Principal subspaces for the affine Lie algebras of type F, E and D

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joint work with Slaven Kožić

PROVEDBA VRHUNSKIH ISTRAŽIVANJA U SKLOPU
ZNANSTVENOG CENTRA IZVRSNOSTI
QuantiXLie
ZA KVANTNE I KOMPLEKSNE SUSTAVE
TE REPREZENTACIJE LIEJEVIH ALGEBRI

Representation Theory XVI

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Plan of this talk

- construction of combinatorial basis of principal subspace of standard module $L(\Lambda_0)$ of the affine Lie algebra of type $F_4^{(1)}$
- characters principal subspaces of affine Lie algebra of type F, E and D

Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

Projekt KK.01.1.1.01.0004

Projekt je sufinancirala Europska unija iz Europskog fonda za regionalni razvoj. Sadržaj ovog seminara isključiva je odgovornost Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu te ne predstavlja nužno stajalište Europske unije.

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Principal subspace

$\mathfrak{g}$ simple Lie algebra of type X_l

$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ affine Kac-Moody Lie algebra of type $X_l^{(1)}$

- $[x(j_1), y(j_2)] = [x, y]\,(j_1 + j_2) + \langle x, y \rangle\, j_1 \delta_{j_1+j_2,0}\, c, \quad [d, x(j)] = jx(j),$ where $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}$ and $j \in \mathbb{Z}$
- $\widetilde{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}], \quad \mathfrak{n}_+ = \oplus_{\alpha \in R_+} \mathfrak{n}_\alpha, \quad \mathfrak{n}_\alpha = \mathbb{C}x_\alpha$

V highest weight $\widetilde{\mathfrak{g}}$-module with highest weight vector v

Principal subspace of V

$$W_V := U(\widetilde{\mathfrak{n}}_+)v$$

Modules of affine Lie algebra

$k \in \mathbb{N}$

$N(k\Lambda_0)$ - generalized Verma module

$$N(k\Lambda_0) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_+)} \mathbb{C} v_{N(k\Lambda_0)} \overset{\text{PBW}}{\cong} U(\tilde{\mathfrak{g}}_-)$$

- $\tilde{\mathfrak{g}}_+ = \bigoplus_{n \geq 0} (\mathfrak{g} \otimes t^n) \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{g}}_- = \bigoplus_{n<0} (\mathfrak{g} \otimes t^n)$ - subalgebras of $\tilde{\mathfrak{g}}$
- $1 \otimes v_{N(k\Lambda_0)} = v_N$ - highest weight vector
- a vertex operator algebra with a vacuum vector v_N

$L(k\Lambda_0)$ - standard (integrable highest weight) $\tilde{\mathfrak{g}}$-module

- v_L - a highest weight vector of $L(k\Lambda_0)$
- simple vertex operator algebra
- every level k standard $\tilde{\mathfrak{g}}$-module is $L(k\Lambda_0)$-module

Quasi-particles

- $\alpha_i, \quad 1 \leq i \leq l,$
$$x_{\alpha_i}(z) = \sum_{m \in \mathbb{Z}} x_{\alpha_i}(m) z^{-m-1} = Y(x_{\alpha_i}(-1)v_N, z)$$
- $x_{\alpha_i}(m)$ - quasi-particle of color i, charge 1 and energy $-m$
- $r \in \mathbb{N}$, $m \in \mathbb{Z}$

Quasi-particle of color i, charge r and energy -m

$$x_{r\alpha_i}(m) = \sum_{\substack{m_1, \ldots, m_r \in \mathbb{Z} \\ m_1 + \cdots + m_r = m}} x_{\alpha_i}(m_r) \cdots x_{\alpha_i}(m_1)$$

$$x_{r\alpha_i}(z) = Y(x_{\alpha_i}(-1)^r v_N, x) = \sum_{m \in \mathbb{Z}} x_{r\alpha_i}(m) z^{-m-r}$$

Character of the principal subspace

- $V = N(k\Lambda_0)$ or $V = L(k\Lambda_0)$
- $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ the Cartan subalgebra
- $(W_V)_{-m\delta + r_1\alpha_1 + \ldots + r_l\alpha_l}$ the weight subspaces of W_V with respect to $\widetilde{\mathfrak{h}}$

Character of the principal subspace

$$ch\, W_V = \sum_{m, r_1, \ldots, r_l \geq 0} dim\, (W_V)_{-m\delta + r_1\alpha_1 + \ldots + r_l\alpha_l} q^m \prod_{i=1}^{l} y_i^{r_i}$$

Motivation

- principal subspaces were first introduced and studied by Feigin-Stoyanovsky

  B. Feigin and A. Stoyanovsky, Quasi-particles models for the representations of Lie algebras and geometry of flag manifold; arXiv:hep-th/9308079.

