A construction of twisted modules for a grading-restricted vertex (super)algebra

Yi-Zhi Huang

Rutgers University

June 24, 2019
Representation Theory XVI
Dubrovnik, Croatia

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Definition

- Frenkel, Lepowsky and Meurman (1988): Twisted modules associated to automorphisms of finite orders of a vertex operator algebra.
- H. (2009): Twisted modules associated general automorphisms of a vertex operator algebra.
- Twisted vertex operators contain logarithm of the variable when the automorphism does not act semisimply.
- V: A grading-restricted vertex superalgebra (a vertex superalgebra with a compatible $\mathbb{Z}$-grading satisfying the two grading restriction properties). .
- g: an automorphism of V.

Definition

- Frenkel, Lepowsky and Meurman (1988): Twisted modules associated to automorphisms of finite orders of a vertex operator algebra.
- H. (2009): Twisted modules associated general automorphisms of a vertex operator algebra.
- Twisted vertex operators contain logarithm of the variable when the automorphism does not act semisimply.
- V: A grading-restricted vertex superalgebra (a vertex superalgebra with a compatible $\mathbb{Z}$-grading satisfying the two grading restriction properties). .
- g: an automorphism of V.

Definition

- Frenkel, Lepowsky and Meurman (1988): Twisted modules associated to automorphisms of finite orders of a vertex operator algebra.
- H. (2009): Twisted modules associated general automorphisms of a vertex operator algebra.
- Twisted vertex operators contain logarithm of the variable when the automorphism does not act semisimply.
- V: A grading-restricted vertex superalgebra (a vertex superalgebra with a compatible $\mathbb{Z}$-grading satisfying the two grading restriction properties). .
- g: an automorphism of V.

Definition

- Frenkel, Lepowsky and Meurman (1988): Twisted modules associated to automorphisms of finite orders of a vertex operator algebra.
- H. (2009): Twisted modules associated general automorphisms of a vertex operator algebra.
- Twisted vertex operators contain logarithm of the variable when the automorphism does not act semisimply.
- V: A grading-restricted vertex superalgebra (a vertex superalgebra with a compatible $\mathbb{Z}$-grading satisfying the two grading restriction properties). .
- g: an automorphism of V.

Definition

- Frenkel, Lepowsky and Meurman (1988): Twisted modules associated to automorphisms of finite orders of a vertex operator algebra.
- H. (2009): Twisted modules associated general automorphisms of a vertex operator algebra.
- Twisted vertex operators contain logarithm of the variable when the automorphism does not act semisimply.
- V: A grading-restricted vertex superalgebra (a vertex superalgebra with a compatible $\mathbb{Z}$-grading satisfying the two grading restriction properties). .
- g: an automorphism of V.

Definition

- Frenkel, Lepowsky and Meurman (1988): Twisted modules associated to automorphisms of finite orders of a vertex operator algebra.
- H. (2009): Twisted modules associated general automorphisms of a vertex operator algebra.
- Twisted vertex operators contain logarithm of the variable when the automorphism does not act semisimply.
- V: A grading-restricted vertex superalgebra (a vertex superalgebra with a compatible $\mathbb{Z}$-grading satisfying the two grading restriction properties). .
- g: an automorphism of V.

Definition

- Data of a lower-bounded generalized g-twisted V-module:
  - A $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space

$$W = \coprod_{n \in \mathbb{C}, s \in \mathbb{Z}_2, [\alpha] \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}.$$

  - A twisted vertex operator map:

$$Y_W^g : V \otimes W \to W\{x\}[\log x],$$
$$v \otimes w \mapsto Y_W^g(v, x)w = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} (Y_W^g)_{n,k} x^n (\log x)^k$$

  - Operators $L_W(0)$ and $L_W(-1)$ on W.
  - An action of g on W.

Definition

- Data of a lower-bounded generalized g-twisted V-module:
  - A $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space

$$W = \coprod_{n\in\mathbb{C}, s\in\mathbb{Z}_2, [\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}.$$

  - A twisted vertex operator map:

$$Y_W^g : V \otimes W \to W\{x\}[\log x],$$
$$v \otimes w \mapsto Y_W^g(v, x)w = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} (Y_W^g)_{n,k} x^n (\log x)^k$$

  - Operators $L_W(0)$ and $L_W(-1)$ on W.
  - An action of g on W.

Definition

- Data of a lower-bounded generalized g-twisted V-module:
  - A $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space

$$W = \coprod_{n \in \mathbb{C}, s \in \mathbb{Z}_2, [\alpha] \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}.$$

  - A twisted vertex operator map:

$$\begin{aligned} Y_W^g : V \otimes W &\to W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} (Y_W^g)_{n,k} x^n (\log x)^k \end{aligned}$$

  - Operators $L_W(0)$ and $L_W(-1)$ on W.
  - An action of g on W.

Definition

- Data of a lower-bounded generalized g-twisted V-module:
  - A $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space

$$W = \coprod_{n\in\mathbb{C}, s\in\mathbb{Z}_2, [\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}.$$

  - A twisted vertex operator map:

$$\begin{aligned} Y_W^g : V \otimes W &\to W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} (Y_W^g)_{n,k} x^n (\log x)^k \end{aligned}$$

  - Operators $L_W(0)$ and $L_W(-1)$ on W.
  - An action of g on W.

Definition

- Data of a lower-bounded generalized g-twisted V-module:
  - A $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space
$$W = \coprod_{n\in\mathbb{C}, s\in\mathbb{Z}_2, [\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}.$$
  - A twisted vertex operator map:
$$Y_W^g : V \otimes W \to W\{x\}[\log x],$$
$$v \otimes w \mapsto Y_W^g(v, x)w = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} (Y_W^g)_{n,k} x^n (\log x)^k$$
  - Operators $L_W(0)$ and $L_W(-1)$ on W.
  - An action of g on W.

Definition

- Data of a lower-bounded generalized g-twisted V-module:
  - A $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space

$$W = \coprod_{n \in \mathbb{C}, s \in \mathbb{Z}_2, [\alpha] \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}.$$

  - A twisted vertex operator map:

$$\begin{aligned} Y_W^g : V \otimes W &\to W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} (Y_W^g)_{n,k} x^n (\log x)^k \end{aligned}$$

  - Operators $L_W(0)$ and $L_W(-1)$ on W.
  - An action of g on W.

Definition

- Axioms:
  - The equivalence property:
$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w.$$
  - The identity property: $Y_W^g(\mathbf{1}, x)w = w$.
  - The duality property: The series
$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle,$$
$$(-1)^{|u||v|}\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle,$$
$$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle$$
are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ (also $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$), respectively, to
$$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}.$$

Definition

- Axioms:
  - The equivariance property:
$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w.$$
  - The identity property: $Y_W^g(\mathbf{1}, x)w = w$.
  - The duality property: The series
$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle,$$
$$(-1)^{|u||v|}\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle,$$
$$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle$$
are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ (also $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$), respectively, to
$$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}.$$

Definition

- Axioms:
  - The equivariance property:
$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w.$$
  - The identity property: $Y_W^g(\mathbf{1}, x)w = w$.
  - The duality property: The series
$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle,$$
$$(-1)^{|u||v|}\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle,$$
$$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle$$
are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ (also $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$), respectively, to
$$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}.$$

Definition

- Axioms:
  - The equivariance property:
$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w.$$
  - The identity property: $Y_W^g(\mathbf{1}, x)w = w$.
  - The duality property: The series
$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle,$$
$$(-1)^{|u||v|}\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle,$$
$$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle$$
are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ (also $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$), respectively, to
$$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}.$$

Definition

- Axioms:
  - The equivariance property:
    $$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w.$$
  - The identity property: $Y_W^g(\mathbf{1}, x)w = w$.
  - The duality property: The series
    $$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w\rangle,$$
    $$(-1)^{|u||v|}\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w\rangle,$$
    $$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w\rangle$$
    are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$ (also $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$), respectively, to
    $$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}.$$

Definition

- More axioms:
  - The Lower-boundedness condition: There exists $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$.
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:
  $$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz}Y_W^g(v, z) + Y_W^g(L_V(0)v, z).$$
  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.
  - The $L(-1)$-commutator formula: $\frac{d}{dx}Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)]$.

Definition

- More axioms:
  - The Lower-boundedness condition: There exists $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$.
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:

$$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz}Y_W^g(v, z) + Y_W^g(L_V(0)v, z).$$

  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.
  - The $L(-1)$-commutator formula: $\frac{d}{dx}Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)]$.

Definition

- More axioms:
  - The Lower-boundedness condition: There exists $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$.
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:

$$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz}Y_W^g(v, z) + Y_W^g(L_V(0)v, z).$$

  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.
  - The $L(-1)$-commutator formula: $\frac{d}{dx}Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)]$.

Definition

- More axioms:
  - The Lower-boundedness condition: There exists $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$.
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:

$$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz} Y_W^g(v, z) + Y_W^g(L_V(0)v, z).$$

  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.
  - The $L(-1)$-commutator formula: $\frac{d}{dx} Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)]$.

Definition

- More axioms:
  - The Lower-boundedness condition: There exists $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$.
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:

$$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz}Y_W^g(v, z) + Y_W^g(L_V(0)v, z).$$

  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.
  - The $L(-1)$-commutator formula: $\frac{d}{dx}Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)]$.

Definition

- More axioms:
  - The Lower-boundedness condition: There exists $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$.
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:
  $$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz}Y_W^g(v, z) + Y_W^g(L_V(0)v, z).$$
  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.
  - The $L(-1)$-commutator formula: $\frac{d}{dx}Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)]$.

Definition

- More axioms:
  - The Lower-boundedness condition: There exists $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$.
  - The $L(0)$-grading condition: For $w \in W_{[n]}$, $(L_W(0) - n)^K w = 0$.
  - The $L(0)$-commutator formula:

$$[L_W(0), Y_W^g(v, z)] = z\frac{d}{dz} Y_W^g(v, z) + Y_W^g(L_V(0)v, z).$$

  - The g-grading condition: $(g - e^{2\pi\alpha i})^\Lambda w = 0$.
  - The g-compatibility condition and $\mathbb{Z}_2$-fermion number compatibility condition: $gY_W^g(u, x)w = Y_W^g(gu, x)gw$ and $|Y_W^g(u, x)w| = |u| + |w|$.
  - The $L(-1)$-commutator formula: $\frac{d}{dx} Y_W^g(v, x) = [L_W(-1), Y_W^g(v, x)]$.

A Jacobi identity

- Jacobi identity for twisted vertex operators (a reformulation of the Jacobi identity obtained by Frenkel-Lepowsky-Meurman, Bakalov, H.-Yang): Write the automorphism g as $g = e^{2\pi i \mathcal{L}_g}$. Then

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1) Y_W^g(v, x_2)$$
$$- (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W^g(v, x_2) Y_W^g(u, x_1)$$
$$= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W^g\left(Y_V\left(\left(\frac{x_2 + x_0}{x_1}\right)^{\mathcal{L}_g} u, x_0\right) v, x_2\right).$$

A Jacobi identity

- Jacobi identity for twisted vertex operators (a reformulation of the Jacobi identity obtained by Frenkel-Lepowsky-Meurman, Bakalov, H.-Yang): Write the automorphism g as $g = e^{2\pi i \mathcal{L}_g}$. Then

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1) Y_W^g(v, x_2)$$
$$- (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W^g(v, x_2) Y_W^g(u, x_1)$$
$$= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W^g\left(Y_V\left(\left(\frac{x_2 + x_0}{x_1}\right)^{\mathcal{L}_g} u, x_0\right) v, x_2\right).$$

A Jacobi identity

- Jacobi identity for twisted vertex operators (a reformulation of the Jacobi identity obtained by Frenkel-Lepowsky-Meurman, Bakalov, H.-Yang): Write the automorphism g as $g = e^{2\pi i \mathcal{L}_g}$. Then

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1) Y_W^g(v, x_2) - (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W^g(v, x_2) Y_W^g(u, x_1) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W^g\left(Y_V\left(\left(\frac{x_2 + x_0}{x_1}\right)^{\mathcal{L}_g} u, x_0\right) v, x_2\right).$$

A Jacobi identity

- Jacobi identity for twisted vertex operators (a reformulation of the Jacobi identity obtained by Frenkel-Lepowsky-Meurman, Bakalov, H.-Yang): Write the automorphism g as $g = e^{2\pi i \mathcal{L}_g}$. Then

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1) Y_W^g(v, x_2)$$
$$- (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W^g(v, x_2) Y_W^g(u, x_1)$$
$$= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W^g\left(Y_V\left(\left(\frac{x_2 + x_0}{x_1}\right)^{\mathcal{L}_g} u, x_0\right) v, x_2\right).$$

Outline

The problem

- The problem: Give a **general**, **direct** and **explicit** construction of (universal lower-bounded generalized) g-twisted V-modules based on algebraic assumptions on a graded space and some operators on the space.
- **General**: Not just for special classes of vertex algebras (lattices, tensor powers of a vertex algebra, etc.) or special classes of automorphisms (permutation automorphisms or automorphisms of finite order, etc.).
- **Direct**: Not from modules for the twisted Zhu's algebras given by Dong-Li-Mason in the finite order case, the twisted zero-mode algebras given by H.-Yang or the twisted Zhu's algebra given by H.-Yang in the general case.
- **Explicit**: Not just a proof of the existence.

The problem

- The problem: Give a **general**, **direct** and **explicit** construction of (universal lower-bounded generalized) g-twisted V-modules based on algebraic assumptions on a graded space and some operators on the space.
- **General**: Not just for special classes of vertex algebras (lattices, tensor powers of a vertex algebra, etc.) or special classes of automorphisms (permutation automorphisms or automorphisms of finite order, etc.).
- **Direct**: Not from modules for the twisted Zhu's algebras given by Dong-Li-Mason in the finite order case, the twisted zero-mode algebras given by H.-Yang or the twisted Zhu's algebra given by H.-Yang in the general case.
- **Explicit**: Not just a proof of the existence.

The problem

- The problem: Give a **general**, **direct** and **explicit** construction of (universal lower-bounded generalized) g-twisted V-modules based on algebraic assumptions on a graded space and some operators on the space.
- **General**: Not just for special classes of vertex algebras (lattices, tensor powers of a vertex algebra, etc.) or special classes of automorphisms (permutation automorphisms or automorphisms of finite order, etc.).
- **Direct**: Not from modules for the twisted Zhu's algebras given by Dong-Li-Mason in the finite order case, the twisted zero-mode algebras given by H.-Yang or the twisted Zhu's algebra given by H.-Yang in the general case.
- **Explicit**: Not just a proof of the existence.

The problem

- The problem: Give a **general**, **direct** and **explicit** construction of (universal lower-bounded generalized) g-twisted V-modules based on algebraic assumptions on a graded space and some operators on the space.
- **General**: Not just for special classes of vertex algebras (lattices, tensor powers of a vertex algebra, etc.) or special classes of automorphisms (permutation automorphisms or automorphisms of finite order, etc.).
- **Direct**: Not from modules for the twisted Zhu's algebras given by Dong-Li-Mason in the finite order case, the twisted zero-mode algebras given by H.-Yang or the twisted Zhu's algebra given by H.-Yang in the general case.
- **Explicit**: Not just a proof of the existence.

