# Classification of moonshine type VOAS generated by Ising vectors of $\sigma$-type

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# Outline

- 3-transposition groups of symplectic type
- Ising vectors of $\sigma$-type
- VOAS generated by Ising vectors of $\sigma$-type
- Main results.

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## Definition 1

A 3-*transposition group* is a pair $(G, I)$ of a group $G$ and a set $I$ of involutions of $G$ satisfying the following conditions.

(1) $G$ is generated by $I$.

(2) $I$ is closed under the conjugation, i.e., if $a$, $b \in I$ then $a^b = aba \in I$.

(3) For any $a$ and $b \in I$, the order of $ab$ is bounded by 3.

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- A 3-transposition group $(G, I)$ is called *indecomposable* if $I$ is a conjugacy class of $G$.
- An indecomposable $(G, I)$ is called *non-trivial* if $I$ is not a singleton, i.e., $G$ is not cyclic.
- Let $(G, I)$ be a 3-transposition group and $a, b \in I$. We define a graph structure on $I$ by $a \sim b$ if and only if $a$ and $b$ are non-commutative.
- It is clear that $I$ is a connected graph if and only if $I$ is a single conjugacy class of $G$.

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- Let $\alpha$, $\beta$ be non-zero complex numbers. Let $B_{\alpha,\beta}(G,I) = \oplus_{i\in I}\mathbb{C}x^i$ be the vector space spanned by a formal basis $\{x^i \mid i \in I\}$ indexed by the set of involutions.
- We define a bilinear product and a bilinear form on $B_{\alpha,\beta}(G,I)$ by

$$x^i \cdot x^j := \begin{cases} 2x^i & \text{if } i = j, \\ \frac{\alpha}{2}(x^i + x^j - x^{iji}) & \text{if } i \sim j, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

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$$(x^i|x^j) := \begin{cases} \frac{\beta}{2} & \text{if } i = j, \\ \frac{\alpha\beta}{8} & i \sim j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

- Then $B_{\alpha,\beta}(G,I)$ is a commutative non-associative algebra with a symmetric invariant bilinear form [Ma05].

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- This algebra is called the *Matsuo algebra* associated with a 3-transposition group $(G, I)$ with accessory parameters $\alpha$ and $\beta$.
- The radical of the bilinear form on $B_{\alpha,\beta}(G)$ forms an ideal. We call the quotient algebra of $B_{\alpha,\beta}(G)$ by the radical of the bilinear is the *non-degenerate quotient*.

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- Suppose $G$ is indecomposable. Then the number $\sharp\{j \in I \mid j \sim i\}$ is independent of $i \in I$ if it is finite. We denote this number by $k$.
- One can verify that

$$\left(\sum_{i\in I} x^i\right) \cdot x^j = \left(\frac{k\alpha}{2} + 2\right) x^j.$$

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- So if $k\alpha + 4$ is non-zero then

$$\omega := \frac{4}{k\alpha + 4} \sum_{i \in I} x^i \tag{3}$$

satisfies $\omega v = 2v$ for $v \in B_{\alpha,\beta}(G)$.

- By the invariance, one has $(x^i|\omega) = (x^i|x^i)$ and $(\omega|\omega) = \frac{2\beta|I|}{k\alpha+4}$.

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**Remark 2**

*A Matsuo algebra $B_{\alpha,\beta}(G)$ corresponds to the Griess algebra of a VOA generated by Virasoro vectors of central charge $\beta$ with binary fusions determined by $\alpha$ [Ma05]. The vector $\omega$ is the conformal vector of such a VOA.*

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We now recall the notation of the Fischer space associated with a 3-transposition group. See [Ma05], [We84], [Ha89-1], [CH95] and [As97] for detail.

- A *partial linear space* is a pair $(X, \mathcal{L})$ with $X$ being the set of points and $\mathcal{L}$ the subsets of $X$ called the set of lines such that any two points lie on at most one line and any line has at least two points.
- Consequently for any lines $l_1$ and $l_2$, we have either $l_1 \cap l_2 = \emptyset$, $|l_1 \cap l_2| = 1$ or $l_1 = l_2$.

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- The *Dual affine plane of order 2* is the partial space $(X, \mathcal{L})$ such that

$$X = \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\}$$

and $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$, where $l_i = \{x_{mn} | i \notin \{m, n\}\}$.

