of conformal weight $\frac{3}{2}$
- $J^{\{a\}}$, $a \in \mathfrak{g}^\natural$, of conformal weight 1
- ω is the conformal vector of central charge

$$c(\mathfrak{g}, k) = \frac{k\,\mathrm{dim}\mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

For $k \neq -h^\vee$, $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ has a unique simple quotient $\mathcal{W}_k(\mathfrak{g}, f_\theta)$.

Bershadsky-Polyakov vertex algebra $\mathcal{W}^k$

Bershadsky-Polyakov vertex algebra $\mathcal{W}^k := \mathcal{W}^k(sl_3, f_\theta)$ is the minimal affine $\mathcal{W}$-algebra obtained by quantum DS reduction from $V^k(sl_3)$.

- $\mathcal{W}^k$ is generated by the fields T, J, G^+, G^-
- we choose a new Virasoro vector

$$L(z) = T(z) + \frac{1}{2} DJ(z)$$

- the fields L, J, G^+, G^- satisfy commutation relations:

$$[J(m), J(n)] = \tfrac{2k+3}{3} m\delta_{m+n,0}, \quad [J(m), G^\pm(n)] = \pm G^\pm(m+n),$$

$$[L(m), J(n)] = -nJ(m+n) - \tfrac{(2k+3)(m+1)m}{6}\delta_{m+n,0},$$

$$[L(m), G^+(n)] = -nG^+(m+n), \quad [L(m), G^-(n)] = (m-n)G^-(m+n),$$
$$[G^+(m), G^-(n)] = 3(J^2)(m+n) + (3(k+1)m - (2k+3)(m+n+1))J(m+n) - (k+3)L(m+n) + \tfrac{(k+1)(2k+3)(m-1)m}{2}\delta_{m+n,0}.$$

Zhu algebra

Let $V = \bigoplus_{n=0}^{\infty} V(n)$ be a $\mathbb{Z}$-graded VOA, and let $\deg a = n$ for $a \in V(n)$.
Define bilinear mappings $* : V \times V \longrightarrow V$, $\circ : V \times V \longrightarrow V$:

$$a * b = Res_z \left(Y(a,z) \frac{(1+z)^{\deg a}}{z} b \right),$$

$$a \circ b = Res_z \left(Y(a,z) \frac{(1+z)^{\deg a}}{z^2} b \right),$$

for $a \in V(n)$, $b \in V$.
Let $O(V) \subset V$ be the linear span of the elements $a \circ b$. The quotient space

$$A(V) = \frac{V}{O(V)}$$

is an associative algebra called the **Zhu algebra** of the VOA V.

Irreducible highest weight $\mathcal{W}^k$-modules

For every $(x, y) \in \mathbb{C}^2$ there exists an irreducible $\mathcal{W}^k$-module $L(x, y)$ generated with a highest weight vector $v_{x,y}$ such that

$$J(0)v_{x,y} = xv_{x,y}, \quad J(n)v_{x,y} = 0 \text{ for } n > 0,$$

$$L(0)v_{x,y} = yv_{x,y}, \quad L(n)v_{x,y} = 0 \text{ for } n > 0,$$

$$G^-(n-1)v_{x,y} = G^+(n)v_{x,y} = 0 \text{ for } n \geq 1.$$

Zhu algebra $A(\mathcal{W}^k)$

Let $A(\mathcal{W}^k)$ denote the Zhu algebra of $\mathcal{W}^k$. Let $[v]$ be the image of $v \in \mathcal{W}^k$ under the mapping $\mathcal{W}^k \mapsto A(\mathcal{W}^k)$.

- $A(\mathcal{W}^k)$ is generated by $[G^+], [G^-], [J], [\omega]$

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- $A(\mathcal{W}^k)$ is generated by $[G^+], [G^-], [J], [\omega]$
- Zhu algebra $A(\mathcal{W}^k)$ is actually a quotient of another associative algebra, called Smith algebra

Smith-type algebra

Let $g(x, y) \in \mathbb{C}[x, y]$ be an arbitrary polynomial. Associative algebra $R(g)$ of **Smith type** is generated by $\{E, F, X, Y\}$ such that Y is a central element and the following relations hold:

$$XE - EX = E, \ XF - FX = -F, \ EF - FE = g(X, Y).$$

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- $R(g)$ is a certain generalization of $U(sl_2)$
- highest weight modules for Smith-type algebra:

$$V(x, y) = R(g) \otimes_B \mathbb{C}v_{x,y},$$

where $B =< X, Y, E >$ is a Borel subalgebra of $R(g)$ and $\mathbb{C}v_{x,y}$ is a B-module such that $Ev_{x,y} = 0$, $Xv_{x,y} = xv_{x,y}$, $Yv_{x,y} = yv_{x,y}$

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- $V(x, y)$ has a unique simple quotient $L(x, y)$

Structure of the Zhu algebra $A(\mathcal{W}^k)$

Denote

$$E = [G^+],\ F = [G^-],\ X = [J],\ Y = [\omega].$$

Proposition

Let $R(g)$ be the Smith-type algebra generated by $\{E, F, X, Y\}$, with

$$g(x, y) = -(3x^2 - (2k + 3)x - (k + 3)y).$$

Then the Zhu algebra $A(\mathcal{W}^k)$ associated to the Bershadsky-Polyakov algebra $\mathcal{W}^k$ is isomorphic to a certain quotient of the Smith algebra $R(g)$.