The problem

- The problem: Give a **general**, **direct** and **explicit** construction of (universal lower-bounded generalized) g-twisted V-modules based on algebraic assumptions on a graded space and some operators on the space.
- **General**: Not just for special classes of vertex algebras (lattices, tensor powers of a vertex algebra, etc.) or special classes of automorphisms (permutation automorphisms or automorphisms of finite order, etc.).
- **Direct**: Not from modules for the twisted Zhu's algebras given by Dong-Li-Mason in the finite order case, the twisted zero-mode algebras given by H.-Yang or the twisted Zhu's algebra given by H.-Yang in the general case.
- **Explicit**: Not just a proof of the existence.

Why is it important to have such a construction?

- The universal g-twisted V-modules are analogues of the Verma modules for finite-dimensional Lie algebras.
- We can use such modules to prove the existence of g-twisted V-modules.
- This construction will be useful even for the construction and study of (untwisted) modules for vertex algebras obtained as kernels of the screening operators.
- In fact, the exponentials of screening operators on a vertex algebra are automorphisms of the vertex algebra one starts with. The kernels of the screening operators are the same as the fixed point subalgebras of these automorphisms. One way to construct modules for the fixed point subalgebras is to construct twisted modules for the larger vertex algebra and then take the generalized eigenspaces for the automorphisms.

Why is it important to have such a construction?

- The universal g-twisted V-modules are analogues of the Verma modules for finite-dimensional Lie algebras.
- We can use such modules to prove the existence of g-twisted V-modules.
- This construction will be useful even for the construction and study of (untwisted) modules for vertex algebras obtained as kernels of the screening operators.
- In fact, the exponentials of screening operators on a vertex algebra are automorphisms of the vertex algebra one starts with. The kernels of the screening operators are the same as the fixed point subalgebras of these automorphisms. One way to construct modules for the fixed point subalgebras is to construct twisted modules for the larger vertex algebra and then take the generalized eigenspaces for the automorphisms.

Why is it important to have such a construction?

- The universal g-twisted V-modules are analogues of the Verma modules for finite-dimensional Lie algebras.
- We can use such modules to prove the existence of g-twisted V-modules.
- This construction will be useful even for the construction and study of (untwisted) modules for vertex algebras obtained as kernels of the screening operators.
- In fact, the exponentials of screening operators on a vertex algebra are automorphisms of the vertex algebra one starts with. The kernels of the screening operators are the same as the fixed point subalgebras of these automorphisms. One way to construct modules for the fixed point subalgebras is to construct twisted modules for the larger vertex algebra and then take the generalized eigenspaces for the automorphisms.

Why is it important to have such a construction?

- The universal g-twisted V-modules are analogues of the Verma modules for finite-dimensional Lie algebras.
- We can use such modules to prove the existence of g-twisted V-modules.
- This construction will be useful even for the construction and study of (untwisted) modules for vertex algebras obtained as kernels of the screening operators.
- In fact, the exponentials of screening operators on a vertex algebra are automorphisms of the vertex algebra one starts with. The kernels of the screening operators are the same as the fixed point subalgebras of these automorphisms. One way to construct modules for the fixed point subalgebras is to construct twisted modules for the larger vertex algebra and then take the generalized eigenspaces for the automorphisms.

Why is it important to have such a construction?

- The universal g-twisted V-modules are analogues of the Verma modules for finite-dimensional Lie algebras.
- We can use such modules to prove the existence of g-twisted V-modules.
- This construction will be useful even for the construction and study of (untwisted) modules for vertex algebras obtained as kernels of the screening operators.
- In fact, the exponentials of screening operators on a vertex algebra are automorphisms of the vertex algebra one starts with. The kernels of the screening operators are the same as the fixed point subalgebras of these automorphisms. One way to construct modules for the fixed point subalgebras is to construct twisted modules for the larger vertex algebra and then take the generalized eigenspaces for the automorphisms.

Orbifold theory conjectures

- **Conjecture**: For a suitable vertex operator algebra V and a finite group G of automorphisms of V, the category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category satisfying additional properties.
- A stronger **Conjecture**: Twisted intertwining operators among g-twisted V-modules for $g \in G$ satisfy associativity, commutativity and modular invariance property.
- In the case of $g = 1$, this stronger conjecture is a theorem (H., 2002, 2003).
- This theorem in the case of $g = 1$ indeed gives a modular tensor category (H., 2005).

Orbifold theory conjectures

- **Conjecture**: For a suitable vertex operator algebra V and a finite group G of automorphisms of V, the category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category satisfying additional properties.
- A stronger **Conjecture**: Twisted intertwining operators among g-twisted V-modules for $g \in G$ satisfy associativity, commutativity and modular invariance property.
- In the case of $g = 1$, this stronger conjecture is a theorem (H., 2002, 2003).
- This theorem in the case of $g = 1$ indeed gives a modular tensor category (H., 2005).

Orbifold theory conjectures

- **Conjecture**: For a suitable vertex operator algebra V and a finite group G of automorphisms of V, the category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category satisfying additional properties.
- A stronger **Conjecture**: Twisted intertwining operators among g-twisted V-modules for $g \in G$ satisfy associativity, commutativity and modular invariance property.
- In the case of $g = 1$, this stronger conjecture is a theorem (H., 2002, 2003).
- This theorem in the case of $g = 1$ indeed gives a modular tensor category (H., 2005).

Orbifold theory conjectures

- **Conjecture**: For a suitable vertex operator algebra V and a finite group G of automorphisms of V, the category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category satisfying additional properties.
- A stronger **Conjecture**: Twisted intertwining operators among g-twisted V-modules for $g \in G$ satisfy associativity, commutativity and modular invariance property.
- In the case of $g = 1$, this stronger conjecture is a theorem (H., 2002, 2003).
- This theorem in the case of $g = 1$ indeed gives a modular tensor category (H., 2005).

Orbifold theory conjectures

- **Conjecture**: For a suitable vertex operator algebra V and a finite group G of automorphisms of V, the category of g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed modular tensor category satisfying additional properties.
- A stronger **Conjecture**: Twisted intertwining operators among g-twisted V-modules for $g \in G$ satisfy associativity, commutativity and modular invariance property.
- In the case of $g = 1$, this stronger conjecture is a theorem (H., 2002, 2003).
- This theorem in the case of $g = 1$ indeed gives a modular tensor category (H., 2005).

Outline

Definition

- For $v \in V$ and $w \in W$,

$$(Y^g)_{WV}^W(w,x)v$$
$$= (-1)^{|v||w|} e^{xL_W(-1)} Y_W^g(v,y)w\Big|_{y^n=e^{\pi n i}x^n,\ \log y=\log x+\pi i}$$

- The twist vertex operator map:

$$(Y^g)_{WV}^W : W \otimes V \to W\{x\}[\log x],$$
$$w \otimes v \mapsto (Y^g)_{WV}^W(w,x)v$$

- Twist vertex operator map is the twisted intertwining operator of type $\binom{W}{WV}$ obtained by applying the skew-symmetry isomorphism Ω_+ to the twisted vertex operator map Y_W^g.

Definition

- For $v \in V$ and $w \in W$,

$$(Y^g)_{WV}^{W}(w, x)v = (-1)^{|v||w|} e^{xL_W(-1)} Y_W^g(v, y)w \Big|_{y^n = e^{\pi n i} x^n,\ \log y = \log x + \pi i}$$

- The twist vertex operator map:

$$(Y^g)_{WV}^{W} : W \otimes V \to W\{x\}[\log x], \quad w \otimes v \mapsto (Y^g)_{WV}^{W}(w, x)v$$

- Twist vertex operator map is the twisted intertwining operator of type $\binom{W}{WV}$ obtained by applying the skew-symmetry isomorphism Ω_+ to the twisted vertex operator map Y_W^g.

Definition

- For $v \in V$ and $w \in W$,

$$(Y^g)_{WV}^W(w,x)v$$
$$= (-1)^{|v||w|} e^{xL_W(-1)} Y_W^g(v,y)w\Big|_{y^n = e^{\pi n i}x^n,\ \log y = \log x + \pi i}$$

- The twist vertex operator map:

$$(Y^g)_{WV}^W : W \otimes V \to W\{x\}[\log x],$$
$$w \otimes v \mapsto (Y^g)_{WV}^W(w,x)v$$

- Twist vertex operator map is the twisted intertwining operator of type $\binom{W}{WV}$ obtained by applying the skew-symmetry isomorphism Ω_+ to the twisted vertex operator map Y_W^g.

Definition

- For $v \in V$ and $w \in W$,

$$(Y^g)^W_{WV}(w,x)v = (-1)^{|v||w|} e^{xL_W(-1)} Y^g_W(v,y)w\Big|_{y^n=e^{\pi ni}x^n,\ \log y=\log x+\pi i}$$

- The twist vertex operator map:

$$(Y^g)^W_{WV} : W \otimes V \to W\{x\}[\log x], \quad w \otimes v \mapsto (Y^g)^W_{WV}(w,x)v$$

- Twist vertex operator map is the twisted intertwining operator of type $\binom{W}{WV}$ obtained by applying the skew-symmetry isomorphism Ω_+ to the twisted vertex operator map Y^g_W.

Main properties

- The duality property: The series

$$\langle w', (Y_W^g)^{p_1}(u, z_1)((Y^g)_{WV}^W)^{p_2}(w, z_2)v\rangle,$$
$$(-1)^{|u||w|}\langle w', ((Y^g)_{WV}^W)^{p_2}(w, z_2)Y_V(u, z_1)v\rangle,$$
$$\langle w', ((Y^g)_{WV}^W)^{p_2}((Y_W^g)^{p_1}(u, z_1 - z_2)w, z_2)v\rangle$$

are absolutely convergent in the regions given by $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to suitable branches of the multivalued analytic function

$$\sum_{j,k,m,n=0}^{N} a_{jkmn} z_1^r z_2^{s_j} (z_1 - z_2)^{t_k} (\log z_2)^m (\log(z_1 - z_2))^n.$$

Main properties

- The duality property: The series

$$\langle w', (Y^g_W)^{p_1}(u, z_1)((Y^g)^W_{WV})^{p_2}(w, z_2)v\rangle,$$
$$(-1)^{|u||w|}\langle w', ((Y^g)^W_{WV})^{p_2}(w, z_2)Y_V(u, z_1)v\rangle,$$
$$\langle w', ((Y^g)^W_{WV})^{p_2}((Y^g_W)^{p_1}(u, z_1 - z_2)w, z_2)v\rangle$$

are absolutely convergent in the regions given by $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to suitable branches of the multivalued analytic function

$$\sum_{j,k,m,n=0}^{N} a_{jkmn} z_1^r z_2^{s_j}(z_1 - z_2)^{t_k}(\log z_2)^m(\log(z_1 - z_2))^n.$$

Main properties

- Jacobi identity: Recall that $g = e^{2\pi i \mathcal{L}_g}$.

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\cdot$$
$$\cdot Y_W^g\left(\left(\frac{x_1 - x_2}{x_0}\right)^{\mathcal{L}_g} u, x_1\right)(Y^g)_{WV}^W(w, x_2)v$$
$$-(-1)^{|u||w|}x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)\cdot$$
$$\cdot (Y^g)_{WV}^W(w, x_2)Y_V\left(\left(\frac{-x_2 + x_1}{x_0}\right)^{\mathcal{L}_g} u, x_1\right)v$$
$$= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)(Y^g)_{WV}^W\left(Y_W^g(u, x_0)w, x_2\right)v.$$

Main properties

- Jacobi identity: Recall that $g = e^{2\pi i \mathcal{L}_g}$.

$$
\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) \cdot \\
& \quad \cdot Y_W^g\left(\left(\frac{x_1 - x_2}{x_0}\right)^{\mathcal{L}_g} u, x_1\right)(Y^g)_{WV}^W(w, x_2)v \\
& - (-1)^{|u||w|} x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) \cdot \\
& \quad \cdot (Y^g)_{WV}^W(w, x_2) Y_V\left(\left(\frac{-x_2 + x_1}{x_0}\right)^{\mathcal{L}_g} u, x_1\right) v \\
& = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)(Y^g)_{WV}^W\left(Y_W^g(u, x_0)\, w, x_2\right) v.
\end{aligned}
$$

Main properties

- The generalized weak commutativity: For $u \in V$ and $w \in W$, there exists $M_{u,w} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{u,w}} Y_W^g((x_1 - x_2)^{\mathcal{L}_g} u, x_1)(Y^g)_{WV}^W(w, x_2)v = (-1)^{|u||w|}(x_1 - x_2)^{M_{u,w}}(Y^g)_{WV}^W(w, x_2) \cdot \cdot Y_V((-x_2 + x_1)^{\mathcal{L}_g} u, x_1)v.$$

- Let $\mathcal{N}_g$ be the nilpotent part of $\mathcal{L}_g$. Then for $u \in V^{[\alpha]}$, the generalized weak commutativity can be rewritten as:

$$(x_1 - x_2)^{\alpha + M_{u,w}}(x_1 - x_2)^{\mathcal{N}_g} Y_W^g(u, x_1)(x_1 - x_2)^{-\mathcal{N}_g}(Y^g)_{WV}^W(w, x_2)v = (-1)^{|u||w|}(-x_2 + x_1)^{\alpha + M_{u,w}}(Y^g)_{WV}^W(w, x_2) \cdot \cdot (-x_2 + x_1)^{\mathcal{N}_g} Y_V(u, x_1)(-x_2 + x_1)^{-\mathcal{N}_g} v,$$

Main properties

- The generalized weak commutativity: For $u \in V$ and $w \in W$, there exists $M_{u,w} \in \mathbb{Z}_+$ such that

$$
\begin{aligned}
(x_1 - x_2)^{M_{u,w}} Y_W^g((x_1 - x_2)^{\mathcal{L}_g} u, x_1)(Y^g)_{WV}^W(w, x_2)v \\
= (-1)^{|u||w|}(x_1 - x_2)^{M_{u,w}}(Y^g)_{WV}^W(w, x_2)\cdot \\
\cdot Y_V((-x_2 + x_1)^{\mathcal{L}_g} u, x_1)v.
\end{aligned}
$$

- Let $\mathcal{N}_g$ be the nilpotent part of $\mathcal{L}_g$. Then for $u \in V^{[\alpha]}$, the generalized weak commutativity can be rewritten as:

$$
\begin{aligned}
(x_1 - x_2)^{\alpha + M_{u,w}}(x_1 - x_2)^{\mathcal{N}_g} Y_W^g(u, x_1)(x_1 - x_2)^{-\mathcal{N}_g}(Y^g)_{WV}^W(w, x_2)v \\
= (-1)^{|u||w|}(-x_2 + x_1)^{\alpha + M_{u,w}}(Y^g)_{WV}^W(w, x_2)\cdot \\
\cdot (-x_2 + x_1)^{\mathcal{N}_g} Y_V(u, x_1)(-x_2 + x_1)^{-\mathcal{N}_g} v,
\end{aligned}
$$

