# Weak Quasi-Hopf Algebras and Vertex Operator Algebras

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# Introduction

- Rational QFTs, when seen from a representation theory point of view, naturally give rise to fusion categories. When the QFT is unitary the corresponding representation category should also be unitary.
- In particular rational chiral CFTs are an important source very interesting fusion categories. In fact it has been conjectured that every unitary modular fusion category comes from a rational chiral CFT.
- Weak quasi-Hopf algebras are a generalization Drinfelds' quasi-Hopf algebras. Every fusion category is tensor equivalent to the representation category of a weak quasi-Hopf algebra.
- In this talk I will discuss some recent results showing that weak quasi-Hopf algebras are a useful and natural tool to understand certain aspects of the representation theory of rational VOAs especially for the unitarity and the relations to the theory of conformal nets.

# Tensor categories

- We denote the objects of a category $\mathcal{C}$ by $X, Y, Z, \dots \in \mathrm{Obj}(\mathcal{C})$ and the corresponding hom-spaces by $\mathrm{Hom}(X, Y) \dots \subset \mathrm{Hom}(\mathcal{C})$.
- In a linear category the hom-spaces are vector spaces (finite-dimensional and over $\mathbb{C}$ in this talk) and the composition is bilinear.
- In a tensor category we have a tensor product of objects $X, Y \mapsto X \otimes Y$ and a corresponding tensor product of arrows $T \in \mathrm{Hom}(X_1, Y_1), S \in \mathrm{Hom}(X_2, Y_2) \mapsto T \otimes S \in \mathrm{Hom}(X_1, \otimes X_2, Y_1 \otimes Y_2)$.
- We have a unit object $\iota \in \mathrm{Obj}(\mathcal{C})$ that is simple, i.e. $\mathrm{Hom}(\iota, \iota) = \mathbb{C}$ and that here we assume to be strict i.e. $\iota \otimes X = X \otimes \iota = X$ for all $X \in \mathrm{Obj}(\mathcal{C})$. Moreover, we have associativity isomorphisms $\alpha_{X,Y,Z} \in \mathrm{Hom}\big((X \otimes Y) \otimes Z), X \otimes (Y \otimes Z)\big)$ satisfying the so called pentagon equation. A tensor category is called strict if the tensor product is (strictly) associative and the associativity isomorphisms are the identity isomorphisms.

- To simplify the exposition I will only consider fusion categories. These are tensor categories with finitely many isomorphism classes of simple objects and which are rigid i.e. every object $X$ has a (two-sided) dual object $X^\vee$. The Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ generated by the isomorphism classes of simple objects is the fusion ring of the fusion category $\mathcal{C}$.
- A fusion category is called braided if it admits a natural family of isomorphisms $c_{X,Y} \in \mathrm{Hom}(X \otimes Y, Y \otimes X)$ satisfyinfg the so called hexagon equations. Braided fusion categories give rise to representations of the braid group.
- A braided fusion category with a compatible twist $X \mapsto \theta_X \in \mathrm{Hom}(X, X)$ is called a ribbon fusion category. A ribbon fusion category with a non-degenerate grading is called a modular fusion category. The latter defines a (projective) representation of the modular group $\mathrm{SL}(2, \mathbb{Z})$ trough the modular matrices $S$, $T$.
- Some examples of fusion categories are: Vec; $\mathrm{Vec}_G$; $\mathrm{Rep}(G)$ ($G$ finite group); $\mathrm{Vec}_G^\omega$ ($\omega$ 3-cocycle on $G$); $\mathrm{Rep}(A)$ ($A$ finite dimensional semisimple Hopf algebra).

# Unitary fusion categories

- A C*-category (with f.d. hom spaces) is a linear category with a *-structure on the home spaces. This means that there is an anti-linear involutive map $\mathrm{Hom}(X, Y) \ni T \mapsto T^* \in \mathrm{Hom}(Y, X)$ such that $(TS)^* = S^* T^*$. Moreover we have the positivity condition $T^* T = 0 \Rightarrow T = 0$.
- A unitary (or C*) fusion category is a fusion category which is also a C*-category and such that $(T \otimes S)^* = T^* \otimes S^*$. Moreover the associativity isomorphisms are unitary, i.e. $\alpha^*_{X,Y,Z} = \alpha^{-1}_{X,Y,Z}$.
- Some examples of unitary fusion categories are: Hilb; $\mathrm{Hilb}_G$; $\mathrm{Hilb}_G^\omega$; $\mathrm{Rep}^u(G)$ ...