Define functions

$$h_i(x, y) = \frac{1}{i}(g(x, y) + g(x + 1, y) + ... + g(x + i - 1, y))$$

Lemma (Arakawa 2013)

If the top level $L(x, y)(0)$ is n-dimensional, then $h_n(x, y) = 0$.

Irreducible highest weight $\mathcal{W}^k$-modules

We will need the following Δ-operator

$$\Delta(-J, z) = z^{-J(0)} exp\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{-J(0)}{kz^k}\right),$$

such that

$$\sum_{n\in\mathbb{Z}}\Psi(a_n)z^{-n-1} = Y(\Delta(-J,z)a, z).$$

Lemma (Arakawa 2013)

Let dim $(L(x,y)(0)) = i$. *Then*

$$\Psi(L(x,y)) \cong L(x+i-1-\frac{2k+3}{3}, y-x-i+1+\frac{2k+3}{3}).$$

Classification of irreducible ordinary $\mathcal{W}_k$-modules for $k = -5/3$

- goal: classify irreducible ordinary $\mathcal{W}_k$–modules($=$ modules with finite dimensional $L(0)$–weight subspaces) for $k = -5/3$ (and some other levels)

Proposition

Let $k = -\frac{5}{3}$. Define

$$\mathcal{S}_k = \{(-\frac{1}{9}, 0), (0,0), (\frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, \frac{2}{3}), (-\frac{4}{9}, \frac{1}{3}), (-\frac{7}{9}, \frac{2}{3}), \}.$$

(i) *For every $(x, y) \in \mathcal{S}_k$, $L(x, y)$ is a $\mathcal{W}_k$–module.*

(ii) *Assume that $L(x, y)$ is an ordinary $\mathcal{W}_k$–module. Then it holds that $(x, y) \in \mathcal{S}_k$.*

- There is a singular vector in $\mathcal{W}_{-5/3}$ of level 4:

$$W_4 = -\frac{62}{9}\overline{L}(-2)^2\mathbb{1} + \frac{14}{3}\overline{L}(-4)\mathbb{1} - 18J(-1)^4\mathbb{1} + 31J(-2)J(-1)^2\mathbb{1} -$$
$$- 118J(-3)J(-1)\mathbb{1} + \frac{133}{9}J(-2)^2\mathbb{1} - \frac{8}{9}J(-4)\mathbb{1} +$$
$$+ \frac{62}{9}\overline{L}(-2)J(-2)\mathbb{1} - 12\overline{L}(-3)J(-1)\mathbb{1} + 46\overline{L}(-2)J(-1)^2\mathbb{1} -$$
$$- G^+(-2)G^-(-2)\mathbb{1} + G^+(-1)G^-(-3)\mathbb{1} -$$
$$- 18J(-1)G^+(-1)G^-(-2)\mathbb{1}.$$

- From this formula, we obtain a relation in the Zhu algebra $A(\mathcal{W}^k)$:

$$[G^+]^2([\omega] + \frac{1}{9}) = 0. \quad (*)$$

- Let $L(x,y) = \bigoplus_{n=0}^{\infty} L(x,y)(n)$ be an irreducible ordinary $\mathcal{W}_k$-module. From $(*)$ it follows that either:
  - (i) $L(x,y)(0)$ is a 1-dimensional or 2-dimensional module for the Smith algebra $R(g)$ and hence $h_1(x,y) = 0$ or $h_2(x,y) = 0$, or
  - (ii) $y = -1/9$ (we show that there are no ordinary modules satisfying this condition)

- Now we consider the modules $\Psi(L(x,y))$. Since $\dim\ (L(x,y)(0)) < \infty$, the module $\Psi(L(x,y)) := L(\hat{x},\hat{y})$ is also a $\mathcal{W}_k$-module with $\dim\ (L(\hat{x},\hat{y})(0)) < \infty$
  $\Longrightarrow$ again, either:
  (i') $L(\hat{x},\hat{y})(0)$ is a 1-dimensional or 2-dimensional module for the Smith algebra $R(g)$ and hence $h_1(\hat{x},\hat{y}) = 0$ or $h_2(\hat{x},\hat{y}) = 0$, or
  (ii') $\hat{y} = -1/9$

