On Indecomposable $\mathbb{N}$-graded Vertex Algebras

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- This talk is based on the joint works with P. Jitjankarn.
- List of Topics
  - Leibniz Algebras
  - 1-Truncated Conformal Algebras and Vertex Algebroids
  - $\mathbb{N}$-graded Vertex Algebras
  - Vertex Algebras Associated with Vertex Algebroids

Leibniz Algebras

- Definition A *left Leibniz algebra* $\mathfrak{L}$ is a $\mathbb{C}$-vector space equipped with a bilinear map $[\ ,\ ] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ satisfying the Leibniz identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

  for all $a, b, c \in \mathfrak{L}$.
- Examples
  - (i) Every Lie algebra is a left Leibniz algebra.
  - (ii) Let G be a Lie algebra and let M be a skew-symmetric G-module (i.e., $[m, g] = 0$ for all $g \in G$, $m \in M$). Then the vector space $Q = G \oplus M$ equipped with the multiplication $[u + m, v + n] = [u, v] + u \cdot n$ is a left Leibniz algebra. Here, $u, v \in G$, $m, n \in M$.

- Definition Let $\mathfrak{L}$ be a left Leibniz algebra over $\mathbb{C}$. Let I be a subspace of $\mathfrak{L}$. I is a *left* (respectively, *right*) *ideal* of $\mathfrak{L}$ if $[\mathfrak{L}, I] \subseteq I$ (respectively, $[I, \mathfrak{L}] \subseteq I$). I is an *ideal* of $\mathfrak{L}$ if it is both a left and a right ideal.
- Example: We define

$$\begin{aligned} Leib(\mathfrak{L}) &= Span\{\ [u,u] \mid u \in \mathfrak{L}\ \} \\ &= Span\{\ [u,v] + [v,u] \mid u, v \in \mathfrak{L}\}. \end{aligned}$$

$Leib(\mathfrak{L})$ is an ideal of $\mathfrak{L}$. Moreover, for $v, w \in Leib(\mathfrak{L})$, $[v, w] = 0$.

- Definition Let $(\mathfrak{L}, [\ ,\ ])$ be a left Leibniz algebra. The series of ideals
$$... \subseteq \mathfrak{L}^{(2)} \subseteq \mathfrak{L}^{(1)} \subseteq \mathfrak{L}$$
where $\mathfrak{L}^{(1)} = [\mathfrak{L}, \mathfrak{L}]$, $\mathfrak{L}^{(i+1)} = [\mathfrak{L}^{(i)}, \mathfrak{L}^{(i)}]$ is called the *derived series* of $\mathfrak{L}$.

  A left Leibniz algebra $\mathfrak{L}$ is *solvable* if $\mathfrak{L}^{(m)} = 0$ for some integer $m \geq 0$.

  As in the case of Lie algebras, any left Leibniz algebra $\mathfrak{L}$ contains a unique maximal solvable ideal $rad(\mathfrak{L})$ called the *radical* of $\mathfrak{L}$ which contains all solvable ideals.
- Example: $Leib(\mathfrak{L})$ is a solvable ideal.

- Definition
  - A left Leibniz algebra $\mathfrak{L}$ is *simple* if $[\mathfrak{L}, \mathfrak{L}] \neq Leib(\mathfrak{L})$, and $\{0\}$, $Leib(\mathfrak{L})$, $\mathfrak{L}$ are the only ideals of $\mathfrak{L}$.
  - A left Leibniz algebra $\mathfrak{L}$ is said to be *semisimple* if $rad(\mathfrak{L}) = Leib(\mathfrak{L})$.

- Remarks [Demir-Misra-Stitzinger]
  - The Leibniz algebra $\mathfrak{L}$ is semisimple if and only if the Lie algebra $\mathfrak{L}/Leib(\mathfrak{L})$ is semisimple.
  - However, if $\mathfrak{L}/Leib(\mathfrak{L})$ is a simple Lie algebra then $\mathfrak{L}$ is not necessarily a simple Leibniz algebra.
  - Also, $\mathfrak{L}$ is not necessary a direct sum of simple Leibniz ideals when $\mathfrak{L}/Leib(\mathfrak{L})$ is a semisimple Lie algebra .

