Cohomology of algebraic structures: from Lie algebra to vertex algebra cohomology

Victor Kac

MIT

In any cohomology theory the notion of a vector superspace and a superalgebra are indispensable.

A vector superspace is a vector space V with a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$. For $v \in V_\alpha, \alpha \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$, one calls $p(v) = \alpha$ the parity of v. A superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra: $V_\alpha V_\beta \subset V_{\alpha+\beta}$.

Basic Example:

$$\operatorname{End} V = (\operatorname{End} V)_{\bar{0}} \oplus (\operatorname{End} V)_{\bar{1}},$$

where $(\operatorname{End} V)_{\bar{0}}$ (resp. $(\operatorname{End} V)_{\bar{1}}$) consists of parity preserving (resp. reversing) endomorphisms.

The Lie bracket on a superalgebra is

$$[a, b] = ab - (-1)^{p(a)p(b)} ba. \tag{1}$$

If a superalgebra is associative, then the Lie bracket (1) on it defines a Lie superalgebra structure. Axioms of a Lie superalgebra

$$\begin{aligned} &\text{(skew-commutivity)} && [a, b] = -(-1)^{p(a)p(b)}[b, a] \\ &\text{(Jacobi identity)} && [a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]] \end{aligned}$$

Recall Koszul rule: sign changes iff odd passes odd

Basic example: $\operatorname{End} V$ with bracket (1) is the general linear Lie superalgebra

Another example: if V carries a structure of a superalgebra, then

$$\operatorname{Der} V = \{D \in \operatorname{End} V \mid D(ab) = D(a)b + (-1)^{p(D)p(a)} aD(b)\}$$

is the Lie superalgebra of derivations of the superalgebra V.

An important special case (W for Witt):

$$W(V) = \operatorname{Der} S(V),$$

the Lie superalgebra of derivations of the (commutative) superalgebra of polynomial functions on V^* (= the Lie superalgebra of polynomial vector fields on V^*).

The Lie superalgebra $W(V)$ carries a natural $\mathbb{Z}$-grading, coming from $S(V)$:

$$W(V) = \bigoplus_{j \geq -1} W^j(V),$$

where

$$W^{-1}(V) = V,\ W^0(V) = \operatorname{End} V,\ W^j(V) = \operatorname{Hom}(S^{j+1}(V), V).$$

Explicit formula for the bracket on $W(V)$:

$$[X, Y] = X \square Y - (-1)^{p(X)p(Y)} Y \square X, \tag{2}$$

where $X \in W^n(V)$, $Y \in W^m(V)$, and

$$(X \square Y)(v_0 \otimes \ldots \otimes v_{m+n}) = \sum_{\substack{i_0 < \ldots < i_m \\ i_{m+1} < \ldots < i_{m+n}}} \epsilon_v(i_0, \ldots, i_{m+n}) X(Y(v_{i_0} \otimes \ldots \otimes v_{i_m}) \otimes v_{i_{m+1}} \otimes \ldots \otimes v_{i_{m+n}}). \tag{3}$$

The summation is over the shuffles in S_{m+n+1}, and $\epsilon_v = (-1)^N$, where $N = \#$ of interchanges of indices of odd v_i's.

Since $W^1(V) = \mathrm{Hom}(S^2V, V)$, even elements of the vector superspace $W^1(V)$ correspond bijectively to commutative superalgebra structures (i.e. $ab = (-1)^{p(a)p(b)}ba$) on V.

But if we want skew-commutative, we need to reverse the parity of V : consider the vector superspace ΠV with the even part $V_{\bar{1}}$ and the odd part $V_{\bar{0}}$, and consider the Lie superalgebra $W(\Pi V)$.