- The *Affine plane of order 3* is the partial space $(X, \mathcal{L})$ such that $X = \{x_{ij} | 0 \leq i \leq j \leq 2\}$ and a 3-set $\{x_{ij}, x_{kl}, x_{mn}\}$ is a line if and only if $(i + k + m, j + l + n) \equiv (0, 0) \bmod 3$ so that $\sharp(\mathcal{L}) = 12$.

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- A partial linear space $(X, \mathcal{L})$ for which the lines consists of three points is called an (abstract) Fischer space if it satisfies the following property [CH95]:
  (FS): For any two intersecting lines $l_1$ and $l_2$, the span of them is isomorphic either to the dual affine plane of order 2 or to the affine plane of order 3.

**Proposition 3 (Fischer)**

*Let $(G, I)$ be a 3-transposition group. Then the partial linear space associated with $(G, I)$ is a Fischer space.*

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- Conversely, let $(X, \mathcal{L})$ be an abstract Fischer space. For $x \in X$, let $\sigma_x$ be the permutation of $X$ defined by

$$\sigma_x(y) = \begin{cases} y & \text{if } x \text{ and } y \text{ are not collinear} \\ z & \text{if } \{x, y, z\} \text{ is a line.} \end{cases}$$

- The pair $(G, I)$ is a centerfree 3-transposition group, where $I = \{\sigma_x | x \in X\}$ and $G = < I >$.

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A 3-transposition group is called *of symplectic type* if the affine plane of order 3 does not occur in the associated Fischer space. Such groups were classified in [Ha89-1] and [Ha89-2].

**Theorem 4 (J.I. Hall)**

*An indecomposable centerfree 3-transposition group of symplectic type is isomorphic to the extension of one of the groups:* $\mathfrak{S}_n(n \geq 3)$; $\mathrm{Sp}_{2n}(2)(n \geq 3)$; $\mathrm{O}^{+}_{2n}(2)(n \geq 4)$; *and* $\mathrm{O}^{-}_{2n}(2)(n \geq 3)$, *by the direct sum of copies of the natural modules.*

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- Here the natural module which will be denoted by $F$, is isomorphic to $2^{2n}$ for $\mathrm{O}^{\pm}_{2n}(2)$ or $\mathrm{Sp}_{2n}(2)$. Note that $\mathfrak{S}_4 \cong 2^2 : \mathfrak{S}_3$.

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Let $V$ be a VOA.

- A Virasoro vector $e \in V$ with central charge $c$ is called *simple* if the subalgebra $< e >$ generated by $e$ is isomorphic to $L(c, 0)$.
- A simple Virasoro vector of central charge $1/2$ is called an *Ising vector*.

Let $e$ be an Ising vector of a VOA $V$ of moonshine-type.

- An Ising vector $e$ is said to be *of $\sigma$-type* if there exists no irreducible $< e >$-submodule of $V$ isomorphic to $L(\frac{1}{2}, \frac{1}{16})$.

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- In this case, we have

$$V = V[0]_e \oplus V[1/2]_e \tag{4}$$

where $V[h]_e$ is the sum of all irreducible $< e >$-submodules isomorphic to $L(1/2, h)$.

- By the fusion rules of $L(1/2, 0)$-modules and based on the decomposition (1), we can define an automorphism by

$$\sigma_e := \begin{cases} 1 & \text{on } V[0]_e, \\ -1 & \text{on } V[1/2]_e. \end{cases} \tag{5}$$

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- The involution $\sigma_e$ is called a *Miyamoto involution* of $\sigma$-type or a $\sigma$*-involution* (cf. [Mi96]).

By the definition, we have the following conjugation.

**Proposition 5**

*Let $e \in V$ be an Ising vector of $\sigma$-type and $g \in \mathrm{Aut}(V)$. Then we have $\sigma_{ge} = g\sigma_e g^{-1}$.*

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The local structures of subalgebras of the Matsuo algebra generated by two Ising vectors of $\sigma$-type are completely determined in [Mi96, Ma05].