# Fusion categories from chiral CFT

- There are two main approaches to chiral (2D) CFT: VOAs and conformal nets. Under suitable rationality conditions they both give rise to modular fusion categories.
- If $V$ is strongly rational VOA then $\mathrm{Rep}(V)$ is a modular fusion categories (Huang 2008).
- A conformal net $\mathcal{A}$ is an inclusion preserving map $S^1 \ni I \mapsto \mathcal{A}(I)$, where each $\mathcal{A}(I)$ is a von Neumann algebra acting on a fixed Hilbert space $\mathcal{H}$. The map is assumed to satisfy various natural assumptions: locality, conformal covariance, positivity of the energy ..... Conformal nets have interesting representation theories.
- If $\mathcal{A}$ is a completely rational then $\mathrm{Rep}(\mathcal{A})$ is a unitary modular fusion category (Kawahigashi, Longo Mueger (2001)).

# Hopf algebras and generalizations

- Original motivation for Hopf algebras: algebraic topology (50s) Further motivations: duality for locally compact groups (G. Kac 60s); quantum groups (Drinfeld-Jimbo, Woronowicz 80s). Here I will focus on the representation theory aspects
- A Hopf algebra is a quadruple $(A, \Delta, \varepsilon, S)$. Here $A$ is a unital associative algebra (over $\mathbb{C}$ in this talk), the coproduct $\Delta : A \to A \otimes A$ is a unital homomorphism, the counit $\varepsilon : A \to \mathbb{C}$ is a nonzero homomorphism and the antipode $S : A \to A$ is an antiautomorphism $+$ axioms
- The coproduct gives a tensor structure on Rep$(A)$. The tensor product $\underline{\otimes}$ on the objects of Rep$(A)$ is then given by $\pi_1 \underline{\otimes} \pi_2 := \pi_1 \otimes \pi_2 \circ \Delta \in \text{Rep}(A)$.
- If $A$ is finite dimensional and semisimple then Rep$(A)$ is a fusion category. In fact the category is strict bacause the coproduct is assumed to be coassociative: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$

- A paradigmatic example is the group algebra $A := \mathbb{C}G$ which admit a natural Hopf algebra structure so that Rep$(A)$ becomes tensor equivalent to Rep$(G)$.
- By relaxing coassociativity one obtain the notion of quasi-Hopf algebra first introduced by Drinfeld. These allows more flexibility in dealing with non strict tensor categories: non-trivial associators $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$. This is done trough a suitable element $\Phi \in A \otimes A \otimes A$ satisfying a 3-cocycle condition related to the pentagon equation. Accordingly the data of a quasi-Hopf algebra is given by a quintuple $(A, \Delta, \varepsilon, S, \Phi)$
- Quasi-Hopf algebras are not sufficiently general to describe many interesting fusion categories related to QFT. This is because, when $A$ is semisimple, the function $D$ on the fusion ring $\mathrm{Gr}(\mathrm{Rep}(A))$ defined by $D([\pi]) := \dim(V_\pi)$, where $V_\pi$ is the representation space of $\pi$, is a positive integral valued dimension function and hence it must agree with the Frobenius-Perron dimension of the category which in general is not integer valued. For example it can take the values $D([\pi]) = 2\cos(\frac{\pi}{n})$, n=3, 4, 5, . . . .

- In the early 90s Mack and Schoumerus suggested the following solution to the above problem: give up to the request that $\Delta$ is unital so that a wak quasi-Hopf algebra is again a quintuple $(A, \Delta, \varepsilon, S, \Phi)$ with a possibly non-unital coproduct.
- In this way $\Delta(1_A)$ is an idempotent in $A \otimes A$ commuting with $\Delta(A)$ but typically different from $1_A \otimes 1_A$.
- The tensor product $\pi_1 \underline{\otimes} \pi_2$ in Rep$(A)$ is now defined by the restriction of $\pi_1 \otimes \pi_2 \circ \Delta$ to $\pi_1 \otimes \pi_2 \circ \Delta(1_A) V_{\pi_1} \otimes V_{\pi_2}$.
- Now, for a given (f.d., semisimple) $A$, the additive function $D : \mathrm{Gr}(\mathrm{Rep}(A)) \to \mathbb{Z}_{>0}$ defined by $D([\pi]) := \dim(V_\pi)$ is only a weak dimension function i.e. it satisfies $D([\pi_1 \underline{\otimes} \pi_2]) \leq D([\pi_1])D([\pi_2])$, $D([\iota]) = 1$ and $D(\overline{\pi}) = D(\pi)$ and this gives no important restrictions.