Main properties

- The generalized weak commutativity: For $u \in V$ and $w \in W$, there exists $M_{u,w} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{u,w}} Y^g_W((x_1 - x_2)^{\mathcal{L}_g} u, x_1)(Y^g)^W_{WV}(w, x_2)v$$
$$= (-1)^{|u||w|}(x_1 - x_2)^{M_{u,w}}(Y^g)^W_{WV}(w, x_2)\cdot$$
$$\cdot\, Y_V((-x_2 + x_1)^{\mathcal{L}_g} u, x_1)v.$$

- Let $\mathcal{N}_g$ be the nilpotent part of $\mathcal{L}_g$. Then for $u \in V^{[\alpha]}$, the generalized weak commutativity can be rewritten as:

$$(x_1 - x_2)^{\alpha+M_{u,w}}(x_1 - x_2)^{\mathcal{N}_g} Y^g_W(u, x_1)(x_1 - x_2)^{-\mathcal{N}_g}(Y^g)^W_{WV}(w, x_2)v$$
$$= (-1)^{|u||w|}(-x_2 + x_1)^{\alpha+M_{u,w}}(Y^g)^W_{WV}(w, x_2)\cdot$$
$$\cdot\,(-x_2 + x_1)^{\mathcal{N}_g} Y_V(u, x_1)(-x_2 + x_1)^{-\mathcal{N}_g} v,$$

Outline

The algebra and the space

- Assume that $V = \coprod_{\alpha \in P_V} V^{[\alpha]}$ where $V^{[\alpha]}$ is the generalized eigenspace for g with eigenvalue $e^{2\pi i\alpha}$ and is generated by $\phi^i(x)$ for $i \in I$ of weights $\mathrm{wt}\, \phi^i$ and $\mathbb{Z}_2$-fermion numbers $|\phi^i|$. Also assume that for $i \in \mathcal{I}$, there exist $\alpha^i \in P_V$ and $\mathcal{N}_g(i) \in I$ such that $e^{2\pi i \mathcal{S}_g} \phi^i(z) e^{-2\pi i \mathcal{S}_g} = e^{2\pi \alpha^i} \phi^i(z)$ and $[\mathcal{N}_g, \phi^i(x)] = \phi^{\mathcal{N}_g(i)}(x)$, where $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Let
$$W = \coprod_{n \in \mathbb{C}, s \in \mathbb{Z}_2, [\alpha] \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}$$
be a $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space such that $W_{[n]} = \coprod_{s \in \mathbb{Z}_2, [\alpha] \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]} = 0$ when the real part of n is sufficiently negative.

The algebra and the space

- Assume that $V=\coprod_{\alpha\in P_V} V^{[\alpha]}$ where $V^{[\alpha]}$ is the generalized eigenspace for g with eigenvalue $e^{2\pi i\alpha}$ and is generated by $\phi^i(x)$ for $i\in I$ of weights $\mathrm{wt}\ \phi^i$ and $\mathbb{Z}_2$-fermion numbers $|\phi^i|$. Also assume that for $i\in\mathcal{I}$, there exist $\alpha^i\in P_V$ and $\mathcal{N}_g(i)\in I$ such that $e^{2\pi i\mathcal{S}_g}\phi^i(z)e^{-2\pi i\mathcal{S}_g}=e^{2\pi\alpha^i}\phi^i(z)$ and $[\mathcal{N}_g,\phi^i(x)]=\phi^{\mathcal{N}_g(i)}(x)$, where $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Let
$$W=\coprod_{n\in\mathbb{C},s\in\mathbb{Z}_2,[\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}$$
be a $\mathbb{C}\times\mathbb{Z}_2\times\mathbb{C}/\mathbb{Z}$-graded vector space such that $W_{[n]}=\coprod_{s\in\mathbb{Z}_2,[\alpha]\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}=0$ when the real part of n is sufficiently negative.

The algebra and the space

- Assume that $V = \coprod_{\alpha \in P_V} V^{[\alpha]}$ where $V^{[\alpha]}$ is the generalized eigenspace for g with eigenvalue $e^{2\pi i \alpha}$ and is generated by $\phi^i(x)$ for $i \in I$ of weights $\mathrm{wt}\, \phi^i$ and $\mathbb{Z}_2$-fermion numbers $|\phi^i|$. Also assume that for $i \in \mathcal{I}$, there exist $\alpha^i \in P_V$ and $\mathcal{N}_g(i) \in I$ such that $e^{2\pi i \mathcal{S}_g} \phi^i(z) e^{-2\pi i \mathcal{S}_g} = e^{2\pi \alpha^i} \phi^i(z)$ and $[\mathcal{N}_g, \phi^i(x)] = \phi^{\mathcal{N}_g(i)}(x)$, where $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Let
$$W = \coprod_{n \in \mathbb{C}, s \in \mathbb{Z}_2, [\alpha] \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]}$$
be a $\mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}$-graded vector space such that $W_{[n]} = \coprod_{s \in \mathbb{Z}_2, [\alpha] \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{s;[\alpha]} = 0$ when the real part of n is sufficiently negative.

The fields and operators

- Generating twisted fields: For $i \in I$, $\phi^i_W(x) \in x^{\alpha^i}(\mathrm{End}\ W)[[x, x^{-1}]][\log x]$ such that for $w \in W$,
$$\phi^i_W(x)w = \sum_{k=0}^{K^i} \sum_{n \in \alpha^i + N - \mathbb{N}} (\phi^i_W)_{n,k} w x^{-n-1}(\log x)^k.$$
- Generator twist fields: For $a \in A$, $\psi^a_W(x) \in \sum_{\alpha \in P_V} x^{\alpha}\mathrm{Hom}(V, W)[[x, x^{-1}]][\log x]$ such that for $v \in V^{[\alpha]}$, $\psi^a_W(x)v = \sum_{k=0}^{K^v} \sum_{n \in N_i - \mathbb{N}} (\psi^a_W)_{n,k} v x^{-n-1}(\log x)^k$.
- An action of g on W and an operator $\mathcal{L}_g$ on W such that $g = e^{2\pi i \mathcal{L}_g}$.
- Two operators $L_W(0)$ and $L_W(-1)$ on W.

The fields and operators

- Generating twisted fields: For $i \in I$, $\phi^i_W(x) \in x^{\alpha^i}(\mathrm{End}\ W)[[x, x^{-1}]][\log x]$ such that for $w \in W$,
$$\phi^i_W(x)w = \sum_{k=0}^{K^i} \sum_{n \in \alpha^i + N - \mathbb{N}} (\phi^i_W)_{n,k} w x^{-n-1} (\log x)^k.$$
- Generator twist fields: For $a \in A$, $\psi^a_W(x) \in \sum_{\alpha \in P_V} x^{\alpha}\mathrm{Hom}(V, W)[[x, x^{-1}]][\log x]$ such that for $v \in V^{[\alpha]}$, $\psi^a_W(x)v = \sum_{k=0}^{K^v} \sum_{n \in N_i - \mathbb{N}} (\psi^a_W)_{n,k} v x^{-n-1} (\log x)^k$.
- An action of g on W and an operator $\mathcal{L}_g$ on W such that $g = e^{2\pi i \mathcal{L}_g}$.
- Two operators $L_W(0)$ and $L_W(-1)$ on W.

The fields and operators

- Generating twisted fields: For $i \in I$, $\phi^i_W(x) \in x^{\alpha^i}(\mathrm{End}\ W)[[x, x^{-1}]][\log x]$ such that for $w \in W$,
$$\phi^i_W(x)w = \sum_{k=0}^{K^i} \sum_{n \in \alpha^i + N - \mathbb{N}} (\phi^i_W)_{n,k} w x^{-n-1}(\log x)^k.$$
- Generator twist fields: For $a \in A$, $\psi^a_W(x) \in \sum_{\alpha \in P_V} x^{\alpha}\mathrm{Hom}(V, W)[[x, x^{-1}]][\log x]$ such that for $v \in V^{[\alpha]}$, $\psi^a_W(x)v = \sum_{k=0}^{K^v} \sum_{n \in N_i - \mathbb{N}} (\psi^a_W)_{n,k} v x^{-n-1}(\log x)^k$.
- An action of g on W and an operator $\mathcal{L}_g$ on W such that $g = e^{2\pi i \mathcal{L}_g}$.
- Two operators $L_W(0)$ and $L_W(-1)$ on W.

The fields and operators

- Generating twisted fields: For $i \in I$, $\phi^i_W(x) \in x^{\alpha^i}(\mathrm{End}\, W)[[x, x^{-1}]][\log x]$ such that for $w \in W$,
$$\phi^i_W(x)w = \sum_{k=0}^{K^i} \sum_{n \in \alpha^i + N - \mathbb{N}} (\phi^i_W)_{n,k} w x^{-n-1}(\log x)^k.$$
- Generator twist fields: For $a \in A$, $\psi^a_W(x) \in \sum_{\alpha \in P_V} x^\alpha \mathrm{Hom}(V, W)[[x, x^{-1}]][\log x]$ such that for $v \in V^{[\alpha]}$, $\psi^a_W(x)v = \sum_{k=0}^{K^v} \sum_{n \in N_i - \mathbb{N}} (\psi^a_W)_{n,k} v x^{-n-1}(\log x)^k$.
- An action of g on W and an operator $\mathcal{L}_g$ on W such that $g = e^{2\pi i \mathcal{L}_g}$.
- Two operators $L_W(0)$ and $L_W(-1)$ on W.

The fields and operators

- Generating twisted fields: For $i \in I$, $\phi^i_W(x) \in x^{\alpha^i}(\mathrm{End}\ W)[[x, x^{-1}]][\log x]$ such that for $w \in W$,
$$\phi^i_W(x)w = \sum_{k=0}^{K^i} \sum_{n\in\alpha^i+N-\mathbb{N}} (\phi^i_W)_{n,k} w x^{-n-1}(\log x)^k.$$
- Generator twist fields: For $a \in A$, $\psi^a_W(x) \in \sum_{\alpha\in P_V} x^{\alpha}\mathrm{Hom}(V, W)[[x, x^{-1}]][\log x]$ such that for $v \in V^{[\alpha]}$, $\psi^a_W(x)v = \sum_{k=0}^{K^v} \sum_{n\in N_i-\mathbb{N}} (\psi^a_W)_{n,k} v x^{-n-1}(\log x)^k$.
- An action of g on W and an operator $\mathcal{L}_g$ on W such that $g = e^{2\pi i\mathcal{L}_g}$.
- Two operators $L_W(0)$ and $L_W(-1)$ on W.

The assumptions

- $L_W(0) = L_W(0)_S + L_W(0)_N$ where $L_W(0)_S$ and $L_W(0)_N$ are semisimple and nilpotent. For $i \in I$, $[L_W(0), \phi^i_W(x)] = x\frac{d}{dx}\phi^i_W(x) + (\mathrm{wt}\ \phi^i)\phi^i_W(x)$. For $a \in A$, there exists $(\mathrm{wt}\ \psi^a_W) \in \mathbb{C}$ and, when $L_W(0)_N\psi^a(x) \neq 0$, there exists $L_W(0)_N(a) \in A$ such that
$$L_W(0)\psi^a_W(x) - \psi^a_W(x)L_V(0)$$
$$= x\frac{d}{dx}\psi^a_W(x) + (\mathrm{wt}\ \psi^a_W)\psi^a(x) + \psi_W^{L_W(0)_N(a)}(x),$$
where $\psi_W^{L_W(0)_N(a)}(x) = 0$ when $L_W(0)_N\psi^a_W(x) = 0$.
- For $i \in I$, $[L^g_W(-1), \phi^i_W(x)] = \frac{d}{dx}\phi^i_W(x)$ and for $a \in A$, $L^g_W(-1)\psi^a_W(x) - \psi^a(x)L_V(-1) = \frac{d}{dz}\psi^a_W(x)$.
- For $a \in A$, $\psi^a_W(x)\mathbf{1} \in W[[x]]$ and its constant terms $\lim_{x\to 0}\psi^a_W(x)\mathbf{1}$ is homogeneous with respect to weights, $\mathbb{Z}_2$-fermion number and g-weights.

The assumptions

- $L_W(0) = L_W(0)_S + L_W(0)_N$ where $L_W(0)_S$ and $L_W(0)_N$ are semisimple and nilpotent. For $i \in I$, $[L_W(0), \phi^i_W(x)] = x\frac{d}{dx}\phi^i_W(x) + (\mathrm{wt}\ \phi^i)\phi^i_W(x)$. For $a \in A$, there exists $(\mathrm{wt}\ \psi^a_W) \in \mathbb{C}$ and, when $L_W(0)_N\psi^a(x) \neq 0$, there exists $L_W(0)_N(a) \in A$ such that
$$\begin{aligned}L_W(0)\psi^a_W(x) - \psi^a_W(x)L_V(0)\\ = x\frac{d}{dx}\psi^a_W(x) + (\mathrm{wt}\ \psi^a_W)\psi^a(x) + \psi_W^{L_W(0)_N(a)}(x),\end{aligned}$$
where $\psi_W^{L_W(0)_N(a)}(x) = 0$ when $L_W(0)_N\psi^a_W(x) = 0$.
- For $i \in I$, $[L^g_W(-1), \phi^i_W(x)] = \frac{d}{dx}\phi^i_W(x)$ and for $a \in A$, $L^g_W(-1)\psi^a_W(x) - \psi^a(x)L_V(-1) = \frac{d}{dz}\psi^a_W(x)$.
- For $a \in A$, $\psi^a_W(x)\mathbf{1} \in W[[x]]$ and its constant terms $\lim_{x\to 0}\psi^a_W(x)\mathbf{1}$ is homogeneous with respect to weights, $\mathbb{Z}_2$-fermion number and g-weights.

The assumptions

- $L_W(0) = L_W(0)_S + L_W(0)_N$ where $L_W(0)_S$ and $L_W(0)_N$ are semisimple and nilpotent. For $i \in I$, $[L_W(0), \phi^i_W(x)] = x\frac{d}{dx}\phi^i_W(x) + (\mathrm{wt}\ \phi^i)\phi^i_W(x)$. For $a \in A$, there exists $(\mathrm{wt}\ \psi^a_W) \in \mathbb{C}$ and, when $L_W(0)_N\psi^a(x) \neq 0$, there exists $L_W(0)_N(a) \in A$ such that
$$L_W(0)\psi^a_W(x) - \psi^a_W(x)L_V(0) = x\frac{d}{dx}\psi^a_W(x) + (\mathrm{wt}\ \psi^a_W)\psi^a(x) + \psi^{L_W(0)_N(a)}_W(x),$$
where $\psi^{L_W(0)_N(a)}_W(x) = 0$ when $L_W(0)_N\psi^a_W(x) = 0$.
- For $i \in I$, $[L^g_W(-1), \phi^i_W(x)] = \frac{d}{dx}\phi^i_W(x)$ and for $a \in A$, $L^g_W(-1)\psi^a_W(x) - \psi^a(x)L_V(-1) = \frac{d}{dz}\psi^a_W(x)$.
- For $a \in A$, $\psi^a_W(x)\mathbf{1} \in W[[x]]$ and its constant terms $\lim_{x\to 0}\psi^a_W(x)\mathbf{1}$ is homogeneous with respect to weights, $\mathbb{Z}_2$-fermion number and g-weights.