- if $\tilde{\mathfrak{g}}$ is of type $A_1^{(1)}$ we have a connection of ch $W_{L(\Lambda_i)}$, $i = 0, 1$ with Rogers-Ramanujan identities

  - ch $W_{L(\Lambda_0)} = \sum_{r\geq 0} \frac{q^{r^2}}{(q;q)_r} = \prod_{i\geq 0} \frac{1}{(1-q^{5i+1})(1-q^{5i+4})}$, where $(q;q)_r = \prod_{i=1}^{r}(1-q^i)$

  - ch $W_{L(\Lambda_1)} = \sum_{r\geq 0} \frac{q^{r^2+r}}{(q;q)_r} = \prod_{i\geq 0} \frac{1}{(1-q^{5i+2})(1-q^{5i+3})}$

Motivation

- G. Georgiev constructed combinatorial bases of principal subspaces of $L(k_0\Lambda_0 + k_j\Lambda_j)$ of $A_l^{(1)}$

G. Georgiev, *Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace*, J. Pure Appl. Algebra **112** (1996), 247–286; arXiv:hep-th/9412054.

$$\mathrm{ch}\ W_{L(k_0\Lambda_0+k_j\Lambda_j)} = \sum_{r_1^{(1)}\geqslant\cdots\geqslant r_1^{(k)}\geqslant 0} \frac{q^{\sum_{t=1}^k r_1^{(t)^2} - \sum_{t=1}^k r_1^{(t)}\delta_{1,j_t}}}{(q;q)_{r_i^{(1)}-r_i^{(2)}}\cdots(q;q)_{r_i^{(k)}}} y_1^{n_1}$$

$$\cdots$$

$$\sum_{r_l^{(1)}\geqslant\ldots\geqslant r_l^{(k)}\geqslant 0} \frac{q^{\sum_{t=1}^k r_l^{(t)^2} - \sum_{t=1}^k r_{l-1}^{(t)} r_l^{(t)} - \sum_{t=1}^k r_l^{(t)}\delta_{l,j_t}}}{(q;q)_{r_l^{(1)}-r_l^{(2)}}\cdots(q;q)_{r_i^{(k)}}} y_l^{n_l},$$

where $n_i = \sum_{t=1}^k r_i^{(t)}$ for $i = 1, \cdots, l$.

Characters of principal subspaces $W_{L(k\Lambda_0)}$

Characters of the principal subspace $W_{L(k\Lambda_0)}$

For any untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$ *and for any integer* $k \geqslant 1$ *we have*

$$chW_{L(k\Lambda_0)} = \sum_{\substack{r_1^{(1)} \geq \cdots \geq r_1^{(k_1)} \geq 0 \\ \vdots \\ r_l^{(1)} \geq \cdots \geq r_l^{(k_l)} \geq 0}} \frac{q^{\sum_{i=1}^{l} \sum_{t=1}^{k_i} r_i^{(t)^2} - \sum_{i=2}^{l} \sum_{t=1}^{k} \sum_{p=0}^{\mu_i - 1} r_{i'}^{(t)} r_i^{(\mu_i t - p)}}}{\prod_{i=1}^{l} (q)_{r_i^{(1)} - r_i^{(2)}} \cdots (q)_{r_i^{(k_i)}}} \prod_{i=1}^{l} y_i^{n_i},$$

where $n_i = \sum_{t \geq 1} r_i^{(t)}$ *for* $i = 1, \ldots, l$, $\mu_i = \frac{k_i}{k_i'}$,

$$k_i = \frac{2k}{\langle \alpha_i, \alpha_i \rangle}$$

and where

$$i' = \begin{cases} l-2, & \text{if } i = l \text{ and } \mathfrak{g} = D_l, \\ 3, & \text{if } i = l \text{ and } \mathfrak{g} = E_6, E_7, \\ 5, & \text{if } i = l \text{ and } \mathfrak{g} = E_8, \\ i-1, & \text{otherwise.} \end{cases}$$

Motivation

- $W_{N(k\Lambda_0)} = U(\widetilde{\mathfrak{n}}_+)v_N$
- $\widetilde{\mathfrak{n}}_+^{<0} = \mathfrak{n}_+ \otimes t^{-1}\mathbb{C}\left[t^{-1}\right]$