Examples [Demir-Misra-Stitzinger]

- For a positive integer m, we set V_m to be an irreducible sl_2-module of dimension m. Then $sl_2 \oplus V_m$ is a simple Leibniz algebra with $Leib(sl_2 \oplus V_m) = V_m$.
- Next, we set $U = V_m \oplus V_n$. A vector space $\mathfrak{L} = sl_2 \oplus U$ is a Leibniz algebra with $Leib(\mathfrak{L}) = U$.
  - Clearly, U, V_m, V_n are all different ideals of $\mathfrak{L}$.
  - Hence, $\mathfrak{L}$ is not a simple Leibniz algebra although $\mathfrak{L}/Leib(\mathfrak{L})$ is a simple Lie algebra.
  - Furthermore, observe that $\mathfrak{L}$ can not be written as a direct sum of simple Leibniz ideals.

- Theorem [Barnes, Demir-Misra-Stitzinger]

  Let $\mathfrak{L}$ be a left Leibniz algebra.

  - (i) There exists a subalgebra S which is a semisimple Lie algebra of $\mathfrak{L}$ such that $\mathfrak{L} = S \dot{+} rad(\mathfrak{L})$. As in the case of a Lie algebra, we call S a Levi subalgebra or a Levi factor of B.
  - (ii) If $\mathfrak{L}$ is a semisimple Leibniz algebra then $\mathfrak{L} = (S_1 \oplus S_2 \oplus ... \oplus S_k) \dot{+} Leib(\mathfrak{L})$, where S_j is a simple Lie algebra for all $1 \leq j \leq k$. Moreover, $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$.
  - (iii) If $\mathfrak{L}$ is a simple Leibniz algebra, then there exists a simple Lie algebra S such that $Leib(\mathfrak{L})$ is an irreducible module over S and $\mathfrak{L} = S \dot{+} Leib(\mathfrak{L})$.

- Definition Let $\mathfrak{L}$ be a left Leibniz algebra. A left $\mathfrak{L}$-module is a vector space M equipped with a $\mathbb{C}$-bilinear map $\mathfrak{L} \times M \to M; (u, m) \mapsto u \cdot m$ such that

$$([u, v]) \cdot m = u \cdot (v \cdot m) - v \cdot (u \cdot m)$$

  for all $u, v \in \mathfrak{L}, m \in M$.

  The usual definitions of the notions of submodule, irreducibility, complete reducibility, homomorphism, isomorphism, etc., hold for left Leibniz modules.
- Remark $Leib(\mathfrak{L})$ acts as zero on M.

1-Truncated Conformal Algebras and Vertex Algebroids

- Definition [Gorbounov-Malikov-Schechtman]

  A *1-truncated conformal algebra* is a graded vector space $C = C_0 \oplus C_1$ equipped with a linear map $\partial : C_0 \to C_1$ and bilinear operations $(u, v) \mapsto u_i v$ for $i = 0, 1$ of degree $-i-1$ on $C = C_0 \oplus C_1$ such that the following axioms hold:

  - (Derivation) for $a \in C_0$, $u \in C_1$,

  $$(\partial a)_0 = 0, \quad (\partial a)_1 = -a_0, \quad \partial(u_0 a) = u_0 \partial a;$$

  - (Commutativity) for $a \in C_0$, $u, v \in C_1$,

  $$u_0 a = -a_0 u, \quad u_0 v = -v_0 u + \partial(u_1 v), \quad u_1 v = v_1 u;$$

  - (Associativity) for $\alpha, \beta, \gamma \in C_0 \oplus C_1$,

  $$\alpha_0 \beta_i \gamma = \beta_i \alpha_0 \gamma + (\alpha_0 \beta)_i \gamma.$$