- Combining these conditions we get:
  (a) if $dim\ L(x,y)(0)=1$,

$$h_1(x,y)=h_1(\hat{x},\hat{y})=0 \Longrightarrow (x,y)=(-1/9,0)$$
$$h_1(x,y)=h_2(\hat{x},\hat{y})=0 \Longrightarrow (x,y)=(-4/9,1/3) \text{ or}$$
$$\hat{y}=-1/9 \Longrightarrow (x,y)=(0,0),(x,y)=(1/3,1/3);$$

  (b) if $dim\ L(x,y)(0)=2$,

$$h_2(x,y)=h_1(\hat{x},\hat{y})=0 \Longrightarrow (x,y)=(-7/9,2/3)$$
$$h_2(x,y)=h_2(\hat{x},\hat{y})=0 \Longrightarrow (x,y)=(-10/9,5/4) \text{ or}$$
$$\hat{y}=-1/9 \Longrightarrow (x,y)=(-1/3,2/3).$$

We need to check if $L(0,0)$, $L(1/3,1/3)$, $L(-1/9,0)$, $L(-4/9,1/3)$, $L(-1/3,2/3)$, $L(-7/9,2/3)$, $L(-10/9,5/4)$ are indeed modules for $\mathcal{W}_k$.

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- First notice that
  - $L(-1/9,0) = \Psi^{-1}(L(0,0))$
  - $L(-4/9,1/3) = \Psi^{-1}(L(-1/3,2/3))$
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- Since $(x,y) \in \mathbb{C}^2$ needs to be a zero of the polynomial $U(x,y) = [W_4] \implies L(-10/9,5/4)$ cannot be a $\mathcal{W}_k$-module

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- Since $(x,y) \in \mathbb{C}^2$ needs to be a zero of the polynomial $U(x,y) = [W_4] \Longrightarrow L(-10/9,5/4)$ cannot be a $\mathcal{W}_k$-module
- We will realize $L(0,0), L(-1/3,2/3), L(1/3,1/3)$ as certain subalgebras of the Weyl vertex algebra

Proposition

Let

$$J = -\frac{1}{3}a^{+}_{-1}a^{-}_{-1}\mathbb{1}, \ \omega = \frac{1}{2}\left(a^{-}_{-2}a^{+}_{-1} - a^{+}_{-2}a^{-}_{-1}\right)\mathbb{1},$$

$$G^{+} = \frac{1}{3}\left(a^{+}_{-1}\right)^{3}\mathbb{1}, \ G^{-} = \frac{1}{9}\left(a^{-}_{-1}\right)^{3}\mathbb{1},$$

where $\{a^{\pm}_{n} : n \in \mathbb{Z}\}$ *are generators of the Weyl vertex algebra* W*. The vertex subalgebra* $\widetilde{\mathcal{W}_k}$ *of the Weyl vertex algebra* W *generated by vectors* $J, \omega, G^{\pm}$ *is isomorphic to a certain quotient of* $\mathcal{W}^k$*.*

- we will show that $\widetilde{\mathcal{W}_k}$ is in fact isomorphic to the simple quotient $\mathcal{W}_k$

Let $g = e^{\frac{2\pi i}{3} J_0}$. Then g is an automorphism of W of order 3 and it holds that

$$W = W^{(0)} + W^{(1)} + W^{(-1)},$$

where

$$W^{(j)} = \{v \in W |\ gv = e^{\frac{-2\pi i}{3} j} v\}, \quad j = 0, 1, 2.$$

Hence $W^{(0)}$ is a simple vertex algebra, and $W^{(\pm 1)}$ are irreducible $W^{(0)}$-modules.

Proposition

Let W be the Weyl vertex algebra, g as above. Then it holds that:

(1) $\mathcal{W}_k = W^{(0)}$.

(2) $W^{(\pm 1)}$ *are irreducible* $\mathcal{W}_k$*–modules of highest weight* $(\frac{1}{3}, \frac{1}{3})$, $(-\frac{1}{3}, \frac{2}{3})$ *respectively.*

Weyl vertex algebra W is a direct sum of three irreducible $\mathcal{W}_k$–modules, with the following highest weights:

- $W^{(0)}$ has a highest weight vector $\mathbb{1}$, with the highest weight $(0,0)$
- $W^{(1)}$ has a highest weight vector $a^+_{-1}\mathbb{1}$, with the highest weight $(1/3, 1/3)$
- $W^{(-1)}$ has a highest weight vector $a^-_{-1}\mathbb{1}$, with the highest weight $(-1/3, 2/3)$.