Isomorphism of $\widetilde{\mathfrak{n}}_+^{<0}$-modules

$$W_{N(k\Lambda_0)} \cong U(\widetilde{\mathfrak{n}}_+^{<0})$$

- if $\widetilde{\mathfrak{g}}$ is of type $A_1^{(1)}$, then
  - ch $W_{N(k\Lambda_0)} = \frac{1}{\prod_{m\geq 0}(1-yq^m)} = 1 + \sum_{\substack{r_1^{(1)}\geq\cdots\geq r_1^{(m)}\geq\cdots\geq 0 \\ m\geq 1}} \frac{q^{\sum_{t=1}^m r^{(t)^2}}}{(q;q)_{r^{(1)}-r^{(2)}}\cdots(q;q)_{r^{(m)}}}y^r$

- generalization of the theorem of Euler and Cauchy:

$$\frac{1}{(yq)_\infty} = \sum_{m=0}^{\infty} \frac{q^{m^2}y^m}{(q)_m(yq)_m}$$

Main result

Identites from the characters of $W_{N(k\Lambda_0)}$

For any untwisted affine Lie algebra $\tilde{\mathfrak{g}}$ we have

$$\frac{1}{\prod_{\alpha\in R_+}(\alpha;q)_\infty} =$$

$$= \sum_{\substack{r_1^{(1)}\geq\cdots\geq r_1^{(m)}\geq\cdots\geq 0\\ \vdots \\ r_l^{(1)}\geq\cdots\geq r_l^{(m)}\geq\cdots\geq 0}} \frac{q^{\sum_{i=1}^{l}\sum_{t\geq 1} r_i^{(t)^2} - \sum_{i=2}^{l}\sum_{t\geq 1}\sum_{p=0}^{\mu_i-1} r_{i'}^{(t)} r_i^{(\mu_i t-p)}}}{\prod_{i=1}^{l}\prod_{j\geq 1}(q)_{r_i^{(j)}-r_i^{(j+1)}}} \prod_{i=1}^{l} y_i^{n_i},$$

where $n_i = \sum_{t\geq 1} r_i^{(t)}$ *for* $i = 1,\ldots,l$, $\mu_i = \frac{k_i}{k_{i'}}$,

$$k_i = \frac{2k}{\langle\alpha_i,\alpha_i\rangle}$$

and where

$$i' = \begin{cases} l-2, & \text{if } i = l \text{ and } \mathfrak{g} = D_l,\\ 3, & \text{if } i = l \text{ and } \mathfrak{g} = E_6, E_7,\\ 5, & \text{if } i = l \text{ and } \mathfrak{g} = E_8,\\ i-1, & \text{otherwise,} \end{cases}$$

$$(\alpha;q)_\infty = (qy_1^{a_1};q)_\infty\cdots(qy_l^{a_l};q)_\infty$$

$$(qy_1^{a_1};q)_\infty = \prod_{r\geq 1}(1-qy_1^{a_1}q^{r-1}),$$

$\alpha = \sum_{i=1}^{l} a_i\alpha_i$, *and the sum on the right hand side of goes over all descending infinite sequences of nonnegative integers with finite support.*

Principal subspace $W_{L(\Lambda_0)}$ of $F_4^{(1)}$

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ} - \underset{\alpha_4}{\circ}$$

- $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
- $\mathfrak{n}_\alpha = \mathbb{C}x_\alpha, \ \alpha \in R_+$
- $\mathfrak{n}_+ = \oplus_{\alpha \in R_+} \mathfrak{n}_\alpha$

$$W_{L(\Lambda_0)} = U(\widetilde{\mathfrak{n}}_+)v_L \subset L(\Lambda_0)$$

Lemma

$$W_{L(\Lambda_0)} = U(\widetilde{\mathfrak{n}}_{\alpha_4})U(\widetilde{\mathfrak{n}}_{\alpha_3})U(\widetilde{\mathfrak{n}}_{\alpha_2})U(\widetilde{\mathfrak{n}}_{\alpha_1})v_L$$

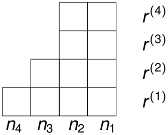
- $\widetilde{\mathfrak{n}}_{\alpha_i} = \mathfrak{n}_{\alpha_i} \otimes \mathbb{C}[t, t^{-1}]$

Quasi-particle monomials

$$x_{n_{r^{(1)},i}\alpha_i}(m_{r^{(1)},i}) \cdots x_{n_{2,i}\alpha_i}(m_{2,i}) x_{n_{1,i}\alpha_1}(m_{1,i})$$