The odd elements $X \in W^1(\Pi V)$ are in bijective correspondence with skew-commutative superalgebra structures on V :

$$[a, b] = (-1)^{p(a)} X(a \otimes b), \ a, b \in V \tag{4}$$

A remarkable fact is that (4) satisfies the Jacobi identity (hence defines a Lie superalgebra on V) if and only if

$$[X, X] = 0. \tag{5}$$

Since, by Jacobi identity, $\mathrm{ad}[X, X] = 2(\mathrm{ad}\, X)^2$, we see that, given a Lie superalgebra structure on V, with the bracket, defined by $X \in W^1(\Pi V)$, we obtain a cohomology complex

$$(C^{\cdot} = \bigoplus_{j \geq 0} C^j, \mathrm{ad}\, X), \text{ where } C^j = W^{j-1}(\Pi V).$$

This is the Chevalley-Eilenberg Lie (super)algebra cohomology complex with coefficients in the adjoint representation.

More generally, given a module M over the Lie superalgebra V, one considers, instead of V, the Lie superalgebra $V \ltimes M$ with M an abelian ideal, and by a simple reduction procedure construct the Chevalley-Eilenberg complex of V with coefficients in M.

Basic idea. Given an algebraic structure A on a vector (super)space V, construct a $\mathbb{Z}$-graded Lie superalgebra

$$W_A(\Pi V) = \bigoplus_{j \geq -1} W_A^j(\Pi V,)$$

such that an odd element $X \in W_A^1(\Pi V)$, satisfying $[X, X] = 0$, defines the algebraic structure A on V. Then we obtain a cohomology complex for the structure A:

$$(C_A^{\cdot} = \bigoplus_{j \geq 0} C_A^j, \, \mathrm{ad}\, X), \text{ where } C_A^j = W_A^{j-1}(\Pi V).$$

For a V-module M one uses a reduction procedure as in the Lie (super)algebra case.

The above construction of the Lie superalgebra $W(V)$ is easy to reformulate in terms of the linear operad $P = \mathrm{Hom}(V)$, where $P(j) = \mathrm{Hom}(V^{\otimes j}, V)$, $j \geq 0$, so that

$$W_P^j = P(j+1)^{S_{j+1}}.$$

A straightforward generalization produces a cohomology theory for any linear symmetric (super)operad P. The art is how to construct the linear operad P, which produces a cohomology theory of a given algebraic structure A.

The first example beyond the Lie (super)algebra is the Lie conformal (super)algebra (LCA). Recall that an LCA structure on a vector superspace V with an even endomorphism ∂ is defined by the λ-bracket (λ indeterminate, $p(\lambda) = \bar{0}$)

$$[\cdot_\lambda \cdot] : V^{\otimes 2} \to V[\lambda],\ a \otimes b \mapsto [a_\lambda b],$$

satisfying the following axioms:

$$\begin{aligned} &\text{(sesquilinearity)} && [\partial a_\lambda b] = -\lambda[a_\lambda b],\ [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \\ &\text{(skew-commutivity)} && [b_\lambda a] = -(-1)^{p(a)p(b)}[a_{-\lambda-\partial} b]. \\ &\text{(Jacobi identity)} && [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]]. \end{aligned}$$

Explanation: writing $[a_\mu b] = \sum_{n\geq 0} \mu^n c_n$, $[a_{-\lambda-\partial} b]$ is $\sum_{n\geq 0} (-\lambda - \partial)^n c_n$.

In order to construct the corresponding $\mathbb{Z}$-graded Lie superalgebra $W_{\mathrm{LCA}}(V)$, let

$$V_n = V[\lambda_1, \dots, \lambda_n]/(\partial + \lambda_1 + \dots + \lambda_n)V[\lambda_1, \dots, \lambda_n]. \tag{6}$$

Then $W^n_{\mathrm{LCA}}(V)$ consists of linear maps $Y_{\lambda_0,\dots,\lambda_n} : V^{\otimes(n+1)} \to V_{n+1}$, which are invariant w.r. to the simultaneous permutation of factors in $V^{\otimes(n+1)}$ and of the λ_i's, and which satisfy the sesquilinearity properties $(0 \leq j \leq n)$

$$Y_{\lambda_0,\dots,\lambda_n}(v_0 \otimes \dots \otimes \partial v_j \otimes \dots \otimes v_n) = -\lambda_j Y_{\lambda_0,\dots,\lambda_n}(v_0 \otimes \dots \otimes v_n).$$