The assumptions

- $L_W(0) = L_W(0)_S + L_W(0)_N$ where $L_W(0)_S$ and $L_W(0)_N$ are semisimple and nilpotent. For $i \in I$, $[L_W(0), \phi^i_W(x)] = x\frac{d}{dx}\phi^i_W(x) + (\mathrm{wt}\ \phi^i)\phi^i_W(x)$. For $a \in A$, there exists $(\mathrm{wt}\ \psi^a_W) \in \mathbb{C}$ and, when $L_W(0)_N\psi^a(x) \neq 0$, there exists $L_W(0)_N(a) \in A$ such that
$$L_W(0)\psi^a_W(x) - \psi^a_W(x)L_V(0) = x\frac{d}{dx}\psi^a_W(x) + (\mathrm{wt}\ \psi^a_W)\psi^a(x) + \psi_W^{L_W(0)_N(a)}(x),$$
where $\psi_W^{L_W(0)_N(a)}(x) = 0$ when $L_W(0)_N\psi^a_W(x) = 0$.
- For $i \in I$, $[L^g_W(-1), \phi^i_W(x)] = \frac{d}{dx}\phi^i_W(x)$ and for $a \in A$, $L^g_W(-1)\psi^a_W(x) - \psi^a(x)L_V(-1) = \frac{d}{dz}\psi^a_W(x)$.
- For $a \in A$, $\psi^a_W(x)\mathbf{1} \in W[[x]]$ and its constant terms $\lim_{x\to 0}\psi^a_W(x)\mathbf{1}$ is homogeneous with respect to weights, $\mathbb{Z}_2$-fermion number and g-weights.

The assumptions

- The vector space W is spanned by elements of the form $(\phi_W^{i_1})_{n_1,l_1}\cdots(\phi_W^{i_k})_{n_k,l_k}(\psi_W^a)_{n,l}v$ for $i_1,\dots,i_k\in I$, $a\in\mathcal{A}$ and $n_1\in\alpha^{i_1}+\mathbb{Z},\dots,n_k\in\alpha^{i_k}+\mathbb{Z}$, $n\in\mathbb{C}$, $l_1,\dots,l_k,l\in\mathbb{N}$, $v\in V$.
- Let $\mathcal{S}_g$ and $\mathcal{N}_g$ be the semisimple and nilpotent parts, respectively, of $\mathcal{L}_g$. (i) For $i\in I$, $g\phi_W^{i;p+1}(z)g^{-1}=\phi_W^{i;p}(z)$. (ii) For $i\in I$, $\phi_W^i(x)=x^{-\mathcal{N}_g}(\phi_W^i)_0(x)x^{\mathcal{N}_g}$ and for $a\in A$, $\psi_W^a(x)=(\psi_W^a)_0(x)e^{-\pi i\mathcal{N}_g}x^{-\mathcal{N}_g}$ where $(\phi_W^i)_0(x)$ and $(\psi_W^a)_0(x)$ are the constant terms of $\phi_W^i(x)$ and $\psi_W^a(x)$, respectively, viewed as a power series of $\log x$. (iii) For $i\in I$, $e^{2\pi i\mathcal{S}_g}\phi_W^i(z)e^{-2\pi i\mathcal{S}_g}=e^{2\pi\alpha^i}\phi_W^i(z)$ and $[\mathcal{N}_g,\phi_W^i(z)]=\phi_W^{\mathcal{N}_g(i)}(z)$. (iv) For $a\in A$, there exists $\alpha^a\in[0,1)$ such that $(\psi_W^a)_{n,0}\mathbf{1}$ for $n\in-\mathbb{N}-1$ are generalized eigenvectors for g with the eigenvalue $e^{2\pi i\alpha^a}$.

The assumptions

- The vector space W is spanned by elements of the form $(\phi_W^{i_1})_{n_1,l_1}\cdots(\phi_W^{i_k})_{n_k,l_k}(\psi_W^a)_{n,l}v$ for $i_1,\dots,i_k\in I$, $a\in\mathcal{A}$ and $n_1\in\alpha^{i_1}+\mathbb{Z},\dots,n_k\in\alpha^{i_k}+\mathbb{Z}$, $n\in\mathbb{C}$, $l_1,\dots,l_k,l\in\mathbb{N}$, $v\in V$.
- Let $\mathcal{S}_g$ and $\mathcal{N}_g$ be the semisimple and nilpotent parts, respectively, of $\mathcal{L}_g$. (i) For $i\in I$, $g\phi_W^{i;p+1}(z)g^{-1}=\phi_W^{i;p}(z)$. (ii) For $i\in I$, $\phi_W^i(x)=x^{-\mathcal{N}_g}(\phi_W^i)_0(x)x^{\mathcal{N}_g}$ and for $a\in A$, $\psi_W^a(x)=(\psi_W^a)_0(x)e^{-\pi i\mathcal{N}_g}x^{-\mathcal{N}_g}$ where $(\phi_W^i)_0(x)$ and $(\psi_W^a)_0(x)$ are the constant terms of $\phi_W^i(x)$ and $\psi_W^a(x)$, respectively, viewed as a power series of $\log x$. (iii) For $i\in I$, $e^{2\pi i\mathcal{S}_g}\phi_W^i(z)e^{-2\pi i\mathcal{S}_g}=e^{2\pi\alpha^i}\phi_W^i(z)$ and $[\mathcal{N}_g,\phi_W^i(z)]=\phi_W^{\mathcal{N}_g(i)}(z)$. (iv) For $a\in A$, there exists $\alpha^a\in[0,1)$ such that $(\psi_W^a)_{n,0}\mathbf{1}$ for $n\in-\mathbb{N}-1$ are generalized eigenvectors for g with the eigenvalue $e^{2\pi i\alpha^a}$.

The assumptions

- The vector space W is spanned by elements of the form $(\phi_W^{i_1})_{n_1,l_1}\cdots(\phi_W^{i_k})_{n_k,l_k}(\psi_W^a)_{n,l}v$ for $i_1,\dots,i_k\in I$, $a\in\mathcal{A}$ and $n_1\in\alpha^{i_1}+\mathbb{Z},\dots,n_k\in\alpha^{i_k}+\mathbb{Z}$, $n\in\mathbb{C}$, $l_1,\dots,l_k,l\in\mathbb{N}$, $v\in V$.
- Let $\mathcal{S}_g$ and $\mathcal{N}_g$ be the semisimple and nilpotent parts, respectively, of $\mathcal{L}_g$. (i) For $i\in I$, $g\phi_W^{i;p+1}(z)g^{-1}=\phi_W^{i;p}(z)$. (ii) For $i\in I$, $\phi_W^i(x)=x^{-\mathcal{N}_g}(\phi_W^i)_0(x)x^{\mathcal{N}_g}$ and for $a\in A$, $\psi_W^a(x)=(\psi_W^a)_0(x)e^{-\pi i\mathcal{N}_g}x^{-\mathcal{N}_g}$ where $(\phi_W^i)_0(x)$ and $(\psi_W^a)_0(x)$ are the constant terms of $\phi_W^i(x)$ and $\psi_W^a(x)$, respectively, viewed as a power series of $\log x$. (iii) For $i\in I$, $e^{2\pi i\mathcal{S}_g}\phi_W^i(z)e^{-2\pi i\mathcal{S}_g}=e^{2\pi\alpha^i}\phi_W^i(z)$ and $[\mathcal{N}_g,\phi_W^i(z)]=\phi_W^{\mathcal{N}_g(i)}(z)$. (iv) For $a\in A$, there exists $\alpha^a\in[0,1)$ such that $(\psi_W^a)_{n,0}\mathbf{1}$ for $n\in-\mathbb{N}-1$ are generalized eigenvectors for g with the eigenvalue $e^{2\pi i\alpha^a}$.

The main assumptions

- For $i, j \in I$, there exists $M_{ij} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{ij}} \phi^i_W(x_1) \phi^j_W(x_2) = (x_1 - x_2)^{M_{ij}} (-1)^{|\phi^i||\phi^j|} \phi^j_W(x_2) \phi^i_W(x_1).$$

- For $i \in I$ and $a \in A$, there exists $M_{ia} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{\alpha_i + M_{ia}} (x_1 - x_2)^{\mathcal{N}_g} \phi^i_W(x_1) (x_1 - x_2)^{-\mathcal{N}_g} \psi^a_W(x_2) = (-x_2 + x_1)^{\alpha_i + M_{ia}} (-1)^{|\phi^i||\psi^a|} \cdot \psi^a_W(x_2) (-x_2 + x_1)^{\mathcal{N}_g} \phi^i(x_1) (-x_2 + x_1)^{-\mathcal{N}_g}.$$

The main assumptions

- For $i, j \in I$, there exists $M_{ij} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{ij}} \phi^i_W(x_1)\phi^j_W(x_2) = (x_1 - x_2)^{M_{ij}}(-1)^{|\phi^i||\phi^j|}\phi^j_W(x_2)\phi^i_W(x_1).$$

- For $i \in I$ and $a \in A$, there exists $M_{ia} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{\alpha_i + M_{ia}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_W(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_W(x_2) = (-x_2 + x_1)^{\alpha_i + M_{ia}}(-1)^{|\phi^i||\psi^a|} \cdot \psi^a_W(x_2)(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}.$$

The main assumptions

- For $i, j \in I$, there exists $M_{ij} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{ij}} \phi^i_W(x_1)\phi^j_W(x_2) \\ = (x_1 - x_2)^{M_{ij}}(-1)^{|\phi^i||\phi^j|}\phi^j_W(x_2)\phi^i_W(x_1).$$

- For $i \in I$ and $a \in A$, there exists $M_{ia} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{\alpha_i + M_{ia}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_W(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_W(x_2) \\ = (-x_2 + x_1)^{\alpha_i + M_{ia}}(-1)^{|\phi^i||\psi^a|}. \\ \cdot \psi^a_W(x_2)(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}.$$

The twisted vertex operator map

Theorem

For $i \in I$ and $p \in \mathbb{Z}$, let $\phi_W^{i;p}(z)$ be the p-th branch of $\phi_W^i(z)$. The series

$$\langle w', \phi_W^{i_1,p}(z_1) \cdots \phi_W^{i_k,p}(z_k) w \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to an analytic function of the form

$$\sum_{n_1,\ldots,n_k=0}^{N} f_{n_1 \cdots n_k}(z_1, \ldots, z_k) e^{-\alpha_{i_1} l_p(z_1)} \cdots e^{-\alpha_{i_k} l_p(z_k)} \cdot (l_p(z_1))^{n_1} \cdots (l_p(z_k))^{n_k},$$

where $f_{n_1 \cdots n_k}(z_1, \ldots, z_k)$ are rational functions with the usual poles, and denoted by $F^p(\langle w', \phi_W^{i_1}(z_1) \cdots \phi_W^{i_k}(z_k) w \rangle)$.

The twisted vertex operator map

Theorem

For $i \in I$ and $p \in \mathbb{Z}$, let $\phi_W^{i;p}(z)$ be the p-th branch of $\phi_W^i(z)$. The series

$$\langle w', \phi_W^{i_1,p}(z_1) \cdots \phi_W^{i_k,p}(z_k) w\rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to an analytic function of the form

$$\sum_{n_1,\ldots,n_k=0}^{N} f_{n_1\cdots n_k}(z_1,\ldots,z_k) e^{-\alpha_{i_1} l_p(z_1)} \cdots e^{-\alpha_{i_k} l_p(z_k)} \cdot (l_p(z_1))^{n_1} \cdots (l_p(z_k))^{n_k},$$

where $f_{n_1\cdots n_k}(z_1,\ldots,z_k)$ are rational functions with the usual poles, and denoted by $F^p(\langle w', \phi_W^{i_1}(z_1) \cdots \phi_W^{i_k}(z_k) w\rangle)$.

The twisted vertex operator map

- For $w' \in W'$, $w \in W$, $i_1, \dots, i_k \in I$, $m_1, \dots, m_k \in \mathbb{Z}$,

$$\begin{aligned}\langle w', (Y_W^g)^p(\phi_{m_1}^{i_1} \cdots \phi_{m_k}^{i_k}\mathbf{1}, z)w\rangle \\ = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0}\xi_1^{m_1} \cdots \xi_k^{m_k} \cdot \\ \cdot F^p(\langle w', \phi_W^{i_1}(\xi_1+z) \cdots \phi_W^{i_k}(\xi_k+z)w\rangle).\end{aligned}$$

- If $\sum_{\mu=1}^{M} \lambda_p \phi_{m_1^\mu}^{i_1^\mu} \cdots \phi_{m_k^\mu}^{i_k^\mu}\mathbf{1} = 0$, then for $w' \in W'$, $w \in W$

$$\begin{aligned}\sum_{\mu=1}^{M} \lambda_\mu \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0}\xi_1^{m_1^\mu} \cdots \xi_k^{m_k^\mu} \cdot \\ \cdot F^p(\langle w', \phi_W^{i_1^\mu}(\xi_1+z) \cdots \phi_W^{i_k^\mu}(\xi_k+z)w\rangle) = 0.\end{aligned}$$

In particular, Y_W^g is well defined.

The twisted vertex operator map

- For $w' \in W'$, $w \in W$, $i_1, \ldots, i_k \in I$, $m_1, \ldots, m_k \in \mathbb{Z}$,

$$\begin{aligned}\langle w', (Y_W^g)^p(\phi_{m_1}^{i_1} \cdots \phi_{m_k}^{i_k} \mathbf{1}, z)w\rangle \\ = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1} \cdots \xi_k^{m_k} \cdot \\ \cdot F^p(\langle w', \phi_W^{i_1}(\xi_1+z) \cdots \phi_W^{i_k}(\xi_k+z)w\rangle).\end{aligned}$$

- If $\sum_{\mu=1}^{M} \lambda_p \phi_{m_1^\mu}^{i_1^\mu} \cdots \phi_{m_k^\mu}^{i_k^\mu} \mathbf{1} = 0$, then for $w' \in W'$, $w \in W$

$$\sum_{\mu=1}^{M} \lambda_\mu \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^\mu} \cdots \xi_k^{m_k^\mu} \cdot \\ \cdot F^p(\langle w', \phi_W^{i_1^\mu}(\xi_1+z) \cdots \phi_W^{i_k^\mu}(\xi_k+z)w\rangle) = 0.$$

In particular, Y_W^g is well defined.