- color-type r_i; $\quad \sum_{p=1}^{r_i^{(1)}} n_{p,i} = r_i$
- charge-type $\left(n_{r^{(1)},i}, \ldots, n_{1,i}\right)$; $\quad 0 \leq n_{r^{(1)},i} \leq \cdots \leq n_{1,i}$,
- dual-charge-type $(r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(s)})$;

$$r_i^{(1)} \geq r_i^{(2)} \geq \ldots \geq r_i^{(s)} \geq 0, \quad \sum_{p=1}^{s} r_i^{(p)} = r_i$$



Principal subspace $W_{L(\Lambda_0)}$ of $F_4^{(1)}$

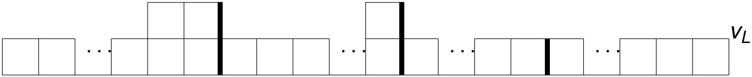
- VOA relations on $L(\Lambda_0)$ (Dong-Lepowsky, Li, Meurman-Primc):

- $x_{2\alpha_i}(z) = 0 \Rightarrow x_{2\alpha_i}(m)v_L = 0, \quad m \in \mathbb{Z}, \quad 1 \leq i \leq 2$
- $x_{3\alpha_i}(z) = 0 \Rightarrow x_{3\alpha_i}(m)v_L = 0, \quad m \in \mathbb{Z}, \quad 3 \leq i \leq 4$

$$b(\alpha_4) \cdots b(\alpha_1) v_L$$

$$x_{n_{r_4^{(1)},4}\alpha_l}(m_{r_4^{(1)},4}) \cdots x_{n_{1,4}\alpha_4}(m_{1,4}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(m_{1,1}) v_L$$

- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i} \leq 1, \quad 1 \leq i \leq 2$
- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i} \leq 2, \quad 3 \leq i \leq 4$



Difference conditions for $i = 1$

$$x_{\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{\alpha_1}(m_{2,1}) x_{\alpha_1}(m_{1,1})$$

- $m_{1,1} \leq -1$ initial condition

Difference conditions for $i = 1$

$$x_{\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{\alpha_1}(m_{2,1}) x_{\alpha_1}(m_{1,1})$$

- $m_{1,1} \leq -1$ initial condition
- $m_{p+1,1} \leq m_{p,1} - 2$, $1 \leq p \leq r_1^{(1)}$ difference two condition

Difference conditions for $i = 1$

$$x_{\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{\alpha_1}(m_{2,1}) x_{\alpha_1}(m_{1,1})$$

- $m_{1,1} \leq -1$ initial condition
- $m_{p+1,1} \leq m_{p,1} - 2$, $1 \leq p \leq r_1^{(1)}$ difference two condition

Relation among quasi-particles of the same color

$$x_{2\alpha}(z) = 0 \Rightarrow$$

$$\Rightarrow \sum_{m_1+m_2=-2m} x_\alpha(m_1)x_\alpha(m_2) = 0, \quad \sum_{m_1+m_2=-2m-1} x_\alpha(m_1)x_\alpha(m_2) = 0$$

Difference conditions for $i = 1$

$$m_{p,1} \leq -1 - 2(p-1), \quad 1 \leq p \leq r_1^{(1)}$$

$$(q;q)_r =$$

$$= \sum_{m \geq 0} \{ \text{ number of partitions of m with at most r parts} \} \, q^m$$

$$\sum_{p=1}^{r_1^{(1)}} (-1 - 2(p-1)) = {r_1^{(1)}}^2$$

- contribution to the character:

$$\sum_{r_1^{(1)} \geq 0} \frac{q^{{r_1^{(1)}}^2}}{(q;q)_{r_1^{(1)}}} y^{r_1^{(1)}}$$

Difference conditions for $i = 2$

$$x_{\alpha_2}(m_{r_2^{(1)},2}) \cdots x_{\alpha_2}(m_{1,2}) x_{\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{\alpha_1}(m_{2,1}) x_{\alpha_1}(m_{1,1})$$

- $m_{p+1,2} \leq m_{p,2} - 2$, $1 \leq p \leq r_2^{(1)}$ difference two condition
- $m_{p,2} \leq -1 + r_1^{(1)}$, $1 \leq p \leq r_2^{(1)}$

Difference conditions for $i = 2$

$$x_{\alpha_2}(m_{r_2^{(1)},2}) \cdots x_{\alpha_2}(m_{1,2}) x_{\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{\alpha_1}(m_{2,1}) x_{\alpha_1}(m_{1,1})$$

- $m_{p+1,2} \leq m_{p,2} - 2$, $1 \leq p \leq r_2^{(1)}$ difference two condition
- $m_{p,2} \leq -1 + r_1^{(1)}$, $1 \leq p \leq r_2^{(1)}$