**Proposition 6 ([Mi96, Ma05])**

*Let $V$ be a VOA of moonshine-type and let $a$ and $b$ be distinct Ising vectors of $\sigma$-type on $V$. Then the Griess subalgebra $B$ generated by $a$ and $b$ is one of the following.*

*(i) $(a|b) = 0$, $a_1 b = 0$ and $B = \mathbb{C}a + \mathbb{C}b$. In this case $\sigma_a$ and $\sigma_b$ are commutative on $V$.*

*(ii) $(a|b) = 2^{-5}$, $\sigma_a b = \sigma_b a$, $4a_1 b = a + b - \sigma_a b$ and $B = \mathbb{C}a + \mathbb{C}b + \mathbb{C}\sigma_a b$. In this case $\sigma_a\sigma_b$ has order three on $V$.*

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- Let $V$ be a moonshine type VOA generated by Ising vectors of $\sigma$-type. For simplicity, we say such VOAS satisfies **Condition 1**. We have

**Lemma 7 ([JLY17])**

*Let $V$ be a simple VOA satisfying* **Condition 1**, *then its Griess algebra is linearly spanned by its Ising vectors of $\sigma$-type.*

Furthermore, we have

**Proposition 8 ([JLY17])**

*Let $V$ be a VOA satisfying* **Condition 1**, *and $e, f \in V$ two Ising vectors of $\sigma$-type such that $(e|f) = \frac{1}{32}$. Denote by $U^{e,f}$ the subVOA generated by $e$ and $f$. We have the following result.*

$$U^{e,f} \cong L(\frac{1}{2},0) \otimes L(\frac{7}{10},0) \oplus L(\frac{1}{2},\frac{1}{2}) \otimes L(\frac{7}{10},\frac{3}{2}).$$

We denote by $E_V$ the set of Ising vectors of $\sigma$-type of $V$ and set $G_V =< \sigma_e | e \in E_V >$. By Proposition 6 and Proposition 8, we have

**Proposition 9 ([Ma05], [JLY17])**

*Let $V$ be a VOA satisfying* **Condition 1**. *Then $G_V$ is a 3-transposition group of symplectic type.*

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The group $G_V$ in the above proposition is said to be $(1/2, 1/2)$-realizable by a VOA.

### Remark 10

*Let $V$ be a VOA satisfying* **Condition 1**. *It is very natural to assume that each indecomposable component of $G_V$ is non-trivial. Then $G_V$ is a center-free 3-transposition group of symplectic type, and the Grieee algebra of $V$ is a quotient of the Matsuo algebra $G_{1/2,1/2}(G_V)$.*

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- Let $R$ be a root lattice with root system $\Phi(R)$ of type ADE.
- Denote by $l$ the rank of $R$ and $h$ the Coxeter number of $R$.
- Denote by $\sqrt{2}R$ the lattice whose norm is twice of $R$'s.
- Let $V_{\sqrt{2}R}^{+}$ be the fixed point subalgebra of $V_{\sqrt{2}R}$ under the lift of (-1)-isometry on $R$. Then $V_{\sqrt{2}R}^{+}$ is moonshine-type.

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- Set

$$s = s_R := \frac{h}{h+2}\omega - \frac{1}{h+2}\sum_{\alpha\in\Phi(R)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}R}^{+},$$

where $\omega$ is the Virasoro vector of $V_{\sqrt{2}R}^{+}$.

- It is shown in [DLMN98] that $s$ is a Virasoro vector with central charge $\frac{lh}{h+2}$.

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- Then

$$\tilde{\omega} = \tilde{\omega}_R := \omega - s = \frac{2}{h+2}\omega + \frac{1}{h+2}\sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha}$$

is also a Virasoro vector with central charge $\frac{2l}{h+2}$ and the decomposition $\omega = s + \tilde{\omega}$ is orthogonal.

- Denote [LSY07]

$$M_R := C_{V_{\sqrt{2}R}^{+}}(Vir(\tilde{\omega})) = Ker_{V_{\sqrt{2}R}^{+}}(\tilde{\omega}_0).$$

- The commutant subalgebra $M_R$ naturally affords an action of the Weyl group $W(R)$ associated to the root system $\Phi(R)$ [LSY07].
- Referee [LSY07], [DLY09], [DLMN98], [KM01], [Gr98], [JL16], [JLY17], etc. for the study of $M_R$.

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- Take $R = A_n$ for example [LY00], [JL16], [La14], [JL14], [DW10],

$$M_{A_n} \cong C_{V_{\sqrt{2}A_n}}(K(sl_2, l))$$

$$\cong C_{L_{\widehat{sl_2}}(1,0)^{\otimes n+1}}(L_{\widehat{sl_2}}(n+1,0)) \cong K(sl_{n+1}, 2).$$

In general for $M_R$ and $\alpha \in R$, let

$$\omega^\alpha = \frac{1}{8}\alpha(-1)\alpha(-1)\mathbf{1} - \frac{1}{4}(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}),$$

then $\omega^\alpha$ is Ising vector of $\sigma$-type and $(M_R)_2$ is linearly spanned by $\{\omega^\alpha | \alpha \in R\}$. Furthermore,

$$E_V = \{\omega^\alpha | \alpha \in R\},$$

except the case that $R = E_8$ [LSY07].