The twisted vertex operator map

- For $w' \in W'$, $w \in W$, $i_1, \dots, i_k \in I$, $m_1, \dots, m_k \in \mathbb{Z}$,

$$\begin{aligned}\langle w', (Y_W^g)^p(\phi_{m_1}^{i_1} \cdots \phi_{m_k}^{i_k}\mathbf{1}, z)w\rangle &= \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1} \cdots \xi_k^{m_k} \cdot \\ &\quad \cdot F^p(\langle w', \phi_W^{i_1}(\xi_1 + z) \cdots \phi_W^{i_k}(\xi_k + z)w\rangle).\end{aligned}$$

- If $\sum_{\mu=1}^{M} \lambda_p \phi_{m_1^\mu}^{i_1^\mu} \cdots \phi_{m_k^\mu}^{i_k^\mu}\mathbf{1} = 0$, then for $w' \in W'$, $w \in W$

$$\begin{aligned}&\sum_{\mu=1}^{M} \lambda_\mu \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^\mu} \cdots \xi_k^{m_k^\mu} \cdot \\ &\quad \cdot F^p(\langle w', \phi_W^{i_1^\mu}(\xi_1 + z) \cdots \phi_W^{i_k^\mu}(\xi_k + z)w\rangle) = 0.\end{aligned}$$

In particular, Y_W^g is well defined.

The main theorem

Theorem

The pair (W, Y_W^g) is a grading-restricted generalized g-twisted V-module generated by $(\psi_W^a)_{n,k}v$ for $a \in A$, $n \in \alpha + \mathbb{Z}$, $v \in V^{[\alpha]}$ and $\alpha \in P_V$. Moreover, this is the unique generalized g-twisted V-module structure on W generated by $(\psi_W^a)_{n,k}v$ for $a \in A$, $n \in \alpha + \mathbb{Z}$, $v \in V^{[\alpha]}$ and $\alpha \in P_V$ such that $Y_W(\phi_{-1}^i \mathbf{1}, z) = \phi_W^i(z)$ for $i \in I$.

The main theorem

Theorem

The pair (W, Y_W^g) is a grading-restricted generalized g-twisted V-module generated by $(\psi_W^a)_{n,k}v$ for $a \in A$, $n \in \alpha + \mathbb{Z}$, $v \in V^{[\alpha]}$ and $\alpha \in P_V$. Moreover, this is the unique generalized g-twisted V-module structure on W generated by $(\psi_W^a)_{n,k}v$ for $a \in A$, $n \in \alpha + \mathbb{Z}$, $v \in V^{[\alpha]}$ and $\alpha \in P_V$ such that $Y_W(\phi_{-1}^i \mathbf{1}, z) = \phi_W^i(z)$ for $i \in I$.

Outline

Twisted affinization

- Let $\hat{V}_\phi^{[g]} = \coprod_{i\in I, k\in\mathbb{N}}(\mathbb{C}\mathcal{N}_g^k\phi^i_{-1}\mathbf{1}\otimes t^{\alpha^i}\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}L_0\oplus\mathbb{C}L_{-1}$, where L_0 and L_{-1} are two abstract elements.
- Let $T(\hat{V}_\phi^{[g]})$ be the tensor algebra of $\hat{V}_\phi^{[g]}$.
- Let $\phi^i_{\hat{V}_\phi^{[g]}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}}((x^{-\mathcal{N}_g}\phi^i_{-1}\mathbf{1})\otimes t^n)x^{-n-1}$ for $i\in I$.
- $\phi^i_{\hat{V}_\phi^{[g]}}(x)$ for $i\in I$ can be viewed as formal series of operators on $T(\hat{V}_\phi^{[g]})$. Also L_0 and L_{-1} can be viewed as operators, on $T(\hat{V}_\phi^{[g]})$.
- We denote L_0 and L_{-1} viewed as operators on $T(\hat{V}_\phi^{[g]})$ by $L_{\hat{V}_\phi^{[g]}}(0)$ and $L_{\hat{V}_\phi^{[g]}}(-1)$.

Twisted affinization

- Let $\hat{V}_{\phi}^{[g]} = \coprod_{i\in I, k\in\mathbb{N}}(\mathbb{C}\mathcal{N}_g^k\phi^i_{-1}\mathbf{1}\otimes t^{\alpha^i}\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}L_0\oplus\mathbb{C}L_{-1}$, where L_0 and L_{-1} are two abstract elements.
- Let $T(\hat{V}_{\phi}^{[g]})$ be the tensor algebra of $\hat{V}_{\phi}^{[g]}$.
- Let $\phi^i_{\hat{V}_{\phi}^{[g]}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}}((x^{-\mathcal{N}_g}\phi^i_{-1}\mathbf{1})\otimes t^n)x^{-n-1}$ for $i\in I$.
- $\phi^i_{\hat{V}_{\phi}^{[g]}}(x)$ for $i\in I$ can be viewed as formal series of operators on $T(\hat{V}_{\phi}^{[g]})$. Also L_0 and L_{-1} can be viewed as operators, on $T(\hat{V}_{\phi}^{[g]})$.
- We denote L_0 and L_{-1} viewed as operators on $T(\hat{V}_{\phi}^{[g]})$ by $L_{\hat{V}_{\phi}^{[g]}}(0)$ and $L_{\hat{V}_{\phi}^{[g]}}(-1)$.

Twisted affinization

- Let $\hat{V}_{\phi}^{[g]} = \coprod_{i\in I, k\in\mathbb{N}}(\mathbb{C}\mathcal{N}_g^k\phi_{-1}^i\mathbf{1}\otimes t^{\alpha^i}\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}L_0\oplus\mathbb{C}L_{-1}$, where L_0 and L_{-1} are two abstract elements.
- Let $T(\hat{V}_{\phi}^{[g]})$ be the tensor algebra of $\hat{V}_{\phi}^{[g]}$.
- Let $\phi^i_{\hat{V}_{\phi}^{[g]}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}}((x^{-\mathcal{N}_g}\phi_{-1}^i\mathbf{1})\otimes t^n)x^{-n-1}$ for $i\in I$.
- $\phi^i_{\hat{V}_{\phi}^{[g]}}(x)$ for $i\in I$ can be viewed as formal series of operators on $T(\hat{V}_{\phi}^{[g]})$. Also L_0 and L_{-1} can be viewed as operators, on $T(\hat{V}_{\phi}^{[g]})$.
- We denote L_0 and L_{-1} viewed as operators on $T(\hat{V}_{\phi}^{[g]})$ by $L_{\hat{V}_{\phi}^{[g]}}(0)$ and $L_{\hat{V}_{\phi}^{[g]}}(-1)$.

Twisted affinization

- Let $\hat{V}_{\phi}^{[g]} = \coprod_{i\in I, k\in\mathbb{N}}(\mathbb{C}\mathcal{N}_g^k\phi^i_{-1}\mathbf{1}\otimes t^{\alpha^i}\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}L_0\oplus\mathbb{C}L_{-1}$, where L_0 and L_{-1} are two abstract elements.
- Let $T(\hat{V}_{\phi}^{[g]})$ be the tensor algebra of $\hat{V}_{\phi}^{[g]}$.
- Let $\phi^i_{\hat{V}_{\phi}^{[g]}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}}((x^{-\mathcal{N}_g}\phi^i_{-1}\mathbf{1})\otimes t^n)x^{-n-1}$ for $i\in I$.
- $\phi^i_{\hat{V}_{\phi}^{[g]}}(x)$ for $i \in I$ can be viewed as formal series of operators on $T(\hat{V}_{\phi}^{[g]})$. Also L_0 and L_{-1} can be viewed as operators, on $T(\hat{V}_{\phi}^{[g]})$.
- We denote L_0 and L_{-1} viewed as operators on $T(\hat{V}_{\phi}^{[g]})$ by $L_{\hat{V}_{\phi}^{[g]}}(0)$ and $L_{\hat{V}_{\phi}^{[g]}}(-1)$.

Twisted affinization

- Let $\hat{V}_{\phi}^{[g]} = \coprod_{i\in I, k\in\mathbb{N}}(\mathbb{C}\mathcal{N}_g^k\phi^i_{-1}\mathbf{1}\otimes t^{\alpha^i}\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}L_0\oplus\mathbb{C}L_{-1}$, where L_0 and L_{-1} are two abstract elements.
- Let $T(\hat{V}_{\phi}^{[g]})$ be the tensor algebra of $\hat{V}_{\phi}^{[g]}$.
- Let $\phi^i_{\hat{V}_{\phi}^{[g]}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}}((x^{-\mathcal{N}_g}\phi^i_{-1}\mathbf{1})\otimes t^n)x^{-n-1}$ for $i \in I$.
- $\phi^i_{\hat{V}_{\phi}^{[g]}}(x)$ for $i \in I$ can be viewed as formal series of operators on $T(\hat{V}_{\phi}^{[g]})$. Also L_0 and L_{-1} can be viewed as operators, on $T(\hat{V}_{\phi}^{[g]})$.
- We denote L_0 and L_{-1} viewed as operators on $T(\hat{V}_{\phi}^{[g]})$ by $L_{\hat{V}_{\phi}^{[g]}}(0)$ and $L_{\hat{V}_{\phi}^{[g]}}(-1)$.

Twisted affinization

- Let $\hat{V}_{\phi}^{[g]} = \coprod_{i\in I, k\in\mathbb{N}}(\mathbb{C}\mathcal{N}_g^k\phi^i_{-1}\mathbf{1}\otimes t^{\alpha^i}\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}L_0\oplus\mathbb{C}L_{-1}$, where L_0 and L_{-1} are two abstract elements.
- Let $T(\hat{V}_{\phi}^{[g]})$ be the tensor algebra of $\hat{V}_{\phi}^{[g]}$.
- Let $\phi^i_{\hat{V}_{\phi}^{[g]}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}}((x^{-\mathcal{N}_g}\phi^i_{-1}\mathbf{1})\otimes t^n)x^{-n-1}$ for $i\in I$.
- $\phi^i_{\hat{V}_{\phi}^{[g]}}(x)$ for $i\in I$ can be viewed as formal series of operators on $T(\hat{V}_{\phi}^{[g]})$. Also L_0 and L_{-1} can be viewed as operators, on $T(\hat{V}_{\phi}^{[g]})$.
- We denote L_0 and L_{-1} viewed as operators on $T(\hat{V}_{\phi}^{[g]})$ by $L_{\hat{V}_{\phi}^{[g]}}(0)$ and $L_{\hat{V}_{\phi}^{[g]}}(-1)$.

Weak commutativity for $\phi^i_{\hat{V}^{[g]}_\phi}(x)$

- For $i, j \in I$, let $M_{i,j}$ be the smallest positive integer such that $x_0^{M_{i,j}} Y_V(\phi^i_{-1}\mathbf{1}, x)\phi^j_{-1}\mathbf{1}$ is a power series in x.
- Let $J(\hat{V}^{[g]}_\phi)$ be the ideal of $T(\hat{V}^{[g]}_\phi)$ generated by the coefficients of the formal series

$$(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2) - (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),$$

$$L_0\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_0 - x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) - (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),$$

$$L_{-1}\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{-1} - \frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x)$$

for $i, j \in I$,

Weak commutativity for $\phi^i_{\hat{V}^{[g]}_\phi}(x)$

- For $i, j \in I$, let $M_{i,j}$ be the smallest positive integer such that $x_0^{M_{i,j}} Y_V(\phi^i_{-1}\mathbf{1}, x)\phi^j_{-1}\mathbf{1}$ is a power series in x.
- Let $J(\hat{V}^{[g]}_\phi)$ be the ideal of $T(\hat{V}^{[g]}_\phi)$ generated by the coefficients of the formal series

$$(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2) - (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),$$

$$L_0\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_0 - x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) - (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),$$

$$L_{-1}\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{-1} - \frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x)$$

for $i, j \in I$,

Weak commutativity for $\phi^i_{\hat{V}^{[g]}_\phi}(x)$

- For $i, j \in I$, let $M_{i,j}$ be the smallest positive integer such that $x_0^{M_{i,j}} Y_V(\phi^i_{-1}\mathbf{1}, x)\phi^j_{-1}\mathbf{1}$ is a power series in x.
- Let $J(\hat{V}^{[g]}_\phi)$ be the ideal of $T(\hat{V}^{[g]}_\phi)$ generated by the coefficients of the formal series

$$(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2) - (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),$$

$$L_0\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_0 - x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) - (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),$$

$$L_{-1}\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{-1} - \frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x)$$

for $i, j \in I$,

Weak commutativity for $\phi^i_{\hat{V}^{[g]}_\phi}(x)$

- Let $U(\hat{V}^{[g]}_\phi) = T(\hat{V}^{[g]}_\phi)/J(\hat{V}^{[g]}_\phi)$.
- In $U(\hat{V}^{[g]}_\phi)$, we have the following weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula:

$$(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2) = (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),$$

$$L_{\hat{V}^{[g]}_\phi}(0)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(0) = x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) + (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),$$

$$L_{\hat{V}^{[g]}_\phi}(-1)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(-1) = \frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x).$$

Weak commutativity for $\phi^i_{\hat{V}^{[g]}_\phi}(x)$

- Let $U(\hat{V}^{[g]}_\phi) = T(\hat{V}^{[g]}_\phi)/J(\hat{V}^{[g]}_\phi)$.
- In $U(\hat{V}^{[g]}_\phi)$, we have the following weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula:

$$(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2) = (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),$$

$$L_{\hat{V}^{[g]}_\phi}(0)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(0) = x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) + (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),$$

$$L_{\hat{V}^{[g]}_\phi}(-1)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(-1) = \frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x).$$

Weak commutativity for $\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)$

- Let $U(\hat{V}^{[g]}_{\phi}) = T(\hat{V}^{[g]}_{\phi})/J(\hat{V}^{[g]}_{\phi})$.
- In $U(\hat{V}^{[g]}_{\phi})$, we have the following weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula:

$$\begin{aligned}(x_1 - x_2)^{M_{ij}} & \phi^{i}_{\hat{V}^{[g]}_{\phi}}(x_1)\phi^{j}_{\hat{V}^{[g]}_{\phi}}(x_2) \\ &= (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}}\phi^{j}_{\hat{V}^{[g]}_{\phi}}(x_2)\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x_1),\end{aligned}$$

$$\begin{aligned}L_{\hat{V}^{[g]}_{\phi}}(0)\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x) & - \phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)L_{\hat{V}^{[g]}_{\phi}}(0) \\ &= x\frac{d}{dx}\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x) + (\mathrm{wt}\ \phi^i)\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x),\end{aligned}$$

$$L_{\hat{V}^{[g]}_{\phi}}(-1)\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x) - \phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)L_{\hat{V}^{[g]}_{\phi}}(-1) = \frac{d}{dx}\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x).$$

Weak commutativity for $\phi^i_{\hat{V}^{[g]}_\phi}(x)$

- Let $U(\hat{V}^{[g]}_\phi) = T(\hat{V}^{[g]}_\phi)/J(\hat{V}^{[g]}_\phi)$.
- In $U(\hat{V}^{[g]}_\phi)$, we have the following weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula:

$$(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2) = (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),$$

$$L_{\hat{V}^{[g]}_\phi}(0)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(0) = x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) + (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),$$

$$L_{\hat{V}^{[g]}_\phi}(-1)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(-1) = \frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x).$$