The box product on $W_{\mathrm{LCA}}(V) = \bigoplus_{j \geq -1} W^j_{\mathrm{LCA}}(V)$ is defined by a formula, similar to (3), but "decorated" by λ_i's:

$$
\begin{aligned}
&(X \Box Y)_{\lambda_0,\ldots,\lambda_{m+n}}(v_0,\ldots,v_{m+n}) \\
&= \sum \epsilon_v(i_0,\ldots,i_{m+n}) X_{\lambda_{i_0}+\ldots+\lambda_{i_n},\lambda_{i_{n+1}},\ldots,\lambda_{i_{m+n}}} \\
&(Y_{\lambda_{i_0},\ldots,\lambda_{i_m}}(v_{i_0}\otimes\ldots\otimes v_{i_m})\otimes v_{i_{m+1}}\otimes\ldots\otimes v_{i_{m+n}}).
\end{aligned} \tag{7}
$$

Here, as in (3), the summation is over the shuffles in S_{m+n+1}, and ϵ_v is the same. Then formula (2) defines a structure of $\mathbb{Z}$-graded Lie superalgebra on $W_{\mathrm{LCA}}(V)$.

As in the Lie (super)algebra case, we consider the $\mathbb{Z}$-graded Lie superalgebra $W_{\mathrm{LCA}}(\Pi V)$. Then $W_{\mathrm{LCA}}^{-1}(\Pi V) = \Pi(V/\partial V)$, $W_{\mathrm{LCA}}^{0}(\Pi V) = \mathrm{End}_{\partial} V$, and odd elements $X \in W_{\mathrm{LCA}}^{1}(\Pi V)$ correspond bijectively to the linear maps $[._\lambda.] : V^{\otimes 2} \to V[\lambda]$, satisfying sesquilinearity and skew-commutativity axioms of LCA. Explicitly, this bijection is given by (cf. with (4)):

$$[a_\lambda b] = (-1)^{p(a)} X_{\lambda,-\lambda-\partial}(a \otimes b). \tag{8}$$

As in the Lie superalgebra case, $[X, X] = 0$ iff (8) satisfies the Jacobi identity.

We obtain an LCA cohomology complex for the adjoint module:

$$(C_{\mathrm{LCA}} = \bigoplus_{j\geq 0} C^j_{\mathrm{LCA}}, \operatorname{ad} X), \text{ where } C^j_{\mathrm{LCA}} = W^{j-1}_{\mathrm{LCA}}(\Pi V).$$

This complex, for any V-module M, was constructed by Bakalov-VK-Voronov in [BKV99], where also its basic properties were studied and cohomology of main examples was computed.

Basic examples of LCA. All simple finitely generated over $\mathbb{C}[\partial]$ Lie conformal algebras were classified by [D'Andrea-VK 98]:

- **Virasoro LCA**: $\mathrm{Vir} = \mathbb{C}[\partial]L, \; [L_\lambda L] = (\partial + 2\lambda)L,$
- **Affine LCA**: $\mathrm{Cur}\,\mathfrak{g} = \mathbb{C}[\partial]\mathfrak{g}, \; [a_\lambda b] = [a, b]$, where $\mathfrak{g}$ is a simple Lie algebra.

Theorem 1 [BKV99]

(a) $\dim H^n_{\mathrm{LCA}}(\mathrm{Vir}, \mathbb{F}) = 1$ for $n = 0, 2, 3;\ = 0$ otherwise.

(b) $H^n_{\mathrm{LCA}}(\mathrm{Cur}\, \mathfrak{g}, \mathbb{F}) \equiv H^n(\mathfrak{g}, \mathbb{F}) \oplus H^{n+1}(\mathfrak{g}, \mathbb{F})$.

Theorem 2 [BKV99]

(a) $H_{\mathrm{LCA}}(\mathrm{Vir}, \mathrm{Vir}) = 0$

(b) $H^n_{\mathrm{LCA}}(\mathrm{Cur}\, \mathfrak{g}, \mathrm{Cur}\, \mathfrak{g}) = H^{n-1}(\mathfrak{g}, \mathbb{F})$.