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We have the following result.

**Theorem 11 ([Ma05], [LSY07])**

*Let $G_V$ be a center-free indecomposable 3-transposition group of symplectic type realizable by a simple vertex operator algebra $V$ which satisfies* **Condition 1** *and carries a positive-definite Hermitian form. Then $(G_V, V)$ is one of the following:*

$$(\mathfrak{S}_{n+1}, M_{A_n}),\ (F:\mathfrak{S}_{n+1}(n\geq 3), V^{+}_{\sqrt{2}A_n}),\ (F^2:\mathfrak{S}_n(n\geq 4), V^{+}_{\sqrt{2}D_n}),$$

$$(\mathrm{O}_6^-(2), M_{E_6}),\ (2^6:\mathrm{O}_6^-(2), V^{+}_{\sqrt{2}E_6}),\ (\mathrm{O}_8^-(2), \mathrm{Com}_{V^{+}_{\sqrt{2}E_8}}(M_{A_2})),$$

$$(\mathrm{Sp}_6(2), M_{E_7}), (2^6:\mathrm{Sp}_6(2), V^{+}_{\sqrt{2}E_7}),\ (\mathrm{O}_8^+(2)\mathrm{or}\ \mathrm{Sp}_8(2):M_{E_8}),$$

$$(2^8:\mathrm{O}_8^+(2)\ \mathrm{or}\ \mathrm{O}_{10}^+(2), V^{+}_{\sqrt{2}E_8}).$$

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## Remark 12

*(1) The natural module* $F$ *for* $\mathfrak{S}_n$ *is defined as* $\mathbb{Z}_2^{n-1}$ *if* $n$ *is odd and as* $\mathbb{Z}_2^{n-2}$ *if* $n$ *is even.*
*(2) For the pairs* $(G_1, G_2) = (\mathrm{O}_8^+(2), \mathrm{Sp}_8(2))$ *and* $(G_1, G_2) = (2^8 : \mathrm{O}_8^+(2), \mathrm{O}_{10}^+(2))$, *we have* $G_1 \le G_2$, *and* $B_{1/2,1/2}(G_1)$ *is a subalgebra of* $B_{1/2,1/2}(G_2)$.
*But the non-degenerate quotients of* $B_{1/2,1/2}(G_1)$ *and* $B_{1/2,1/2}(G_2)$ *are the same. So they are realized by the same VOA.*

## Our Goal:

- Classify VOAS satisfying **Condition 1**.
- Eliminate the positivity-condition in the classification of center-free indecomposable 3-transpaosition groups $(1/2, 1/2)$-realizable by a VOA in [Ma05].

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### Theorem 13 ([JLY17])

*Let $V$ be a simple moonshine VOA generated by Ising vectors of $\sigma$-type. Then the VOA structure of $V$ is uniquely determined by its Griess algebra.*

### Theorem 14 ([JLY17])

*Let $V$ be a moonshine VOA generated by Ising vectors of $\sigma$-type such that $G_V = \mathfrak{S}_{n+1}$. Then $V$ is simple and isomorphic to $M_{A_n}$.*

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**Theorem 15 ([JLY17])**

*Let $V$ be a simple moonshine type VOA generated by Ising vectors of $\sigma$-type and let $V_{\mathbb{R}}$ be the real VOA generated by the set $E_V$ of Ising vectors of $V$ of $\sigma$-type. If the non-degenerate quotient of the real Matsuo algebra $B_{1/2,1/2}(G_V)_{\mathbb{R}}$ associated with $G_V$ is positive definite, then $V_{\mathbb{R}}$ is a compact real form of $V$. In this case a non-trivial indecomposable component of the 3-transposition group $G_V$ is isomorphic to one of the groups listed in Theorem 1 of [Ma05].*

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## Conjecture 16

*Let $V$ be a moonshine type VOA generated by Ising vectors of $\sigma$-type is simple and must be one of those listed in Theorem 11.*

## Conjecture 17

*Let $V$ be a simple moonshine type VOA generated by Ising vectors of $\sigma$-type. Then the bilinear form on the $\mathbb{R}$-span of $E_V$ is positive definite, i.e., the non-degenerate quotient of the real Matsuo algebra $B_{1/2,1/2}(G_V)_{\mathbb{R}}$ associated with $G_V$ is positive definite.*