Weak commutativity for $\phi^i_{\hat{V}^{[g]}_\phi}(x)$

- Let $U(\hat{V}^{[g]}_\phi) = T(\hat{V}^{[g]}_\phi)/J(\hat{V}^{[g]}_\phi)$.
- In $U(\hat{V}^{[g]}_\phi)$, we have the following weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula:

$$
\begin{aligned}
&(x_1 - x_2)^{M_{ij}} \phi^i_{\hat{V}^{[g]}_\phi}(x_1)\phi^j_{\hat{V}^{[g]}_\phi}(x_2)\\
&\qquad\qquad = (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}} \phi^j_{\hat{V}^{[g]}_\phi}(x_2)\phi^i_{\hat{V}^{[g]}_\phi}(x_1),\\
&L_{\hat{V}^{[g]}_\phi}(0)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(0)\\
&\qquad\qquad = x\frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x) + (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}^{[g]}_\phi}(x),
\end{aligned}
$$

$$
L_{\hat{V}^{[g]}_\phi}(-1)\phi^i_{\hat{V}^{[g]}_\phi}(x) - \phi^i_{\hat{V}^{[g]}_\phi}(x)L_{\hat{V}^{[g]}_\phi}(-1) = \frac{d}{dx}\phi^i_{\hat{V}^{[g]}_\phi}(x).
$$

Weak commutativity for $\phi^i_{\hat{V}_\phi^{[g]}}(x)$

- Let $U(\hat{V}_\phi^{[g]}) = T(\hat{V}_\phi^{[g]})/J(\hat{V}_\phi^{[g]})$.
- In $U(\hat{V}_\phi^{[g]})$, we have the following weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula:

$$\begin{aligned}(x_1 - x_2)^{M_{ij}} & \phi^i_{\hat{V}_\phi^{[g]}}(x_1)\phi^j_{\hat{V}_\phi^{[g]}}(x_2) \\ &= (-1)^{|\phi^i||\phi^j|}(x_1 - x_2)^{M_{ij}}\phi^j_{\hat{V}_\phi^{[g]}}(x_2)\phi^i_{\hat{V}_\phi^{[g]}}(x_1),\end{aligned}$$

$$\begin{aligned}L_{\hat{V}_\phi^{[g]}}(0)\phi^i_{\hat{V}_\phi^{[g]}}(x) & - \phi^i_{\hat{V}_\phi^{[g]}}(x)L_{\hat{V}_\phi^{[g]}}(0) \\ &= x\frac{d}{dx}\phi^i_{\hat{V}_\phi^{[g]}}(x) + (\mathrm{wt}\ \phi^i)\phi^i_{\hat{V}_\phi^{[g]}}(x),\end{aligned}$$

$$L_{\hat{V}_\phi^{[g]}}(-1)\phi^i_{\hat{V}_\phi^{[g]}}(x) - \phi^i_{\hat{V}_\phi^{[g]}}(x)L_{\hat{V}_\phi^{[g]}}(-1) = \frac{d}{dx}\phi^i_{\hat{V}_\phi^{[g]}}(x).$$

A space of generators

- Let M be a $\mathbb{Z}_2$-graded vector space (graded by $\mathbb{Z}_2$-fermion numbers).
- Assume that g acts on M and there is an operator $L_M(0)$ on M.
- Assume that there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$ such that on M, $g = e^{2\pi i \mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Also assume that $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$ and that the real parts of the eigenvalues of $L_M(0)$ has a lower bound. In particular, M is a direct sum of generalized eigenspaces for the operator $L_M(0)$.

A space of generators

- Let M be a $\mathbb{Z}_2$-graded vector space (graded by $\mathbb{Z}_2$-fermion numbers).
- Assume that g acts on M and there is an operator $L_M(0)$ on M.
- Assume that there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$ such that on M, $g = e^{2\pi i \mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Also assume that $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$ and that the real parts of the eigenvalues of $L_M(0)$ has a lower bound. In particular, M is a direct sum of generalized eigenspaces for the operator $L_M(0)$.

A space of generators

- Let M be a $\mathbb{Z}_2$-graded vector space (graded by $\mathbb{Z}_2$-fermion numbers).
- Assume that g acts on M and there is an operator $L_M(0)$ on M.
- Assume that there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$ such that on M, $g = e^{2\pi i \mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Also assume that $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$ and that the real parts of the eigenvalues of $L_M(0)$ has a lower bound. In particular, M is a direct sum of generalized eigenspaces for the operator $L_M(0)$.

A space of generators

- Let M be a $\mathbb{Z}_2$-graded vector space (graded by $\mathbb{Z}_2$-fermion numbers).
- Assume that g acts on M and there is an operator $L_M(0)$ on M.
- Assume that there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$ such that on M, $g = e^{2\pi i \mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Also assume that $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$ and that the real parts of the eigenvalues of $L_M(0)$ has a lower bound. In particular, M is a direct sum of generalized eigenspaces for the operator $L_M(0)$.

A space of generators

- Let M be a $\mathbb{Z}_2$-graded vector space (graded by $\mathbb{Z}_2$-fermion numbers).
- Assume that g acts on M and there is an operator $L_M(0)$ on M.
- Assume that there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$ such that on M, $g = e^{2\pi i \mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$.
- Also assume that $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$ and that the real parts of the eigenvalues of $L_M(0)$ has a lower bound. In particular, M is a direct sum of generalized eigenspaces for the operator $L_M(0)$.

A space of generators

- We call the eigenvalue of a generalized eigenvector $w \in M$ for $L_M(0)$ the *weight* of w and denote it by $\mathrm{wt}\ w$.
- Let $\{w^a\}_{a\in A}$ be a basis of M consisting of vectors homogeneous in weights, $\mathbb{Z}_2$-fermion numbers and g-weights (eigenvalues of $\mathcal{L}_g$) such that for $a \in A$, either $L_M(0)_N w^a = 0$ or there exists $L_M(0)_N(a) \in A$ such that $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For simplicity, when $L_M(0)_N w^a = 0$, we shall use $w^{L_M(0)_N(a)}$ to denote 0. Then for $a \in A$, we always have $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For $a \in A$, let $\alpha^a \in \mathbb{C}$ such that $\Re(\alpha^a) \in [0, 1)$ and $e^{2\pi i\alpha^a}$ is the eigenvalue of g for the generalized eigenvector w^a.

A space of generators

- We call the eigenvalue of a generalized eigenvector $w \in M$ for $L_M(0)$ the *weight* of w and denote it by $\mathrm{wt}\ w$.
- Let $\{w^a\}_{a\in A}$ be a basis of M consisting of vectors homogeneous in weights, $\mathbb{Z}_2$-fermion numbers and g-weights (eigenvalues of $\mathcal{L}_g$) such that for $a \in A$, either $L_M(0)_N w^a = 0$ or there exists $L_M(0)_N(a) \in A$ such that $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For simplicity, when $L_M(0)_N w^a = 0$, we shall use $w^{L_M(0)_N(a)}$ to denote 0. Then for $a \in A$, we always have $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For $a \in A$, let $\alpha^a \in \mathbb{C}$ such that $\Re(\alpha^a) \in [0, 1)$ and $e^{2\pi i\alpha^a}$ is the eigenvalue of g for the generalized eigenvector w^a.

A space of generators

- We call the eigenvalue of a generalized eigenvector $w \in M$ for $L_M(0)$ the *weight* of w and denote it by $\mathrm{wt}\ w$.
- Let $\{w^a\}_{a \in A}$ be a basis of M consisting of vectors homogeneous in weights, $\mathbb{Z}_2$-fermion numbers and g-weights (eigenvalues of $\mathcal{L}_g$) such that for $a \in A$, either $L_M(0)_N w^a = 0$ or there exists $L_M(0)_N(a) \in A$ such that $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For simplicity, when $L_M(0)_N w^a = 0$, we shall use $w^{L_M(0)_N(a)}$ to denote 0. Then for $a \in A$, we always have $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For $a \in A$, let $\alpha^a \in \mathbb{C}$ such that $\Re(\alpha^a) \in [0, 1)$ and $e^{2\pi i \alpha^a}$ is the eigenvalue of g for the generalized eigenvector w^a.

A space of generators

- We call the eigenvalue of a generalized eigenvector $w \in M$ for $L_M(0)$ the *weight* of w and denote it by $\mathrm{wt}\ w$.
- Let $\{w^a\}_{a\in A}$ be a basis of M consisting of vectors homogeneous in weights, $\mathbb{Z}_2$-fermion numbers and g-weights (eigenvalues of $\mathcal{L}_g$) such that for $a \in A$, either $L_M(0)_N w^a = 0$ or there exists $L_M(0)_N(a) \in A$ such that $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For simplicity, when $L_M(0)_N w^a = 0$, we shall use $w^{L_M(0)_N(a)}$ to denote 0. Then for $a \in A$, we always have $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For $a \in A$, let $\alpha^a \in \mathbb{C}$ such that $\Re(\alpha^a) \in [0, 1)$ and $e^{2\pi i\alpha^a}$ is the eigenvalue of g for the generalized eigenvector w^a.

A space of generators

- We call the eigenvalue of a generalized eigenvector $w \in M$ for $L_M(0)$ the *weight* of w and denote it by $\mathrm{wt}\ w$.
- Let $\{w^a\}_{a\in A}$ be a basis of M consisting of vectors homogeneous in weights, $\mathbb{Z}_2$-fermion numbers and g-weights (eigenvalues of $\mathcal{L}_g$) such that for $a \in A$, either $L_M(0)_N w^a = 0$ or there exists $L_M(0)_N(a) \in A$ such that $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For simplicity, when $L_M(0)_N w^a = 0$, we shall use $w^{L_M(0)_N(a)}$ to denote 0. Then for $a \in A$, we always have $L_M(0)_N w^a = w^{L_M(0)_N(a)}$.
- For $a \in A$, let $\alpha^a \in \mathbb{C}$ such that $\Re(\alpha^a) \in [0, 1)$ and $e^{2\pi i\alpha^a}$ is the eigenvalue of g for the generalized eigenvector w^a.

The graded space

- Let $\widetilde{M}^{[g]} = \coprod_{\alpha \in P_V} U(\hat{V}_{\phi}^{[g]}) \otimes (M \otimes t^{\alpha}\mathbb{C}[t, t^{-1}]) \otimes V^{[\alpha]}$. Then $\widetilde{M}^{[g]}$ is a left $U(\hat{V}_{\phi}^{[g]})$-module.
- for $i \in I$, we denote the action of $\phi^{i}_{\hat{V}_{\phi}^{[g]}}(x)$ on $\widetilde{M}^{[g]}$ by $\phi^{i}_{\widetilde{M}^{[g]}}(x)$. Then $\phi^{i}_{\widetilde{M}^{[g]}}(x)$ for $i \in I$ satisfy the same weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula as those for $\phi^{i}_{\hat{V}_{\phi}^{[g]}}(x)$.
- For $i \in I$, let $K^i \in \mathbb{N}$ such that $\mathcal{N}_g^{K^i+1}\phi^i_{-1}\mathbf{1} = 0$ and we denote the actions of the elements $\frac{(-1)^k}{k!}(\mathcal{N}_g^k \phi^i_{-1}\mathbf{1}) \otimes t^n$ for $n \in \alpha + \mathbb{Z}$ and $k = 0, \ldots, K^i$ of $U(\hat{V}_{\phi}^{[g]})$ on $\widetilde{M}_{\ell}^{[g]}$ by $(\phi^i_{\widetilde{M}^{[g]}})_{n,k}$.

The graded space

- Let $\widetilde{M}^{[g]} = \coprod_{\alpha \in P_V} U(\hat{V}^{[g]}_{\phi}) \otimes (M \otimes t^{\alpha}\mathbb{C}[t, t^{-1}]) \otimes V^{[\alpha]}$. Then $\widetilde{M}^{[g]}$ is a left $U(\hat{V}^{[g]}_{\phi})$-module.
- for $i \in I$, we denote the action of $\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)$ on $\widetilde{M}^{[g]}$ by $\phi^{i}_{\widetilde{M}^{[g]}}(x)$. Then $\phi^{i}_{\widetilde{M}^{[g]}}(x)$ for $i \in I$ satisfy the same weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula as those for $\phi^{i}_{\hat{V}^{[g]}_{\phi}}(x)$.
- For $i \in I$, let $K^i \in \mathbb{N}$ such that $\mathcal{N}_g^{K^i+1}\phi^i_{-1}\mathbf{1} = 0$ and we denote the actions of the elements $\frac{(-1)^k}{k!}(\mathcal{N}_g^k\phi^i_{-1}\mathbf{1}) \otimes t^n$ for $n \in \alpha + \mathbb{Z}$ and $k = 0, \ldots, K^i$ of $U(\hat{V}^{[g]}_{\phi})$ on $\widetilde{M}^{[g]}_{\ell}$ by $(\phi^i_{\widetilde{M}^{[g]}})_{n,k}$.

The graded space

- Let $\widetilde{M}^{[g]} = \coprod_{\alpha \in P_V} U(\hat{V}_\phi^{[g]}) \otimes (M \otimes t^\alpha \mathbb{C}[t, t^{-1}]) \otimes V^{[\alpha]}$. Then $\widetilde{M}^{[g]}$ is a left $U(\hat{V}_\phi^{[g]})$-module.
- for $i \in I$, we denote the action of $\phi^i_{\hat{V}_\phi^{[g]}}(x)$ on $\widetilde{M}^{[g]}$ by $\phi^i_{\widetilde{M}^{[g]}}(x)$. Then $\phi^i_{\widetilde{M}^{[g]}}(x)$ for $i \in I$ satisfy the same weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula as those for $\phi^i_{\hat{V}_\phi^{[g]}}(x)$.
- For $i \in I$, let $K^i \in \mathbb{N}$ such that $\mathcal{N}_g^{K^i+1} \phi^i_{-1}\mathbf{1} = 0$ and we denote the actions of the elements $\frac{(-1)^k}{k!}(\mathcal{N}_g^k \phi^i_{-1}\mathbf{1}) \otimes t^n$ for $n \in \alpha + \mathbb{Z}$ and $k = 0, \dots, K^i$ of $U(\hat{V}_\phi^{[g]})$ on $\widetilde{M}_\ell^{[g]}$ by $(\phi^i_{\widetilde{M}^{[g]}})_{n,k}$.

The graded space

- Let $\widetilde{M}^{[g]} = \coprod_{\alpha \in P_V} U(\hat{V}_\phi^{[g]}) \otimes (M \otimes t^\alpha \mathbb{C}[t, t^{-1}]) \otimes V^{[\alpha]}$. Then $\widetilde{M}^{[g]}$ is a left $U(\hat{V}_\phi^{[g]})$-module.
- for $i \in I$, we denote the action of $\phi^i_{\hat{V}_\phi^{[g]}}(x)$ on $\widetilde{M}^{[g]}$ by $\phi^i_{\widetilde{M}^{[g]}}(x)$. Then $\phi^i_{\widetilde{M}^{[g]}}(x)$ for $i \in I$ satisfy the same weak commutativity, $L(0)$-commutator formula and $L(-1)$-commutator formula as those for $\phi^i_{\hat{V}_\phi^{[g]}}(x)$.
- For $i \in I$, let $K^i \in \mathbb{N}$ such that $\mathcal{N}_g^{K^i+1} \phi^i_{-1} \mathbf{1} = 0$ and we denote the actions of the elements $\frac{(-1)^k}{k!}(\mathcal{N}_g^k \phi^i_{-1} \mathbf{1}) \otimes t^n$ for $n \in \alpha + \mathbb{Z}$ and $k = 0, \ldots, K^i$ of $U(\hat{V}_\phi^{[g]})$ on $\widetilde{M}_\ell^{[g]}$ by $(\phi^i_{\widetilde{M}^{[g]}})_{n,k}$.