Corollary Since $H^i_{\mathrm{LCA}}(V, V) =$ Casimirs for $i = 0$, =derivations modulo inner derivations for $i = 1$, and = 1st order deformations for $i = 2$, we obtain that all Casimirs of Vir and $\mathrm{Cur}\, \mathfrak{g}$ are trivial, all derivations are inner, and all their 1-st order deformation are the obvious ones.

Theorem 3 [BKV99] (cf. [Ritt50] for $n = 2$) For the Vir-module $M_\Delta = \mathbb{F}[\partial]v$, $L_\lambda v = (\partial + \Delta\lambda)v$, $\Delta \in \mathbb{F}$, on has:

$$\dim H^n_{\mathrm{LCA}}(\mathrm{Vir}, M_{1-\frac{3r^2\pm r}{2}}) = \begin{cases} 2 & \text{if } n = r+1, \text{ and } r \in \mathbb{Z}_+ \\ 1 & \text{if } n = r, r+2, \text{ and } r \in \mathbb{Z}_+, \end{cases}$$

and $H^n_{\mathrm{LCA}}(\mathrm{Vir}, M_\Delta) = 0$ otherwise

The main tool for computing the LCA cohomology is the basic complex

$$(\widetilde{C}_{\mathrm{LCA}} = \bigoplus_{n \geq 0} \widetilde{C}^n_{\mathrm{LCA}}, d_X),$$

obtained from from the definition of C_{LCA}, by replacing in the definition of $W^n_{\mathrm{LCA}}(V)$ the space V_n, defined by (6), by $V[\lambda_1, \dots, \lambda_n]$. One gets thereby an LCA $\widetilde{W}_{\mathrm{LCA}}(V)$ with canonically defined representation of $W_{\mathrm{LCA}}(V)$ on it, giving, in particular the action d_X of X, and an exact sequence of maps:

$$0 \to \partial \widetilde{W}_{\mathrm{LCA}}(V) \to \widetilde{W}_{\mathrm{LCA}}(V) \xrightarrow{\pi} W_{\mathrm{LCA}}(V).$$

In good situations, e.g. if V is free as an $\mathbb{F}[\partial]$-module, the map π is surjective. Hence this short exact sequence induces a cohomology long exact sequence.

The basic LCA complex is more like a Lie (super)algebra complex, in particular, Cartan's formula holds ($a \in \mathrm{LCA}$):

$$a_\lambda = [\iota_\lambda(a), d_X], \tag{9}$$

where $\iota_\lambda(a)$ is defined by

$$(\iota_\lambda(a)\widetilde{Y})_{\lambda_0,\dots,\lambda_k}(a_0 \otimes \dots \otimes a_k) = \pm \widetilde{Y}_{\lambda,\lambda_0,\dots,\lambda_k}(a \otimes a_0 \otimes \dots \otimes a_k). \tag{10}$$

Using Cartan's formula one can often compute the basic LCA cohomology or put severe restrictions on it. Using that the complexes $(\partial\widetilde{C}_{\mathrm{LCA}}, d_X)$ and $(\widetilde{C}_{\mathrm{LCA}}, d_X)$ are isomorphic, and the long exact sequence, one gets hold on the LCA cohomology.

The next characters in our story are Poisson vertex (super)algebras (PVA). They are similar to Poisson algebras in the same way as Lie conformal algebras are similar to Lie algebras. Namely a PVA is a unital commutative associative algebra with an even derivation ∂ (called a differential algebra), endowed with a structure of LCA, such that these two structures are related by

$$\text{(Leibniz rule)} \quad [a_\lambda bc] = [a_\lambda b]c + (-1)^{p(a)p(b)} b[a_\lambda c],$$

which, due to skew-commutativity, implies

$$\text{(right Leibniz rule)} \quad [ab_\lambda c] = (e^{\partial \partial_\lambda} a)[b_\lambda c] + (-1)^{p(a)p(b)} (e^{\partial \partial_\lambda} b)[a_\lambda c].$$