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## Main Theorem 1 (J-Lam-Yamauchi 19)

*(1) Let $V$ be a moonshine type VOA generated by Ising vectors of $\sigma$-type. Then $V$ is simple and isomorphic to one or tensor product of the vertex operator algebras:*

$$M_{A_n},\ M_{E_6},\ M_{E_7},\ M_{E_8},\ \mathrm{Com}_{V^+_{\sqrt{2}E_8}}(M_{A_2}),$$

$$V^+_{\sqrt{2}E_6},\ V^+_{\sqrt{2}D_n},\ V^+_{\sqrt{2}A_n},\ V^+_{\sqrt{2}E_7},\ V^+_{\sqrt{2}E_8}.$$

*(2) All the above vertex operator algebras are rational and $C_2$-cofinite.*

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**Main Theorem 2 (J-Lam-Yamauchi 19)**

*Let $G_V$ be a center-free indecomposable $3$-transposition group of symplectic type realizable by a moonshine type VOA $V$ generated by Ising vectors of $\sigma$-type. Then $(G_V, V)$ is one of the pairs listed in Theorem 11.*

- M. Aschbacher, 3-transposition groups. Cambridge tracts in Mathematics 124, Cambridge University Press, 1997.
- T. Creutzig, S. Kanade, A.R. Linshaw and D. Ridout, Schur-Weyl duality for Heisenberg Cosets, *Transform. Group*(2018), DOI:10.1007/s00031-018-9497-2.
- H. Cuypers and J.I. Hall, The $3$-transposition groups with trivial center. *J. Algebra* **178** (1995), 149–193.
- C. Dong, C. Jiang and X. Lin, Rationality of vertex operator algebra $V_L^+$ : higher rank, *Proc. London Math. Soc.* 104 (2012) 799-826.

上海交通大學 Shanghai Jiao Tong University


- C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Progress in Math., Vol. 112, Birkhäuser, Boston, 1993.
- C. Dong, C.H. Lam, Q. Wang and H. Yamada, The structure of parafermion vertex operator algebras. *J. Algebra* **323** (2010), 371-381.
- C. Dong, C.H. Lam and H. Yamada, W-algebras related to parafermion algebras. *J. Algebra* **322** (2009), 2366-2403.
- C. Dong, H. Li and G. Mason, Compact automorphism groups of vertex operator algebras. *Internat. Math. Res. Notices* **18** (1996), 913–921.

C. Dong, H. Li, G. Mason and S.P. Norton, Associative subalgebras of Griess algebra and related topics. Proc. of the Conference on the Monster and Lie algebra at the Ohio State University, May 1996, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin - New York, 1998, pp. 27–42.

C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module. Proc. Symp. Pure. Math., American Math. Soc. **56 II** (1994), 295–316.

上海交通大學 Shanghai Jiao Tong University


- C. Dong and Q. Wang, The structure of parafermion vertex operator algebras: general case, *Commun. Math. Phys.* **299** (2010), 783-792.
- I. B. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Memoirs American Math. Soc. 104 (1993).

Shanghai Jiao Tong University


- I. B. Frenkel, J. Lepowsky and A. Meurman,Vertex Operator Algebras and the Monster, Pure and Applied Math. Vol. 134, Academic Press, 1988.
- I.B. Frenkel and Y. Zhu, Vertex operator algebras associated to representation of affine and Virasoro algebras. *Duke Math. J.* **66** (1992), 123–168.
- P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro and super-Virasoro algebra. *Commun. Math. Phys.* **103** (1986), 105–119.

上海交通大学 Shanghai Jiao Tong University

M. Geck and G. Pfeiffer, Characters of Finite Coxeter Groups and Iwahori-Heckee Algebras. Oxford University Press, New York, 2000.

J.I. Hall, Graphs, geometry, 3-transpositions, and symplectic $\mathbb{F}_2$-transvection groups, *Proc. London Math. Soc.* **58** (1989), 89-111.

R.L. Griess, Jr: A vertex operator algebra related to E8 with automorphism group O+(10,2). The Monster and Lie algebras, 43 – 58, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.