The graded space

- For $a \in A$, $\alpha \in P_V$, $n \in \alpha + \mathbb{Z}$ and $k \in \mathbb{N}$, we denote the linear map $v \mapsto \frac{(-1)^k}{k!}(w^a \otimes t^n) \otimes \mathcal{N}_g^k v$ from $V^{[\alpha]}$ to $\widetilde{M}^{[g]}$ by $(\psi^a_{\widetilde{M}^{[g]}})_{n,k}$ and extend it to a linear map from V to $\widetilde{M}^{[g]}$ by mapping $V^{[\alpha']}$ to 0 for $\alpha' \neq \alpha$.
- Then $\widetilde{M}^{[g]}$ is spanned by elements of the form

$$(\phi^{i_1}_{\widetilde{M}^{[g]}})_{n_1,k_1} \cdots (\phi^{i_l}_{\widetilde{M}^{[g]}})_{n_l,k_l} (L_{\widetilde{M}^{[g]}}(m))^q)(\psi^a_{\widetilde{M}^{[g]}})_{n,k} v,$$

for $i_1, \ldots, i_l \in I$, $n_1 \in \alpha^{i_1} + \mathbb{Z}, \ldots, n_l \in \alpha^{i_l} + \mathbb{Z}$, $0 \leq k_1 \leq K^{i_1}, \ldots, 0 \leq k_l \leq K^{i_l}$, $m = 0, -1$, $q \in \mathbb{N}$, $a \in A$, $n \in \alpha + \mathbb{Z}$, $0 \leq k \leq K$, $v \in V^{[\alpha]}$, $\alpha \in P_V$, where $K \in \mathbb{N}$ satisfying $\mathcal{N}_g^{K+1} v = 0$.

The graded space

- For $a \in A$, $\alpha \in P_V$, $n \in \alpha + \mathbb{Z}$ and $k \in \mathbb{N}$, we denote the linear map $v \mapsto \frac{(-1)^k}{k!}(w^a \otimes t^n) \otimes \mathcal{N}_g^k v$ from $V^{[\alpha]}$ to $\widetilde{M}^{[g]}$ by $(\psi^a_{\widetilde{M}^{[g]}})_{n,k}$ and extend it to a linear map from V to $\widetilde{M}^{[g]}$ by mapping $V^{[\alpha']}$ to 0 for $\alpha' \neq \alpha$.
- Then $\widetilde{M}^{[g]}$ is spanned by elements of the form

$$(\phi^{i_1}_{\widetilde{M}^{[g]}})_{n_1,k_1} \cdots (\phi^{i_l}_{\widetilde{M}^{[g]}})_{n_l,k_l} (L_{\widetilde{M}^{[g]}}(m))^q)(\psi^a_{\widetilde{M}^{[g]}})_{n,k} v,$$

for $i_1, \ldots, i_l \in I$, $n_1 \in \alpha^{i_1} + \mathbb{Z}, \ldots, n_l \in \alpha^{i_l} + \mathbb{Z}$, $0 \leq k_1 \leq K^{i_1}, \ldots, 0 \leq k_l \leq K^{i_l}$, $m = 0, -1$, $q \in \mathbb{N}$, $a \in A$, $n \in \alpha + \mathbb{Z}$, $0 \leq k \leq K$, $v \in V^{[\alpha]}$, $\alpha \in P_V$, where $K \in \mathbb{N}$ satisfying $\mathcal{N}_g^{K+1} v = 0$.

The graded space

- For $a \in A$, $\alpha \in P_V$, $n \in \alpha + \mathbb{Z}$ and $k \in \mathbb{N}$, we denote the linear map $v \mapsto \frac{(-1)^k}{k!}(w^a \otimes t^n) \otimes \mathcal{N}_g^k v$ from $V^{[\alpha]}$ to $\widetilde{M}^{[g]}$ by $(\psi^a_{\widetilde{M}^{[g]}})_{n,k}$ and extend it to a linear map from V to $\widetilde{M}^{[g]}$ by mapping $V^{[\alpha']}$ to 0 for $\alpha' \neq \alpha$.
- Then $\widetilde{M}^{[g]}$ is spanned by elements of the form

$$(\phi^{i_1}_{\widetilde{M}^{[g]}})_{n_1,k_1} \cdots (\phi^{i_l}_{\widetilde{M}^{[g]}})_{n_l,k_l}(L_{\widetilde{M}^{[g]}}(m))^q)(\psi^a_{\widetilde{M}^{[g]}})_{n,k} v,$$

for $i_1, \dots, i_l \in I$, $n_1 \in \alpha^{i_1} + \mathbb{Z}, \dots, n_l \in \alpha^{i_l} + \mathbb{Z}$, $0 \leq k_1 \leq K^{i_1}, \dots, 0 \leq k_l \leq K^{i_l}$, $m = 0, -1$, $q \in \mathbb{N}$, $a \in A$, $n \in \alpha + \mathbb{Z}$, $0 \leq k \leq K$, $v \in V^{[\alpha]}$, $\alpha \in P_V$, where $K \in \mathbb{N}$ satisfying $\mathcal{N}_g^{K+1} v = 0$.

The graded space

- Let $\psi^a_{\widetilde{M}^{[g]}}(x)v = \sum_{k=0}^{K^v} \sum_{n\in\alpha+\mathbb{Z}} (\psi^a_{\widetilde{M}^{[g]}})_{n,k} v x^{-n-1}(\log x)^k$.
- Let $B \in \mathbb{R}$ such that $B \leq \Re(\mathrm{wt}\ w^a)$ for $a \in A$. Let $J_B(\widetilde{M}^{[g]})$ be the $U(\hat{V}^{[g]}_\phi)$-submodule of $\widetilde{M}^{[g]}$ generated by elements of the following forms: (i) $(\psi^a_{\widetilde{M}^{[g]}})_{n,0}\mathbf{1}$ for $a \in A$, and $n \notin -\mathbb{N} - 1$; (ii) homogeneous elements such that the real parts of their weights are less than B.
- Since $J_B(\widetilde{M}^{[g]})$ is spanned by homogeneous elements, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is also graded. In addition, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is lower bounded with respect to the weight grading with a lower bound B.

The graded space

- Let $\psi^{a}_{\widetilde{M}^{[g]}}(x)v = \sum_{k=0}^{K^v} \sum_{n \in \alpha+\mathbb{Z}} (\psi^{a}_{\widetilde{M}^{[g]}})_{n,k} v x^{-n-1} (\log x)^k$.
- Let $B \in \mathbb{R}$ such that $B \leq \Re(\mathrm{wt}\ w^a)$ for $a \in A$. Let $J_B(\widetilde{M}^{[g]})$ be the $U(\hat{V}^{[g]}_{\phi})$-submodule of $\widetilde{M}^{[g]}$ generated by elements of the following forms: (i) $(\psi^{a}_{\widetilde{M}^{[g]}})_{n,0}\mathbf{1}$ for $a \in A$, and $n \notin -\mathbb{N} - 1$; (ii) homogeneous elements such that the real parts of their weights are less than B.
- Since $J_B(\widetilde{M}^{[g]})$ is spanned by homogeneous elements, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is also graded. In addition, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is lower bounded with respect to the weight grading with a lower bound B.

The graded space

- Let $\psi^a_{\widetilde{M}^{[g]}}(x)v = \sum_{k=0}^{K^v} \sum_{n\in\alpha+\mathbb{Z}} (\psi^a_{\widetilde{M}^{[g]}})_{n,k} v x^{-n-1}(\log x)^k$.
- Let $B \in \mathbb{R}$ such that $B \leq \Re(\mathrm{wt}\ w^a)$ for $a \in A$. Let $J_B(\widetilde{M}^{[g]})$ be the $U(\hat{V}^{[g]}_\phi)$-submodule of $\widetilde{M}^{[g]}$ generated by elements of the following forms: (i) $(\psi^a_{\widetilde{M}^{[g]}})_{n,0}\mathbf{1}$ for $a \in A$, and $n \notin -\mathbb{N} - 1$; (ii) homogeneous elements such that the real parts of their weights are less than B.
- Since $J_B(\widetilde{M}^{[g]})$ is spanned by homogeneous elements, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is also graded. In addition, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is lower bounded with respect to the weight grading with a lower bound B.

The graded space

- Let $\psi^a_{\widetilde{M}^{[g]}}(x)v = \sum_{k=0}^{K^v} \sum_{n\in\alpha+\mathbb{Z}} (\psi^a_{\widetilde{M}^{[g]}})_{n,k} v x^{-n-1}(\log x)^k$.
- Let $B \in \mathbb{R}$ such that $B \leq \Re(\mathrm{wt}\ w^a)$ for $a \in A$. Let $J_B(\widetilde{M}^{[g]})$ be the $U(\hat{V}^{[g]}_\phi)$-submodule of $\widetilde{M}^{[g]}$ generated by elements of the following forms: (i) $(\psi^a_{\widetilde{M}^{[g]}})_{n,0}\mathbf{1}$ for $a \in A$, and $n \notin -\mathbb{N} - 1$; (ii) homogeneous elements such that the real parts of their weights are less than B.
- Since $J_B(\widetilde{M}^{[g]})$ is spanned by homogeneous elements, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is also graded. In addition, $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ is lower bounded with respect to the weight grading with a lower bound B.

The graded space

- For $i \in I$ and $a \in A$, let $M_{i,a} \in \mathbb{Z}_+$ be the smallest of $m \in \mathbb{Z}$ such that $m > \mathrm{wt}\ \phi^i - 1 + \Re(\mathrm{wt}\ w^a) - B - \Re(\alpha^i)$.
- Let $J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$ be the $U(\hat{V}_\phi^{[g]})$-submodule of $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ generated by the coefficients of the series

$$(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widetilde{M}^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widetilde{M}^{[g]}}(x_2)v$$
$$-(-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot$$
$$\cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,$$

$$L_{\widetilde{M}^{[g]}}(0)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v$$
$$-x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v - (\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,$$

$$L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v - \frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v$$

The graded space

- For $i \in I$ and $a \in A$, let $M_{i,a} \in \mathbb{Z}_+$ be the smallest of $m \in \mathbb{Z}$ such that $m > \mathrm{wt}\ \phi^i - 1 + \Re(\mathrm{wt}\ w^a) - B - \Re(\alpha^i)$.
- Let $J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$ be the $U(\hat{V}_\phi^{[g]})$-submodule of $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ generated by the coefficients of the series

$$\begin{aligned}
&(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widetilde{M}^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widetilde{M}^{[g]}}(x_2)v\\
&\quad -(-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot\\
&\qquad \cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,
\end{aligned}$$

$$\begin{aligned}
&L_{\widetilde{M}^{[g]}}(0)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v\\
&\quad -x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v - (\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,
\end{aligned}$$

$$L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v - \frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v$$

The graded space

- For $i \in I$ and $a \in A$, let $M_{i,a} \in \mathbb{Z}_+$ be the smallest of $m \in \mathbb{Z}$ such that $m > \mathrm{wt}\ \phi^i - 1 + \Re(\mathrm{wt}\ w^a) - B - \Re(\alpha^i)$.
- Let $J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$ be the $U(\hat{V}^{[g]}_\phi)$-submodule of $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ generated by the coefficients of the series

$$
\begin{aligned}
(x_1-x_2)^{\alpha^i+M_{i,a}}&(x_1-x_2)^{\mathcal{N}_g}\phi^i_{\widetilde{M}^{[g]}}(x_1)(x_1-x_2)^{-\mathcal{N}_g}\psi^a_{\widetilde{M}^{[g]}}(x_2)v\\
&-(-1)^{|u||w|}(-x_2+x_1)^{\alpha^i+M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot\\
&\qquad\cdot(-x_2+x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2+x_1)^{-\mathcal{N}_g}v,
\end{aligned}
$$

$$
\begin{aligned}
L_{\widetilde{M}^{[g]}}(0)&\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v\\
&-x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v-(\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,
\end{aligned}
$$

$$
L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v-\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v
$$

The graded space

- For $i \in I$ and $a \in A$, let $M_{i,a} \in \mathbb{Z}_+$ be the smallest of $m \in \mathbb{Z}$ such that $m > \mathrm{wt}\ \phi^i - 1 + \Re(\mathrm{wt}\ w^a) - B - \Re(\alpha^i)$.
- Let $J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$ be the $U(\hat{V}_{\phi}^{[g]})$-submodule of $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ generated by the coefficients of the series

$$(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widetilde{M}^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widetilde{M}^{[g]}}(x_2)v$$

$$-(-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot$$

$$\cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,$$

$$L_{\widetilde{M}^{[g]}}(0)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v$$

$$-x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v - (\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,$$

$$L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v - \frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v$$

The graded space

- For $i \in I$ and $a \in A$, let $M_{i,a} \in \mathbb{Z}_+$ be the smallest of $m \in \mathbb{Z}$ such that $m > \mathrm{wt}\ \phi^i - 1 + \Re(\mathrm{wt}\ w^a) - B - \Re(\alpha^i)$.
- Let $J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$ be the $U(\hat{V}^{[g]}_{\phi})$-submodule of $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ generated by the coefficients of the series

$$
\begin{aligned}
&(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widetilde{M}^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widetilde{M}^{[g]}}(x_2)v\\
&\qquad -(-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot\\
&\qquad\qquad \cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,\\
&L_{\widetilde{M}^{[g]}}(0)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v\\
&\qquad -x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v - (\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,\\
&L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v - \psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v - \frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v
\end{aligned}
$$