Given a differential superalgebra $\mathcal{V}$, the Lie superalgebra $W_{\mathrm{PVA}}(\mathcal{V})$ is a $\mathbb{Z}$-graded superalgebra of $W_{\mathrm{LCA}}(\mathcal{V})$, where $W^n_{\mathrm{PVA}} \subset W^n_{\mathrm{LCA}}$ consists of maps, satisfying the Leibniz rules (cf. the right Leibniz rule above):

$$\begin{aligned} &Y_{\lambda_0,\dots,\lambda_n}(a_0 \otimes \dots \otimes b_i c_i \otimes \dots \otimes a_n) \\ &= \pm(e^{\partial \partial_\lambda} b_i) Y_{\lambda_0,\dots,\lambda_n}(a_0 \otimes \dots \otimes c_i \otimes \dots \otimes a_n) \\ &\pm (e^{\partial \partial_\lambda} c_i) Y_{\lambda_0,\dots,\lambda_n}(a_0 \otimes \dots \otimes b_i \otimes \dots \otimes a_n). \end{aligned}$$

Again, an odd $X \in W^1_{\mathrm{PVA}}(\Pi\mathcal{V})$ defines by formula (8) a λ-bracket, satisfying all axioms of a PVA, except for the Jacobi identity, and the Jacobi identity is equivalent to the equation $[X, X] = 0$.

This point of view on LCA and PVA cohomology was developed in [De Sole, VK, 2013] for the needs of the theory of integrable Hamiltonian PDE, but we were unable to extend it to vertex algebras (VA) at the time.

Why PVA cohomology is important for integrable systems of Hamiltonian PDE? Given a PVA $\mathcal{V}$ and $\int h \in \mathcal{V}/\partial\mathcal{V}$, the corresponding Hamiltonian PDE is

$$\frac{df}{dt} = \{\textstyle\int h \;{}_\lambda\; f\}\,\big|_{\lambda=0,}\; f \in \mathcal{V}. \tag{11}$$

It is called integrable if there are infinitely many integrals of motion in involution $\int h_n, n \in \mathbb{Z}_+$:

$$\{\textstyle\int h_m \;{}_\lambda\; \int h_n\}\big|_{\lambda=0} = 0,\; m, n \in \mathbb{Z}_+,\; \int h_0 = \int h. \tag{12}$$

Theorem

Let $\mathcal{V}$ be a PVA with λ-bracket $\{._\lambda.\}$. Suppose that $H^1(\mathcal{V},\mathcal{V}) = 0$ and $\{\int h_0\, {}_\lambda \mathcal{V}\}|_{\lambda=0} = 0$, i.e. $\int h_0$ is a Casimir element. Suppose $\mathcal{V}$ is bi-Poisson, i.e. there is "another" PVA λ-bracket $\{._\lambda.\}_1$ such that any its linear combination with $\{._\lambda.\}$ is again a PVA λ-bracket. Then one can construct $\int h_n, n \in \mathbb{Z}_+$, such that (12) holds.

We have succeeded recently, in collaboration with Bakalov and Heluani [BDSHK18], to built a cohomology of vertex algebras along these lines, by analyzing the construction of the Beilinson-Drinfeld chiral operad, associated to a $\mathcal{D}$-module on a smooth curve X. We showed that for $X = \mathbb{F}$ and the $\mathcal{D}$-module translation invariant, this operad admits a simple description, which is an enhancement of the operad, used for the construction of the LCA cohomology.

To describe this construction, let

$$\mathcal{O}^{*,T}_{n+1} = \mathbb{F}[z_i - z_j, (z_i - z_j)^{-1}]_{0 \le i < j \le n}.$$

Given a vector superspace V with an even endomorphism ∂, the $\mathbb{Z}$-graded Lie superalgebra

$$W_{\mathrm{VA}}(V) = \bigoplus_{j \ge -1} W^j_{\mathrm{VA}}(V)$$

is constructed as follows. Let $W^n_{\mathrm{VA}}(V)$ be the superspace of linear maps

$$\begin{aligned} &Y : V^{\otimes(n+1)} \otimes \mathcal{O}^{*T}_{n+1} \to V_{n+1}, \\ &v_0 \otimes \ldots \otimes v_n \otimes f \mapsto Y_{\lambda_0,\ldots,\lambda_n}(v_0 \otimes \ldots \otimes v_n \otimes f), \end{aligned} \tag{13}$$