# Tannakian results

- The following result are due mainly due to Häring-Oldenburg (1997).
- Let $\mathcal{C}$ be a fusion category and $D : \mathrm{Gr}(\mathcal{C}) \to \mathbb{Z}_{\geq 0}$ be an integral weak dimension then there exists a finite dimensional semisimple weak quasi-Hopf algebra $(A, \Delta, \varepsilon, S, \Phi)$ and a tensor equivalence $\mathscr{F} : \mathcal{C} \to \mathrm{Rep}(A)$ such that $D([X]) = \dim(V_{\mathscr{F}(X)})$ for all $X \in \mathrm{Obj}(\mathcal{C})$.
- Extra structure on $\mathcal{C}$ gives extra structure on $A$: brading $\leftrightarrow$ $R$-matrix ; C*-tensor structure on $\mathcal{C} \leftrightarrow \Omega$ - involutive structure on $A$ (in particular $A$ is a C*-algebra).
- The weak quasi-Hopf algebra associated to a fusion category $\mathcal{C}$ is highly non-unique: it depends on the choice of $D$ and, once $D$ is fixed is only defined up to a "twist".

# From VOAs to conformal nets

- A general connection between VOAs and conformal nets has been recently considered by Carpi, Kawahigashi, Longo and Weiner (2018).
- One first need to consider only unitary VOAs (explicitly defined by Dong, Lin and CKLW).
- For sufficiently nice (simple) unitary VOAs called strongly local one can define a map $V \mapsto \mathcal{A}_V$ into the class of conformal nets.
- Conjecture 1: The map $V \mapsto \mathcal{A}_V$ gives a one-to-one correspondence between the class of simple unitary VOAs and the class of conformal nets.
- Conjecture 2: The map $V \mapsto \mathcal{A}_V$ gives gives a one-to-one correspondence between the class of strongly rational unitary VOAs and the class of completely rational conformal nets. Moreover, if $V$ is completely rational we have a tensor equivalence $\mathrm{Rep}(V) \simeq \mathrm{Rep}(\mathcal{A}_V)$.

- Recently it has been suggested by Carpi, Weiner and Xu (in preparation) to consider a strong integrability condition on unitary VOA-modules of a strongly local $V$ which allows to define a map $M \mapsto \pi_M$ from $V$-modules to representations of $\mathcal{A}_V$. In certain cases this gives an isomorphism of linear C*-categories $\mathscr{F} : \mathrm{Rep}^u(V) \to \mathrm{Rep}(\mathcal{A}_V)$ where $\mathrm{Rep}^u(V)$ is the linear C*-category of unitary $V$-modules. Further results in this direction have been recently obtained by Bin Gui and by James Tener.
- Conjecture 3: Assume that $V$ is strongly rational and strongly local. Then $\mathrm{Rep}^u(V)$ can be upgraded to a unitary modular tensor category such that the forgetful functor : $\mathrm{Rep}^u(V) \to \mathrm{Rep}(V)$ is a braided tensor equivalence. Moreover, the functor $\mathscr{F} : \mathrm{Rep}^u(V) \to \mathrm{Rep}(\mathcal{A}_V)$ discussed above admits a unitary tensor structure.

# From VOAs to unitary fusion categories

- The following result has been obtained using weak quasi-Hopf algebra techniques.
- Theorem (Carpi, Ciamprone, Pinzari): Let $V$ be a strongly rational VOA. Assume that every $V$-module is unitarizable and that $\mathrm{Rep}(V)$ is tensor equivalent to a unitary fusion category. Then, $\mathrm{Rep}^u(V)$ can be upgraded to a unitary fusion category such that the forgetful functor : $\mathrm{Rep}^u(V) \to \mathrm{Rep}(V)$ is a tensor equivalence. Moreover, the corresponding unitary tensor structure on $\mathrm{Rep}^u(V)$ is unique up to unitary tensor equivalence.