- Definition [Bressler, Gorbounov-Malikov-Schechtman]
  Let $(A, *)$ be a unital commutative associative algebra over $\mathbb{C}$ with the identity 1. A *vertex A-algebroid* is a $\mathbb{C}$-vector space Γ equipped with
  1. a $\mathbb{C}$-bilinear map $A \times \Gamma \to \Gamma, \quad (a, v) \mapsto a \cdot v$ such that $1 \cdot v = v$ (i.e. a nonassociative unital A-module),
  2. a structure of a Leibniz $\mathbb{C}$-algebra $[\ ,\ ] : \Gamma \times \Gamma \to \Gamma$,
  3. a homomorphism of Leibniz $\mathbb{C}$-algebra $\pi : \Gamma \to Der(A)$,
  4. a symmetric $\mathbb{C}$-bilinear pairing $\langle\ ,\ \rangle : \Gamma \otimes_{\mathbb{C}} \Gamma \to A$,
  5. a $\mathbb{C}$-linear map $\partial : A \to \Gamma$ such that $\pi \circ \partial = 0$ which satisfying the following conditions:

$$
\begin{aligned}
&a \cdot (a' \cdot v) - (a * a') \cdot v = \pi(v)(a) \cdot \partial(a') + \pi(v)(a') \cdot \partial(a), \\
&[u, a \cdot v] = \pi(u)(a) \cdot v + a \cdot [u, v], \ [u, v] + [v, u] = \partial(\langle u, v \rangle), \\
&\pi(a \cdot v) = a\pi(v), \ \langle a \cdot u, v \rangle = a * \langle u, v \rangle - \pi(u)(\pi(v)(a)), \\
&\pi(v)(\langle v_1, v_2 \rangle) = \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle, \\
&\partial(a * a') = a \cdot \partial(a') + a' \cdot \partial(a), \\
&[v, \partial(a)] = \partial(\pi(v)(a)), \ \langle v, \partial(a) \rangle = \pi(v)(a)
\end{aligned}
$$

  for $a, a' \in A, \ u, v, v_1, v_2 \in \Gamma$.

Proposition[Li-Y.]

Let $(A, *)$ be a unital commutative associative algebra and let B be a module for A as a nonassociative algebra . Then a vertex A-algebroid structure on B exactly amounts to a 1-truncated conformal algebra structure on $C = A \oplus B$ with

$$a_i a' = 0,\ \ u_0 v = [u, v],\ \ u_1 v = \langle u, v\rangle,$$
$$u_0 a = \pi(u)(a),\ \ a_0 u = -u_0 a$$

for $a, a' \in A$, $u, v \in B$, $i = 0, 1$ such that

$$a \cdot (a' \cdot u) - (a * a') \cdot u = (u_0 a) \cdot \partial a' + (u_0 a') \cdot \partial a,$$
$$u_0(a \cdot v) - a \cdot (u_0 v) = (u_0 a) \cdot v,$$
$$u_0(a * a') = a * (u_0 a') + (u_0 a) * a',$$
$$a_0(a' \cdot v) = a' * (a_0 v)$$
$$(a \cdot u)_1 v = a * (u_1 v) - u_0 v_0 a,$$
$$\partial(a * a') = a \cdot \partial(a') + a' \cdot \partial(a).$$

From now, we assume that

(i) $(A, *)$ is a finite dimensional unital commutative associative algebra with the identity $\hat{1}$.

(ii) B is a finite dimensional vertex A-algebroid.

(iii) A is not a trivial module of the Leibniz algebra B.

Theorem

Let B be a simple Leibniz algebra such that $Leib(B) \neq \{0\}$.

- Assume that its Levi factor $S = Span\{e, f, h\}$ such that $e_0 f = h$, $h_0 e = 2e$, $h_0 f = -2f$, and $e_1 f = k\hat{1} \in (\mathbb{C}\hat{1})\backslash\{0\}$.
- Then
  - (i) $e_1 e = f_1 f = e_1 h = f_1 h = 0$, $k = 1$, $h_1 h = 2\hat{1}$.
  - (ii) $Ker(\partial) = \mathbb{C}\hat{1}$
  - (iii)
    - $Leib(B)$ is an irreducible sl_2-module of dimension 2.
    - A is a local algebra. Moreover, as a sl_2-module, A is a direct sum of a trivial module and an irreducible sl_2-module of dimension 2.

- (iv) Let $A_{\neq 0}$ be an irreducible sl_2-submodule of A that has dimension 2.
  - Let a_0 be the highest weight vector of $A_{\neq 0}$ of weight 1 and let $a_1 = f_0 a_0$. Hence, the set $\{a_0, a_1\}$ forms a basis of $A_{\neq 0}$, the set $\{\hat{1}, a_0, a_1\}$ is a basis of A, and the set $\{\partial(a_0), \partial(a_1)\}$ is a basis of $Leib(B)$.