which are invariant w.r. to simultaneous permutation of factors in $V^{\otimes(n+1)}$, the λ_i's and the z_i's, and which satisfy the following two sesquilinearity properties:

$$
\begin{aligned}
&Y^{z_0,\ldots,z_n}_{\lambda_0,\ldots,\lambda_n}(v_1 \otimes \ldots \otimes (\partial + \lambda_i)v_i \otimes \ldots \otimes v_n \otimes f) \\
&= Y^{z_0,\ldots,z_n}_{\lambda_0,\ldots,\lambda_n}\left(v_0 \otimes \ldots \otimes v_n \otimes \frac{\partial f}{\partial z_i}\right), \\
&Y^{z_0,\ldots,z_n}_{\lambda_0,\ldots,\lambda_n}(v_1 \otimes \cdots \otimes v_n \otimes (z_i - z_j)f) \\
&= \left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}\right) Y^{z_0,\ldots,z_n}_{\lambda_0,\ldots,\lambda_n}(v_0 \otimes \ldots \otimes v_n \otimes f).
\end{aligned}
$$

Note that for $f = 1$ the first sesquilinearity property turns into the one for $W_{\mathrm{LCA}}(V)$, so that $W_{\mathrm{LCA}}(V) \subset W_{\mathrm{VA}}(V)$.

The bracket on $W_{\mathrm{VA}}(V)$ is defined through the box product as in (2), and the box product is defined by a formula, similar to (8), a typical term is further decorated by the z_i's as follows.

Let $X \in W^n_{\mathrm{VA}}(V)$, $Y \in W^m_{\mathrm{VA}}(V)$, and $f \in \mathcal{O}^{*T}_{m+n+1}$. We can write f in the form $f(z_0, \dots, z_{m+n}) = h(z_0, \dots, z_m)g(z_0, \dots, z_{m+n})$, where $h \in \mathcal{O}^{*T}_{m+1}$, $g \in \mathcal{O}^{*T}_{m+n+1}$, and g has no poles at $z_i = z_j$ for $0 \leq j < j \leq n$. Then a typical term in the box product is (cf. (7))

$$X^{z_{i_0},\dots,z_{i_{m+n}}}_{\lambda_{i_0}+\dots+\lambda_{i_n},\lambda_{i_{n+1}},\dots,\lambda_{i_{m+n}}} \Big(Y^{z_{i_0},\dots,z_{i_m}}_{\lambda_{i_0}-\partial_{z_{i_0}},\dots,\lambda_{i_m}-\partial_{z_{i_m}}} \left(v_{i_0} \otimes \dots v_{i_m} \otimes \right.$$
$$\left. h(z_{i_0}, \dots, z_{i_m}) \right) \otimes v_{i_{m+1}} \otimes \dots \otimes v_{i_{m+n}} \otimes g(z_{i_0}, \dots, z_{i_{m+n}}) \big)$$
$$\Big| z_{i_0} = \dots = z_{i_m} = 0 \Big) .$$

Then an odd element $X \in W^1_{\mathrm{VA}}(\Pi V) \subset W_{\mathrm{VA}}(\Pi V)$ defines on V an integral of λ-bracket (cf. (3)):

$$\int^{\lambda} [u_\sigma v] d\sigma = (-1)^{p(u)} X^{z_0,z_1}_{\lambda,\lambda-\partial} \left(u \otimes v \otimes \frac{1}{z_1 - z_0} \right) . \tag{14}$$

Recall that, according to [De Sole,VK06], one of the equivalent definitions of a (non-unital) vertex algebra structure on a vector superspace V with an even endomorphism ∂ is given by an integral

$$\int^{\lambda} [u_\sigma v] \, d\sigma =: uv : + \int_0^{\lambda} [u_\sigma v] \, d\sigma,$$

which satisfies the axioms of an LCA under the integral. We show that formula (14) satisfies the axioms of sesquilinearity and skew-commutativity of the λ-bracket under the integral, and the equation $[X, X] = 0$ is equivalent to the Jacobi identity under the integral.