- Let $\mathfrak{g}$ be a complex simple Lie algebra, let $k$ be a positive integer and let $V_{\mathfrak{g}_k}$ be the corresponding simple level $k$ affine VOA. It is known that $V_{\mathfrak{g}_k}$ is a unitary strongly rational VOA and that every $V_{\mathfrak{g}_k}$-module is unitarizable.
- By a result of Finkelberg (1996) based on the work Kazhdan and Lusztig we know that $\mathrm{Rep}(V_{\mathfrak{g}_k})$ is tensor equivalent to the "semisimplified" category $\widetilde{\mathrm{Rep}(G_q)}$ associated to the representations of the quantum group $G_q$, with $G$ the simply connected compact Lie group associated to $\mathfrak{g}$ and $q = e^{\frac{i\pi}{d(k+h^\vee)}}$, $h^\vee =$ dual Coxeter number, $d = 1$ if $\mathfrak{g}$ is ADE, $d = 2$ if $\mathfrak{g}$ is BCF and $d = 3$ if $\mathfrak{g}$ is $G_2$.

- It was shown by Wenzl and Xu (1998) that $\widetilde{\mathrm{Rep}}(G_q)$ is tensor equivalent to a unitary fusion category.
- As a consequence we get that $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ admits an essentially unique structure of unitary fusion category.
- An equivalent result has been proved by Gui in a series of papers appeared in the arXiv between 2017 and 2018 for the Lie types $A, B, C, D, G_2$, by a completely different method based on Connes fusions for bimodules and a deep analysis of the analytic properties of the smeared intertwiners operators for VOA modules. Besides these unitarity results Gui also proved Conjecture 3 in a remarkable class of examples (including unitary affine VOAs for Lie types $A, C, G_2$).
- Our method works also for many other VOAs such as e.g. lattice VOAs, holomorphic orbifolds, . . . . We hope that it could be useful in order to prove Conjecture 3 in the cases not covered in the work of Gui.

# The Zhu algebra as a weak quasi-Hopf algebra

- Let $V$ be strongly rational. In 1998 Zhu introduced a finite-dimensional semisimple algebra $A(V)$ gave a linear equivalence $\mathscr{F}_V : \mathrm{Rep}(V) \to \mathrm{Rep}(A(V))$.
- If $D_V([M]) := \dim(\mathscr{F}_V(M))$ defines a weak dimension function then, it follows from the previously described Tannakian results that $A(V)$ can be upgraded to a weak-quasi Hopf algebra and $\mathscr{F}_V : \mathrm{Rep}(V) \to \mathrm{Rep}(A(V))$ to an equivalence of fusion categories.
- $D_V$ is not always a weak dimension function. A counterexample is given e.g. by the Ising VOA ($c = \frac{1}{2}$ Virasoro). However $D_V$ is a weak dimension function in many interesting cases e.g. if $V$ is a unitary affine VOA.

# Classification of type A ribbon fusion categories

- As another application of the theory of weak quasi-Hopf algebra we give classification of pseudo-unitary type A fusion categories.
- The starting point is the work of Kazhdan and Wenzl (1993) on the classification of type A tensor categories.
- As a consequence of our results we have in particular the following: Let $\mathcal{C}$ be a modular fusion category with modular matrices $S$, $T$ coinciding with the Kac-Peterson matrices for the $\mathfrak{sl}(n)$ affine Lie algebra at positive integer level $k$ . Then $\mathcal{C}$ is ribbon equivalent to $\mathrm{Rep}(V_{\mathfrak{sl}(n)_k})$

- Consequence 1. Conjecture 3 is true for unitary affine VOAs of type A. In fact we have a unitary ribbon equivalence $\mathscr{F}: \mathrm{Rep}(V_{\mathfrak{sl}(n)_k}) \to \mathrm{Rep}(\mathcal{A}_{V_{\mathfrak{sl}(n)_k}})$. As already mentioned the same result has been independently obtained by Bin Gui by different methods (direct analytic proof instead of classification).
- Consequence 2. We have a proof of Finkelberg equivalence in the type A case.

THANK YOU!