  Relationships among $a_0, a_1, e, f, h, \partial(a_0), \partial(a_1)$ are desribed below:

$$
\begin{aligned}
&(\partial(a_0))_1 e = 0, \ (\partial(a_0))_1 f = a_1, \ (\partial(a_0))_1 h = a_0,\\
&(\partial(a_1))_1 e = a_0, \ (\partial(a_1))_1 f = 0, \ (\partial(a_1))_1 h = -a_1,\\
&a_0 \cdot e = 0, \ a_0 \cdot f = \partial(a_1), \ a_0 \cdot h = \partial(a_0), \ a_0 \cdot \partial(a_i) = 0,\\
&a_1 \cdot e = \partial(a_0), \ a_1 \cdot f = 0, \ a_1 \cdot h = -\partial(a_1), \ a_1 \cdot \partial(a_i) = 0,\\
&a_i * a_j = 0 \text{ for all } i, j \in \{0, 1\}.
\end{aligned}
$$

Theorem Suppose that B is a semisimple Leibniz algebra such that $Leib(B) \neq \{0\}$, and $Ker(\partial) = \{a \in A \mid u_0 a = 0 \text{ for all } u \in B\}$.

- Assume that the Levi factor $S = Span\{e, f, h\}$ such that $e_0 f = h, h_0 e = 2e, h_0 f = -2f$ and $e_1 f = k\hat{1} \in \mathbb{C}\hat{1} \backslash \{0\}$.
- We set $A = \mathbb{C}\hat{1} \oplus_{j=1}^{l} N^j$ where each N^j is an irreducible sl_2-submodule of A.
- Then
  (i) $e_1 e = f_1 f = e_1 h = f_1 h = 0,\ k = 1,\ h_1 h = 2\hat{1}$;
  (ii) $Ker(\partial) = \mathbb{C}\hat{1}$;
  (iii) For $j \in \{1, ..., l\}$ dim $N^j = 2$, and dim $Leib(B) = 2l$;

(iv) A is a local algebra.
For each j, we let $a_{j,0}$ be a highest weight vector of N^j and $a_{j,1} = f_0(a_{j,0})$. Then $\{\hat{1}, a_{j,i} \mid j \in \{1, ..., l\},\ i \in \{0,1\}\}$ is a basis of A, and $\{\partial(a_{j,i}) \mid j \in \{1, ..., l\},\ i \in \{0,1\}\}$ is a basis of $Leib(B)$. Relations among $a_{j,i}, e, f, h, \partial(a_{j,i})$ are described below:

$$
\begin{aligned}
& a_{j,i} * a_{j',i'} = 0, \\
& a_{j,0} \cdot e = 0,\ a_{j,1} \cdot e = \partial(a_{j,0}), \\
& a_{j,0} \cdot f = \partial(a_{j,1}),\ a_{j,1} \cdot f = 0, \\
& a_{j,0} \cdot h = \partial(a_{j,0}),\ a_{j,1} \cdot h = -\partial(a_{j,1}), \\
& a_{j,i} \cdot \partial(a_{j',i'}) = 0, \\
& \partial(a_{j,i})_1 e = e_0 a_{j,i} = (2-i) a_{j,i-1}, \\
& \partial(a_{j,i})_1 f = f_0 a_{j,i} = (i+1) a_{j,i+1}, \\
& \partial(a_{j,i})_1 h = h_0 a_{j,i} = (1-2i) a_{j,i}.
\end{aligned}
$$

We set $Ann_B(A) = \{\ b \in B \mid b_0 a = 0 \text{ for all } a \in A\}$.

Corollary

- $Ann_B(A) = \partial(A) = Leib(B)$;
- $B/Ann_B(A)$ is isomorphic to sl_2 as a Lie algebra.

$\mathbb{N}$-graded Vertex Algebras

Proposition [Gorbounov-Malikov-Schechtman]

If $V = \coprod_{n \in \mathbb{N}} V_{(n)}$ is a $\mathbb{N}$-graded vertex algebra then

- (i) $V_{(0)}$ is a commutative associative algebra with the identity $\mathbf{1}$ and $V_{(1)}$ is a Leibniz algebra.
- (ii) In fact, $V_{(0)} \oplus V_{(1)}$ is a 1-truncated conformal algebra.
- (iii) Moreover, $V_{(1)}$ is a vertex $V_{(0)}$-algebroid.