Thus, again we obtain a cohomology complex

$$\left(C^{\cdot}_{\mathrm{VA}} = \bigoplus_{j\geq 0} C^{j}_{\mathrm{VA}}, \operatorname{ad} X \right), \text{ where } C^{j}_{\mathrm{VA}} = W^{j-1}_{\mathrm{VA}}(\Pi V).$$

It cohomology is called the vertex algebra cohomology. We show that this cohomology satisfies the expected properties [BDSHK12]. In particular, $H^{i}_{\mathrm{VA}}(V,V)$ for $i = 0,1,2$ describe the Casimirs, derivations of V modulo inner derivations, and first order deformations, respectively.

We developed a method of computing the VA cohomology via majorizing of it by the corresponding PVA cohomology, which goes as follows.

Recall that if V is a vertex algebra with an increasing filtration by $\mathbb{F}[\partial]$-submodules

$$0 = F^0 V \subset F^1(V) \subset F^2(V) \subset \dots, \tag{15}$$

such that

$$:(F^i V)(F^j V): \subset F^{i+j} V, \ \ [F^i V_\lambda F^j V] \subset (F^{i+j-1} V)[\lambda], \tag{16}$$

then the associated graded $\operatorname{gr} V$ is a PVA.

Taking the increasing filtration of $\mathcal{O}_n^{*T}$ by the number of divisors, combining with (15), we obtain an increasing filtration of $V^{\otimes n} \otimes \mathcal{O}_n^{*,T}$. This filtration induces a decreasing filtration of the Lie superalgebra $W_{\mathrm{VA}}(\Pi V)$.

On the other hand, in [BDSHK18] we introduced, for a vector superspace V with an even derivation ∂, the closely related $\mathbb{Z}$-graded Lie superalgebra $W_{c\ell}(V)$, which "governs" the PVA structures on V. The vector superspace $W_{c\ell}^{n-1}(V)$ consists of linear maps (cf. (13))

$$Y : V^{\otimes n} \otimes \mathcal{G}(n) \to V_n,\, v \otimes \Gamma \mapsto Y^{\Gamma}(v),$$

where $\mathcal{G}(n)$ is the space with even parity spanned by oriented graphs with n vertices, subject to certain conditions. Then odd elements $X \in W_{c\ell}^{1}(\Pi V)$ with $[X, X] = 0$ parameterize the PVA structures on V by (cf. (8)):

$$ab = (-1)^{p(a)} X^{\bullet\!\mapsto\!\bullet}(a \otimes b),\, [a_\lambda b] = (-1)^{p(a)} X^{\bullet\,\bullet}_{\lambda,-\lambda-\partial}(a \otimes b).$$

Assuming that V is endowed with an increasing filtration by $\mathbb{F}[\partial]$-modules, we obtain a canonical Lie superalgebra map

$$\operatorname{gr} W_{\mathrm{VA}}(\Pi V) \to W_{c\ell}(\operatorname{gr} \Pi V).$$

We prove that this map is always injective, and that it is an isomorphism, provided that the filtration on V is induced by a grading by $\mathbb{F}[\partial]$-modules. If, in addition, the filtration on V is such that $\operatorname{gr} V$ inherits from the vertex algebra structure on V, given by X, a PVA structure on $\operatorname{gr} V$ (see (16)), we obtain that

$$\dim H^n_{\mathrm{VA}}(V, V) \leq \dim H^n_{c\ell}(\operatorname{gr} V, \operatorname{gr} V). \qquad (17)$$

Finally, the obvious inclusion of Lie superalgebras $W_{\mathrm{PVA}}(\Pi V) \to W_{c\ell}(\Pi V)$, by taking disconnected graphs, induces an injective map on cohomology, and we prove that this map is an isomorphism, provided that as a differential algebra, V is an algebra of differential polynomials. We prove this, using the HKR theorem and Harrison cohomology. Hence, under the above condition, we obtain from (17):

$$\dim H^n_{\mathrm{VA}}(V, V) \leq \dim H^n_{\mathrm{PVA}}(\operatorname{gr} V, \operatorname{gr} V). \tag{18}$$

Thus, we obtain the following

Theorem

Let V be a vertex algebra with a filtration by $\mathbb{F}[\partial]$-modules, such that $\operatorname{gr} V$ is PVA, which, as a differential algebra, is an algebra of differential polynomials, then (18) holds.