The graded space

- For $i \in I$ and $a \in A$, let $M_{i,a} \in \mathbb{Z}_+$ be the smallest of $m \in \mathbb{Z}$ such that $m > \mathrm{wt}\ \phi^i - 1 + \Re(\mathrm{wt}\ w^a) - B - \Re(\alpha^i)$.
- Let $J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$ be the $U(\hat{V}_\phi^{[g]})$-submodule of $\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]})$ generated by the coefficients of the series

$$(x_1-x_2)^{\alpha^i+M_{i,a}}(x_1-x_2)^{\mathcal{N}_g}\phi^i_{\widetilde{M}^{[g]}}(x_1)(x_1-x_2)^{-\mathcal{N}_g}\psi^a_{\widetilde{M}^{[g]}}(x_2)v$$
$$-(-1)^{|u||w|}(-x_2+x_1)^{\alpha^i+M_{i,a}}\psi^a_{\widetilde{M}^{[g]}}(x_2)\cdot$$
$$\cdot(-x_2+x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2+x_1)^{-\mathcal{N}_g}v,$$
$$L_{\widetilde{M}^{[g]}}(0)\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^a_{\widetilde{M}^{[g]}}(x)L_V(0)v$$
$$-x\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v-(\mathrm{wt}\ w^a)\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^{L_M(0)_N a}_{\widetilde{M}^{[g]}}(x)v,$$
$$L_{\widetilde{M}^{[g]}}(-1)\psi^a_{\widetilde{M}^{[g]}}(x)v-\psi^a_{\widetilde{M}^{[g]}}(x)L_V(-1)v-\frac{d}{dx}\psi^a_{\widetilde{M}^{[g]}}(x)v$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$
\begin{aligned}
&(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v\\
&\quad = (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot\\
&\qquad\qquad \cdot (-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,\\
&L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v\\
&\quad = x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),\\
&L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.
\end{aligned}
$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$
\begin{aligned}
(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v \\
= (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot \\
\cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,
\end{aligned}
$$

$$
\begin{aligned}
L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v \\
= x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),
\end{aligned}
$$

$$
L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.
$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$
\begin{aligned}
(x_1-x_2)^{\alpha^i+M_{i,a}}(x_1-x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1-x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v
&= (-1)^{|u||w|}(-x_2+x_1)^{\alpha^i+M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot \\
&\quad \cdot(-x_2+x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2+x_1)^{-\mathcal{N}_g}v,
\end{aligned}
$$

$$
\begin{aligned}
L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v
&= x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),
\end{aligned}
$$

$$
L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.
$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$\begin{aligned}
&(x_1-x_2)^{\alpha^i+M_{i,a}}(x_1-x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1-x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v\\
&\quad=(-1)^{|u||w|}(-x_2+x_1)^{\alpha^i+M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot\\
&\qquad\cdot(-x_2+x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2+x_1)^{-\mathcal{N}_g}v,
\end{aligned}$$

$$\begin{aligned}
&L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v-\psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v\\
&\quad=x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)+(\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x)+\psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),
\end{aligned}$$

$$L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v-\psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v=\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$(x_1 - x_2)^{\alpha^i + M_{i,a}}(x_1 - x_2)^{\mathcal{N}_g}\phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g}\psi^a_{\widehat{M}_B^{[g]}}(x_2)v$$
$$= (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}}\psi^a_{\widehat{M}_B^{[g]}}(x_2)\cdot$$
$$\cdot(-x_2 + x_1)^{\mathcal{N}_g}\phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g}v,$$
$$L_{\widehat{M}_B^{[g]}}(0)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(0)v$$
$$= x\frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\ w^a)\psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x),$$
$$L_{\widehat{M}_B^{[g]}}(-1)\psi^a_{\widehat{M}_B^{[g]}}(x)v - \psi^a_{\widehat{M}_B^{[g]}}(x)L_V(-1)v = \frac{d}{dx}\psi^a_{\widehat{M}_B^{[g]}}(x)v.$$

The graded space

- Let $\widehat{M}_B^{[g]} = (\widetilde{M}_\ell^{[g]}/J_B(\widetilde{M}^{[g]}))/J(\widetilde{M}^{[g]}/J_B(\widetilde{M}^{[g]}))$.
- In $\widehat{M}_B^{[g]}$, we have

$$
\begin{aligned}
(x_1 - x_2)^{\alpha^i + M_{i,a}} & (x_1 - x_2)^{\mathcal{N}_g} \phi^i_{\widehat{M}_B^{[g]}}(x_1)(x_1 - x_2)^{-\mathcal{N}_g} \psi^a_{\widehat{M}_B^{[g]}}(x_2) v \\
&= (-1)^{|u||w|}(-x_2 + x_1)^{\alpha^i + M_{i,a}} \psi^a_{\widehat{M}_B^{[g]}}(x_2) \cdot \\
&\qquad \cdot (-x_2 + x_1)^{\mathcal{N}_g} \phi^i(x_1)(-x_2 + x_1)^{-\mathcal{N}_g} v, \\
L_{\widehat{M}_B^{[g]}}(0) \psi^a_{\widehat{M}_B^{[g]}}(x) v & - \psi^a_{\widehat{M}_B^{[g]}}(x) L_V(0) v \\
&= x \frac{d}{dx} \psi^a_{\widehat{M}_B^{[g]}}(x) + (\mathrm{wt}\, w^a) \psi^a_{\widehat{M}_B^{[g]}}(x) + \psi^{L_M(0)_N a}_{\widehat{M}_B^{[g]}}(x), \\
L_{\widehat{M}_B^{[g]}}(-1) \psi^a_{\widehat{M}_B^{[g]}}(x) v & - \psi^a_{\widehat{M}_B^{[g]}}(x) L_V(-1) v = \frac{d}{dx} \psi^a_{\widehat{M}_B^{[g]}}(x) v.
\end{aligned}
$$

The lower-bounded generalized twisted module

Theorem

The twisted fields $\phi^i_{\widehat{M}^{[g]}_B}$ for $i \in I$ generate a twisted vertex operator map

$$Y^g_{\widehat{M}^{[g]}_B} : V \otimes \widehat{M}^{[g]}_B \to \widehat{M}^{[g]}_B\{x\}[\log x]$$

such that $(\widehat{M}^{[g]}_B, Y^g_{\widehat{M}^{[g]}_B})$ is a lower-bounded generalized g-twisted V-module. Moreover, this is the unique generalized g-twisted V-module structure on $\widehat{M}^{[g]}_B$ generated by the coefficients of $(\psi^a_{\widehat{M}^{[g]}_B})(x)v$ for $a \in A$ and $v \in V$ such that $Y^g_{\widehat{M}^{[g]}_B}(\phi^i_{-1}\mathbf{1}, z) = \phi^i_{\widehat{M}^{[g]}_B}(z)$ for $i \in I$.

Outline

The universal property $\widehat{M}_B^{[g]}$

Theorem

Let (W, Y_W^g) be a lower-bounded generalized g-twisted V-module and M_0 a $\mathbb{Z}_2$-graded subspace of W invariant under the actions of g, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$. Let $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$. Assume that there is a linear map $f : M \to M_0$ preserving the $\mathbb{Z}_2$-fermion number grading and commuting with the actions of g, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$ $(L_{\widehat{M}_B^{[g]}}(0))$, $L_W(0)_S$ $(L_{\widehat{M}_B^{[g]}}(0)_S)$ and $L_W(0)_N$ $(L_{\widehat{M}_B^{[g]}}(0)_N)$. Then there exists a unique module map $\tilde{f} : \widehat{M}_B^{[g]} \to W$ such that $\tilde{f}|_M = f$. If f is surjective and (W, Y_W^g) is generated by the coefficients of $(Y^g)_{WV}^W(w_0, x)v$ for $w_0 \in M_0$ and $v \in V$, where $(Y^g)_{WV}^W$ is the twist vertex operator map obtained from Y_W^g, then $\tilde{f}$ is surjective.

Lower-bounded generalized twisted modules as quotients

Corollary

Let (W, Y_W^g) be a lower-bounded generalized g-twisted V-module generated by the coefficients of $(Y^g)_{WV}^W(w, x)v$ for $w \in M$, where $(Y^g)_{WV}^W$ is the twist vertex operator map obtained from Y_W^g and M is a $\mathbb{Z}_2$-graded subspace of W invariant under the actions of g, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$. Let $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$. Then there is a generalized g-twisted V-submodule J of $\widehat{M}_B^{[g]}$ such that W is equivalent as a lower-bounded generalized g-twisted V-module to the quotient module $\widehat{M}_B^{[g]}/J$.

A linearly independent set of generators and consequences

Theorem

The lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ is in fact generated by $(\psi^a_{\widehat{M}_B^{[g]}})_{-k-1,0}\mathbf{1} = L_{\widehat{M}_B^{[g]}}(-1)^k(\psi^a_{\widehat{M}_B^{[g]}})_{-1,0}\mathbf{1}$ for $a \in A$ and $k \in \mathbb{N}$. Moreover, this set of generators is linearly independent.

Corollary

The lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ is not 0. In particular, there exists nonzero lower-bounded generalized g-twisted V-modules.

A linearly independent set of generators and consequences

Theorem

The lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ *is in fact generated by* $(\psi^a_{\widehat{M}_B^{[g]}})_{-k-1,0}\mathbf{1} = L_{\widehat{M}_B^{[g]}}(-1)^k(\psi^a_{\widehat{M}_B^{[g]}})_{-1,0}\mathbf{1}$ *for* $a \in A$ *and* $k \in \mathbb{N}$*. Moreover, this set of generators is linearly independent.*

Corollary

The lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ *is not* 0*. In particular, there exists nonzero lower-bounded generalized g-twisted V-modules.*

A linearly independent set of generators and consequences

Theorem

The lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ is in fact generated by $(\psi^a_{\widehat{M}_B^{[g]}})_{-k-1,0}\mathbf{1} = L_{\widehat{M}_B^{[g]}}(-1)^k(\psi^a_{\widehat{M}_B^{[g]}})_{-1,0}\mathbf{1}$ for $a \in A$ and $k \in \mathbb{N}$. Moreover, this set of generators is linearly independent.

Corollary

The lower-bounded generalized g-twisted V-module $\widehat{M}_B^{[g]}$ is not 0*. In particular, there exists nonzero lower-bounded generalized g-twisted V-modules.*

Existence of irreducible lower-bounded generalized g-twisted V-modules

Corollary

The twisted Zhu's algebra $A_g(V)$ or the twisted zero-mode algebra $Z_g(V)$ is not 0.

Theorem

Let W be a lower-bounded generalized g-twisted V-module generated by a nonzero element w (for example, $\widehat{M}_B^{[g]}$ when M is a one dimensional space and B is less than or equal to the real part of the weight of the elements of M). Then there exists a maximal submodule J of W such that J does not contain w and the quotient W/J is irreducible.

Existence of irreducible lower-bounded generalized g-twisted V-modules

Corollary

The twisted Zhu's algebra $A_g(V)$ or the twisted zero-mode algebra $Z_g(V)$ is not 0.

Theorem

Let W be a lower-bounded generalized g-twisted V-module generated by a nonzero element w (for example, $\widehat{M}_B^{[g]}$ when M is a one dimensional space and B is less than or equal to the real part of the weight of the elements of M). Then there exists a maximal submodule J of W such that J does not contain w and the quotient W/J is irreducible.

Existence of irreducible lower-bounded generalized g-twisted V-modules

Corollary

The twisted Zhu's algebra $A_g(V)$ or the twisted zero-mode algebra $Z_g(V)$ is not 0.

Theorem

Let W be a lower-bounded generalized g-twisted V-module generated by a nonzero element w (for example, $\widehat{M}_B^{[g]}$ when M is a one dimensional space and B is less than or equal to the real part of the weight of the elements of M). Then there exists a maximal submodule J of W such that J does not contain w and the quotient W/J is irreducible.

Existence of irreducible grading-restricted generalized and ordinary g-twisted V-modules

- In the case that V is simple, C_2-cofinite and g is of finite order, Dong, Li and Mason proved in 1997 the existence of an irreducible ordinary g-twisted V-module.
- Their proof used genus-one 1-point functions. Thus the simplicity and C_2-cofiniteness of V and the finiteness of the order of g are necessary in their approach.
- Using our construction of the universal lower-bounded generalized g-twisted V-modules, we successfully removed the simplicity and C_2-cofininess of V and the finiteness of the order of g, under some very weak conditions.

Existence of irreducible grading-restricted generalized and ordinary g-twisted V-modules

- In the case that V is simple, C_2-cofinite and g is of finite order, Dong, Li and Mason proved in 1997 the existence of an irreducible ordinary g-twisted V-module.
- Their proof used genus-one 1-point functions. Thus the simplicity and C_2-cofiniteness of V and the finiteness of the order of g are necessary in their approach.
- Using our construction of the universal lower-bounded generalized g-twisted V-modules, we successfully removed the simplicity and C_2-cofininess of V and the finiteness of the order of g, under some very weak conditions.

Existence of irreducible grading-restricted generalized and ordinary g-twisted V-modules

- In the case that V is simple, C_2-cofinite and g is of finite order, Dong, Li and Mason proved in 1997 the existence of an irreducible ordinary g-twisted V-module.
- Their proof used genus-one 1-point functions. Thus the simplicity and C_2-cofiniteness of V and the finiteness of the order of g are necessary in their approach.
- Using our construction of the universal lower-bounded generalized g-twisted V-modules, we successfully removed the simplicity and C_2-cofininess of V and the finiteness of the order of g, under some very weak conditions.

Existence of irreducible grading-restricted generalized and ordinary g-twisted V-modules

- In the case that V is simple, C_2-cofinite and g is of finite order, Dong, Li and Mason proved in 1997 the existence of an irreducible ordinary g-twisted V-module.
- Their proof used genus-one 1-point functions. Thus the simplicity and C_2-cofiniteness of V and the finiteness of the order of g are necessary in their approach.
- Using our construction of the universal lower-bounded generalized g-twisted V-modules, we successfully removed the simplicity and C_2-cofininess of V and the finiteness of the order of g, under some very weak conditions.

Existence of irreducible grading-restricted generalized and ordinary g-twisted V-modules

Theorem

Let V be a Möbius vertex superalgebra (a grading-restricted vertex algebra with a compatible $\mathfrak{sl}_2$-module structure and g an automorphism of V. Assume that the set of real parts of the numbers in $P(V)$ has no accumulation point in $\mathbb{R}$. If the twisted Zhu's algebra $A_g(V)$ or the twisted zero-mode algebra $Z_g(V)$ is finite dimensional, then there exists an irreducible grading-restricted generalized g-twisted V-module. Such an irreducible grading-restricted generalized g-twisted V-module is an irreducible ordinary g-twisted V-module if g acts on it semisimply. In particular, if g is of finite order, there exists an irreducible ordinary g-twisted V-module.

Existence of irreducible grading-restricted generalized and ordinary g-twisted V-modules

Theorem

Let V be a Möbius vertex superalgebra (a grading-restricted vertex algebra with a compatible $\mathfrak{sl}_2$-module structure and g an automorphism of V. Assume that the set of real parts of the numbers in $P(V)$ has no accumulation point in $\mathbb{R}$. If the twisted Zhu's algebra $A_g(V)$ or the twisted zero-mode algebra $Z_g(V)$ is finite dimensional, then there exists an irreducible grading-restricted generalized g-twisted V-module. Such an irreducible grading-restricted generalized g-twisted V-module is an irreducible ordinary g-twisted V-module if g acts on it semisimply. In particular, if g is of finite order, there exists an irreducible ordinary g-twisted V-module.

Twisted extensions of modules for the fixed-point subalgebra

Theorem

Let W_0 be a lower-bounded generalized V^g-module (in particular, W_0 has a lower-bounded grading by $\mathbb{C}$ (graded by weights) and a grading by $\mathbb{Z}_2$ (graded by fermion numbers)). Assume that g acts on W_0 and there are semisimple and nilpotent operators $\mathcal{S}_g$ and $\mathcal{N}_g$, respectively, on W_0 such that $g = e^{2\pi i \mathcal{L}_g}$ where $\mathcal{L}_g = \mathcal{S}_g + \mathcal{N}_g$. Then W_0 can be extended to a lower-bounded generalized g-twisted V-module, that is, there exists a lower-bounded generalized g-twisted V-module W and an injective module map $f : W_0 \to W$ of V^g-modules.