上海交通大学
Shanghai Jiao Tong University


- J.I. Hall, Some 3-transposition groups with non-central 2-subgroups, *Proc. London Math. Soc.* **58** (1989), 112-136.
- J.I. Hall, F. Rehren and S. Shpectorov, Primitive axial algebras of Jordan type. *J. Algebra* **437** (2015), 79–115.
- Y.-Z. Huang, A. Kirillov, and J. Lepowsky, Braided tensor categories and extensions of vertex operator algebras. *Comm. Math. Phys.* **337** (2015), 1143-1159.

上海交通大學
Shanghai Jiao Tong University


- J. Humphreys, Introduction to Lie algebras and Representation theory. Springer-Verlag, Third edition, 1980.
- K. Iohara and Y. Koga, Representation Theory of the Virasoro Algebra. Springer Monographs in Mathematics. Springer-Verlag, London, 2011.
- G. James, The representation theory of the symmetric groups, Springer-Verlag Berlin-Heidelberg-New York, 1978.

上海交通大學 Shanghai Jiao Tong University


- C. Jiang and Z. Lin, The commutant of $L_{\widehat{\mathfrak{sl}_2}}(n, 0)$ in the vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(1, 0)^{\otimes n}$. *Adv. Math.* **301** (2016), 227–257.
- C. Jiang and Z. Lin, Tensor decomposition, parafermions, level-rank duality, and reciprocity law for vertex operator algebras, arXiv: 14064191

上海交通大学 Shanghai Jiao Tong University


- C. Jiang, C. Lam and H. Yamauchi, Vertex operator algebras generated by Ising vectors of $\sigma$-type. *Math. Z.* **?** (2019), ???-???.
- C. Jiang and Q. Wang, Representations of $\mathbb{Z}_2$-orbifold of the parafermion vertex operator algebra $K(sl_2, k)$, arXiv: 1712.07277.
- C. Jiang and Q. Wang, Fusion rules for $\mathbb{Z}_2$-orbifolds of affine and parafermion vertex operator algebras, arXiv:

- V. Kac and A. Raina, Highest Weight Representations of Infinite Dimensional Lie Algebras. World Scientific. Adv. Ser. In Math. Phys., Singapore, 1987.
- M. Kitazume and M. Miyamoto: 3-transposition automorphism groups of VOA. Groups and combinatorics – in memory of Michio Suzuki, 315 – 324, Adv. Stud. Pure Math., 32, Mathematical Society of Japan, Tokyo, 2001.

上海交通大学 Shanghai Jiao Tong University


- C.H. Lam, A level-rank duality for parafermion vertex operator algebras of type A. *Proc. Amer. Math. Soc.*, **142** (2014), no. 12, 4133-4142. (Reviewer: Elizabeth Graf Jurisich) 17B69
- C.H. Lam and S. Sakuma, On a class of vertex operator algebras having a faithful $S_{n+1}$-action. *Taiwanese Journal of Mathematics* **12** (2008), 2465–2488.
- C.H. Lam, S. Sakuma and H. Yamauchi, Ising vectors and automorphism groups of commutant subalgebras related to root systems. *Math. Z.* **255**(3) (2007), 597–626.
- C. Lam and H. Yamada, $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes and vertex operator algebras. *J. Algebra* **224** (2000), 268–291.

上海交通大学 Shanghai Jiao Tong University


- C. Lam and H. Yamada, Decomposition of the lattice vertex operator algebra $V_{\sqrt{2}A_l}$. *J. Algebra* **272** (2004), 614-624.
- J. Lepowsky and H. Li, Introduction to Vertex Operator Algebras and Their Representations. Progress in Mathematics, Vol. 227, Birkhäser Boston, Inc., Boston, MA, 2004.
- H. Li, Symmetric invariant bilinear forms on vertex operator algebras. *J. Pure Appl. Alg.* **96** (1994), 279–297.

- S. Kanade, A.R. Linshaw, Universal two-parameter even spin $\mathcal{W}_\infty$-algebra. arXiv: 1805.11031.
- A. Matsuo, $3$-transposition groups of symplectic type and vertex operator algebras. *J. Math. Soc. Japan* **57**(3) (2005), 639–649.
- M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras. *J. Algebra* **179** (1996), 528–548.
- W. Wang, Rationality of Virasoro vertex operator algebras. *Internat. Math. Res. Notices* **71** (1993), 197–211.

上海交通大学 Shanghai Jiao Tong University


- R. Weiss, 3-transpositions in infinite groups. *Math. Proc. Cambridge Philos. Soc.* **96** (1984), 371-377.
- Y. Zhu, Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.* **9** (1996), 237–302.

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# Thank you for your attention