- Proposition

  Let $V = \oplus_{n=0}^{\infty} V_{(n)}$ be a $\mathbb{N}$-graded vertex algebra such that $V_{(0)}$ is a finite dimensional commutative associative algebra and $\dim V_{(0)} \geq 2$.

  If $V_{(0)}$ is a local algebra then V is indecomposable.

- Remark It was shown by Dong and Mason that if V is a $\mathbb{N}$-graded vertex operator algebra then

  V is indecomposable if and only if V_0 is a commutative associative local algebra.

  Note that in order to prove this statement one needs to have a Virasoro element.

Proposition

Let $V = \oplus_{n=0}^{\infty} V_{(n)}$ be a $\mathbb{N}$-graded vertex algebra that satisfies the following properties

- (a) $2 \leq \dim \ V_{(0)} < \infty$, $1 \leq \dim \ V_{(1)} < \infty$;
- (b) $V_{(0)}$ is not a trivial module for a Leibniz algebra $V_{(1)}$, $u_0 u \neq 0$ for some $u \in V_{(1)}$;
- (c) the Levi factor of $V_{(1)}$ equals $Span\{e, f, h\}$, $e_0 f = h$, $h_0 e = 2e$, $h_0 f = -2f$, and $e_1 f = k\hat{1}$. Here, $k \in \mathbb{C} \backslash \{0\}$.

Assume that one of the following statements hold.

- (I) $V_{(1)}$ is a simple Leibniz algebra;
- (II) $V_{(1)}$ is a semisimple Leibniz algebra and $Ker(D) \cap V_{(0)} = \{a \in V_{(0)} \mid b_0 a = 0 \text{ for all } b \in V_{(1)}\}$. Here, D is a linear operator on V such that $D(v) = v_{-2}\mathbf{1}$ for $v \in V$.

Then V is indecomposable.

Next, we recall definitions of a Lie algebroid and its module.

- Definitions
  - Let A be a commutative associative algebra. A Lie A-algebroid is a Lie algebra $\mathfrak{g}$ equipped with an A-module structure and a module action on A by derivation such that

$$[u, av] = a[u, v] + (ua)v, \quad a(ub) = (au)b$$

    for all $u, v \in \mathfrak{g}, a, b \in A$.
  - A module for a Lie A-algebroid $\mathfrak{g}$ is a vector space W equipped with a $\mathfrak{g}$-module structure and an A-module structure such that

$$u(aw) - a(uw) = (ua)w, \; a(uw) = (au)w$$

    for $a \in A$, $u \in \mathfrak{g}$, $w \in W$.

Example Let $V = \oplus_{n=0}^{\infty} V_{(n)}$ be a $\mathbb{N}$-graded vertex algebra such that $\dim V_{(0)} \geq 2$.

- $V_{(1)}/Ann_{V_{(1)}}(V_{(0)})$ is a Lie $V_{(0)}$-algebroid.
- $V_{(0)}$ is a module of the Lie $V_{(0)}$-algebroid $V_{(1)}/Ann_{V_{(1)}}(V_{(0)})$.

Theorem

Let $V = \oplus_{n=0}^{\infty} V_{(n)}$ be a $\mathbb{N}$-graded vertex algebra such that

- $2 \leq \dim V_{(0)} < \infty$, $1 \leq \dim V_{(1)} < \infty$, and
- V is generated by $V_{(0)}$ and $V_{(1)}$.

If V is simple then $V_{(0)}$ is a simple module for a Lie $V_{(0)}$-algebroid $V_{(1)}/Ann_{V_{(1)}}(V_{(0)})$.

Theorem

Let $V = \oplus_{n=0}^{\infty} V_{(n)}$ be a $\mathbb{N}$-graded vertex algebra that satisfies the following properties

- (a) $2 \leq \dim V_{(0)} < \infty$, $1 \leq dim V_{(1)} < \infty$, V is generated by $V_{(0)}$ and $V_{(1)}$;
- (b) $V_{(0)}$ is not a trivial module of a Leibniz algebra $V_{(1)}$, $u_0 u \neq 0$ for some $u \in V_{(1)}$;
- (c) the Levi Factor of $V_{(1)}$ equals $Span\{e, f, h\}$, $e_0 f = h$, $h_0 e = 2e$, $h_0 f = -2f$ and $e_1 f = k\mathbf{1}$. Here, $k \in \mathbb{C} \backslash \{0\}$.