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$\Longrightarrow$ hence $L(0, 0)$, $L(1/3, 1/3)$ and $L(-1/3, 2/3)$ are irreducible $\mathcal{W}_k$–modules.

Irreducible ordinary $\mathcal{W}_k$-modules for integer levels k

Let $L(x, y)$ be the irreducible highest weight $\mathcal{W}_k$-module of weight $(x, y) \in \mathbb{C}^2$.

- for $k \in \mathbb{Z}$, $k \geq -1$, we show that the highest weights (x, y) are zeroes of polynomials

$$h_i(x, y) = 0, \quad 1 \leq i \leq k + 2,$$

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- for $k \in \mathbb{Z}$, $k \geq -1$, we show that the highest weights (x,y) are zeroes of polynomials
$$h_i(x,y) = 0, \quad 1 \leq i \leq k+2,$$
- vectors
$$(G^+(-1))^n \mathbb{1}, \ (G^-(-2))^n \mathbb{1}$$
are singular in $\mathcal{W}^k$ for $n = k+2$, where $k \in \mathbb{Z}$.

Similar expressions for singular vectors also appeared in [A13], in the case of $k = \frac{p}{2} - 3$, $p = 3, 5, 7, \dots$

Irreducible ordinary $\mathcal{W}_k$-modules for integer levels k

Let

$$\overline{\mathcal{W}_k} := \mathcal{W}_k / < (G^+(-1))^{k+2}\mathbb{1}, (G^-(-2))^{k+2}\mathbb{1} > .$$

Proposition

Let $k \in \mathbb{Z}$, $k \geq -1$. Isomorphism classes of irreducible $\overline{\mathcal{W}_k}$-modules are contained in the set

$$\mathcal{S}_k = \{L(x,y) \,|h_i(x,y) = 0,\ 1 \leq i \leq k+2\}.$$

- question: are modules from the set $\mathcal{S}_k$ indeed $\mathcal{W}_k$-modules?

Classification of irreducible ordinary $\mathcal{W}_k$-modules for $k=-1$

- $k=-1$ is a *collapsing level* for the Bershadsky-Polyakov algebra $\mathcal{W}_k(sl(3), f_\theta)$ (cf. talks of A. Moreau, P. Papi), hence

$$\mathcal{W}_{-1} \cong M(1).$$

Classification of irreducible ordinary $\mathcal{W}_k$-modules for $k = -1$

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$$\mathcal{W}_{-1} \cong M(1).$$

Proposition

All irreducible $\mathcal{W}_{-1}$-modules are contained in the set

$$\mathcal{S}_{-1} = \{L(x, y) \mid h_1(x, y) = 0\}.$$

Classification of irreducible ordinary $\mathcal{W}_k$-modules for $k = 0$

Let $V_L = M(1) \otimes \mathbb{C}[L]$ be the lattice vertex algebra associated with the lattice $L = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$, where

$$\langle \alpha_i, \alpha_j \rangle = \delta_{i,j}, \quad i, j = 1, 2.$$

- we consider the subalgebra $V[D] = M(1) \otimes \mathbb{C}[D]$ of V_L, where $D = (\alpha_1 + \alpha_2)$.
- for every $x \in \mathbb{C}$, $i = 0, 1$,

$$V[D - i\alpha_1 - x(\alpha_1 - \alpha_2)] = V[D].e^{-i\alpha_1 - x(\alpha_1 - \alpha_2)}$$

  is an irreducible $V[D]$–module.

Classification of irreducible ordinary $\mathcal{W}_k$-modules for $k = 0$

Theorem

(1) *The simple vertex algebra $\mathcal{W}_0(= \mathcal{W}_0(sl(3), f_\theta))$ can be realized as a vertex subalgebra of $V[D]$ generated by vectors*

$$
\begin{aligned}
J &\mapsto \alpha_2(-1) \\
L &\mapsto \frac{1}{2}\left(\alpha_1(-1)^2 - \alpha_1(-2) + \alpha_2(-1)^2 + \alpha_2(-2)\right) \\
G^+ &\mapsto \sqrt{3} e^{\alpha_1+\alpha_2} \\
G^- &\mapsto -\sqrt{3}\alpha_1(-1) e^{-\alpha_1-\alpha_2}.
\end{aligned}
$$

(2) *$\mathcal{W}_0$ has two families of irreducible highest weight modules $U_i(x)$, $i = 0, 1$, $x \in \mathbb{C}$, which are realized as*

$$U_i(x) = \mathcal{W}_0(sl(3), f_\theta).e^{-i\alpha_1 - x(\alpha_1-\alpha_2)},$$

Highest weights of $U_i(x)$ with respect to (J_0, L_0) are $(x, x^2 + (i-1)x)$.

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Thank you