In our recent paper [BDSK19] we developed methods of computing PVA cohomology, similar to that for the LCA cohomology. The additional ingredient is the Virasoro element, which exists in all of the most important examples.

Namely, given an LCA R, the symmetric superalgebra $S(R)$ over the vector superspace R inherits, by Leibniz rules, a PVA structure (Kirillov-Kostant like structure).

Basic examples of PVA:

- **Virasoro PVA of central charge** $c \in \mathbb{F} : \mathrm{Vir}^c = S(\widehat{\mathrm{Vir}})/(C - c)$
- **Affine PVA of level** $k \in \mathbb{F} : \mathcal{V}^k(\mathfrak{g}) = S(\widehat{\mathrm{Cur}\mathfrak{g}})/(K - k)$, where

  "hat" stands for the universal central extension of the LCA.

A Virasoro element L in a PVA satisfies:

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3, \ [L_\lambda a]|_{\lambda=0} = \partial a, \ E := \frac{d}{d\lambda}[L_\lambda .]|_{\lambda=0}$$

is diagonalizable. (Note that $\widehat{\mathrm{Vir}} = \mathbb{F}[\partial]L + \mathbb{F}C$, where C is a central element.)

The operator E, called the energy operator, acts consistently on both the PVA cohomology and the basic PVA cohomology, and is diagonizable there.

Theorem [BDSK19]

(a) *Zero is the only eigenvalue of E on the basic PVA cohomology*

(b) *The only eigenvalues of E on the reduced PVA cohomology are 0 and 1.*

Proof.

(a) follows from Cartan's formula (9). (b) follows from (a) and the cohomology long exact sequence. $\square$

From this theorem we can deduce the following results.

Theorem

(a) *Let a PVA $\mathcal{V}$ be, as a differential algebra, a finitely generated algebra of differential polynomials, and let L be a Virasoro element, such that E has positive eigenvalues on $\mathcal{V}$, except for $E1 = 0$. Then* $\dim H^n_{PVA}(\mathcal{V}, M) < \infty$ *for all* $n \geq 0$.

(b) $\dim H^n_{PVA}(\mathrm{Vir}^c, \mathrm{Vir}^c) = 1$ *for* $n = 0, 2, 3;\ = 0$ *otherwise.*

(c) $H^n_{PVA}(\mathcal{V}^k(\mathfrak{g}), \mathcal{V}^k(\mathfrak{g})) = H^n(\mathfrak{g}, \mathbb{F}) \oplus H^{n+1}(\mathfrak{g}, \mathbb{F}), n \geq 0,$ *if* $k \neq 0$.

Proof.

(a) is deduced from the previous theorem. The proof of (b) uses the Virasoro element L. The proof of (c) uses $L = \frac{1}{2k}\sum_i a_i^2$ (the Sugawara construction), where $\{a_i\}$ is an orthonormal basis of $\mathfrak{g}$.

□

Using the inequality (18), we obtain results on the vertex algebra cohomology.

Theorem

(a) *Let V be a freely and finitely generated vertex algebra by elements of positive conformal weight, including a Virasoro element (in particular, $V = W^k(\mathfrak{g}, f)$). Then*

$$\dim H^n_{VA}(V, V) < \infty \text{ for all } n \geq 0.$$

(b) *Cohomology of the freely generated vertex algebras* Vir^c *and* $V^k(\mathfrak{g})$ *with $k \neq 0$ is the same as for the corresponding PVA. In particular, all their Casimirs are trivial, all derivations are inner, and all first order deformations are the obvious ones.*

So computation of cohomology of freely generated vertex algebras is by now well understood, but for their simple quotients, or the lattice VA, the problem is widely open.

The simplest example of a vertex algebra is when $\partial = 0$. Then a vertex algebra V is just a unital commutative associative algebra. Let M be a module over this algebra.

Theorem

$H_{VA}(V, M) = HH(V, M)$.

If $T \neq 0$, but V is commutative the situation is more complicated.