Assume that one of the following statements hold.

- (i) $V_{(1)}$ is a simple Leibniz algebra;
- (ii) $V_{(1)}$ is a semisimple Leibniz algebra and $Ker(D) \cap V_{(0)} = \{a \in V_{(0)} \mid b_0 a = 0 \text{ for all } b \in V_{(1)}\}$. Here, D is a linear operator on V such that $D(v) = v_{-2}\mathbf{1}$ for $v \in V$.

Then V is indecomposable but not a simple vertex algebra.

Vertex Algebras associated with Vertex Algebroids

- Let A be a commutative associative algebra with the identity $\hat{1}$ and let B be a vertex A-algebroid.
- Let V_B be a $\mathbb{N}$-graded vertex algebra associated with the vertex A-algebroid B constructed by Gorbounov, Malikov and Schechtman.
- We have $(V_B)_{(0)} = A$ and $(V_B)_{(1)} = B$, and V_B as a vertex algebra is generated by $A \oplus B$. Furthermore, for any $n \geq 1$,

$$\begin{aligned}(V_B)_{(n)} \quad &= span\{b^1_{-n_1}.....b^k_{-n_k}\mathbf{1} \mid b^i \in B, \\ &\qquad n_1 \geq ... \geq n_k \geq 1, n_1 + ... + n_k = n\}.\end{aligned}$$

Proposition [Li-Y.]

The set of representatives of equivalence classes of simple $\mathbb{N}$-graded V_B-modules is equivalent to the set of representatives of equivalence classes of simple modules for the Lie A-algebroid $B/A\partial A$. Here $A\partial(A) = Span\{a \cdot \partial(a') \mid a, a' \in A\}$.

Theorem

Let A be a finite-dimensional commutative associative algebra with the identity $\hat{1}$ and $dim\ A \geq 2$.

Let B be a finite-dimensional vertex A-algebroid such that A is not a trivial module of the Leibniz algebra B.

Let S be its Levi factor such that $S = Span\{e, f, h\}$, $e_0 f = h$, $h_0 e = 2e$, $h_0 f = -2f$, and $e_1 f = k\hat{1}$. Here, $k \in \mathbb{C} \setminus \{0\}$.

Assume that one of the following statements hold.

- (I) B is simple Leibniz algebra;
- (II) B is a semisimple Leibniz algebra and $Ker(\partial) = \{a \in A \mid b_0 a = 0 \text{ for all } b \in B\}$.

We then have the following results:

- (i) V_B is indecomposable but not a simple $\mathbb{N}$-graded vertex algebra.
- (ii) The set of representatives of equivalence classes of finite-dimensional simple sl_2-modules is equivalent to the set of representatives of equivalence classes of simple $\mathbb{N}$-graded V_B-modules $N = \oplus_{n=0}^{\infty} N_{(n)}$ such that $dim\ N_{(0)} < \infty$.

Theorem

Let A be a finite-dimensional commutative associative algebra with the identity $\hat{1}$ and $dim\ A \geq 2$.

Let B be a finite-dimensional vertex A-algebroid that satisfies the given conditions in the previous theorem.

Let $(e_{-1}e)$ be an ideal of V_B that is generated by $e_{-1}e$.

Then

- (i) $(e_{-1}e) \cap A = \{0\}$, and $(e_{-1}e) \cap B = \{0\}$. Consequently,

$$(V_B/(e_{-1}e))_{(0)} = A, \text{ and } (V_B/(e_{-1}e))_{(1)} = B.$$

- (ii) $V_B/(e_{-1}e)$ is an indecomposable but not simple $\mathbb{N}$-graded vertex algebra.
- (iii) $V_B/(e_{-1}e)$ satisfies the C_2-condition.
- (iv) There are only two representatives of equivalence classes of simple $\mathbb{N}$-graded V_B-modules $N = \oplus_{n=0}^{\infty} N_{(n)}$ such that *dim* $N_{(0)} < \infty$.