Relation among quasi-particles of different colors

$$\prod_{p=1}^{r_2^{(1)}} \prod_{q=1}^{r_1^{(1)}} (1 - \frac{z_{q,1}}{z_{p,2}}) x_{\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{\alpha_2}(z_{1,2}) x_{\alpha_1}(z_{r_1^{(1)},1}) \cdots x_{\alpha_1}(z_{1,1}) v_L$$

$$\in \prod_{p=1}^{r_2^{(1)}} z_{p,2}^{-r_1^{(1)}} W_{L(\Lambda_0)} \left[\left[z_{1,1}, \ldots, z_{r_2^{(1)},2}\right]\right]$$

Difference conditions for $i = 2$

$$m_{p,2} \leq -1 - 2(p-1) + r_1^{(1)}, \quad 1 \leq p \leq r_2^{(1)}$$

- contribution to the character:

$$\sum_{r_2^{(1)} \geq 0} \frac{q^{{r_2^{(1)}}^2 - r_1^{(1)} r_2^{(1)}}}{(q)_{r_2^{(1)}}} y^{r_2^{(1)}}$$

Difference conditions for $i = 3$

$$x_{n_{r_3^{(1)},3}\alpha_3}(m_{r_3^{(1)},3}) \cdots x_{n_{1,3}\alpha_3}(m_{1,3}) x_{\alpha_2}(m_{r_2^{(1)},2}) \cdots x_{\alpha_2}(m_{2,2}) x_{\alpha_2}(m_{1,2})$$

- $m_{p,3} \leq -n_{p,3}$

Difference conditions for $i = 3$

$$x_{n_{r_3^{(1)},3}\alpha_3}(m_{r_3^{(1)},3}) \cdots x_{n_{1,3}\alpha_3}(m_{1,3}) x_{\alpha_2}(m_{r_2^{(1)},2}) \cdots x_{\alpha_2}(m_{2,2}) x_{\alpha_2}(m_{1,2})$$

- $m_{p,3} \leq -n_{p,3}$
- $x_{3\alpha_3}(z) = 0$

Difference conditions for $i = 3$

Relation among quasi-particles of different colors

$$\prod_{p=1}^{r_3^{(1)}} \prod_{q=1}^{r_2^{(1)}} \left(1 - \frac{z_{q,1}}{z_{p,2}}\right)^{\min\{2n_{q,2}, n_{p,3}\}} x_{\alpha_3}(z_{r_3^{(1)},3}) \cdots x_{\alpha_3}(z_{1,3})$$

$$x_{\alpha_2}(z_{r_2^{(1)},2}) \cdots x_{\alpha_2}(z_{1,2}) v_L$$

$$\in \prod_{p=1}^{r_3^{(1)}} z_{p,2}^{-\sum_{q=1}^{r_2^{(1)}} \min\{2n_{q,2}, n_{p,3}\}} W_{L(\Lambda_0)} \left[\left[z_{1,2}, \ldots, z_{r_3^{(1)},3}\right]\right]$$

$$m_{p,3} \leq -n_{p,3} + \sum_{q=1}^{r_2^{(1)}} \min\left\{2n_{q,2}, n_{p,3}\right\}, \quad 1 \leq p \leq r_3^{(1)}$$

Difference conditions for $i = 3$

Theorem (Feigin-Stoyanovsky, Georgiev, Jerković-Primc)

For charges n_1 and n_2 such that $n_2 \leqslant n_1$, we have:

$$\left(\frac{d^p}{dz^p}x_{n_2\alpha_i}(z)\right)x_{n_1\alpha_i}(z) = A_p(z)x_{(n_1+1)\alpha_i}(z) + B_p(z)\frac{d^p}{dz^p}x_{(n_1+1)\alpha_i}(z),$$

where $p = 0, 1, \ldots, 2n_2 - 1$ and $A_p(z), B_p(z)$ are some formal series with coefficients in the set of quasi-particle polynomials.

Difference conditions for $i = 3$

- express 2 monomials of the form

$$x_{\alpha_3}(m_{p+1,3})x_{2\alpha_3}(m_{p,3}), x_{\alpha_3}(m_{p+1,3}-1)x_{2\alpha_i}(m_{p,3}+1)$$

  as a linear combination of monomials

$$x_{\alpha_3}(j_2)x_{2\alpha_i}(j_1) \quad \text{such that} \quad j_2 \leqslant m_{p+1,3}-2, \quad j_1 \geqslant m_{p,3}+2 \text{ and}$$

$$j_1 + j_2 = m_1 + m_2$$

- for $n_{p+1} = n_p$ one can express $2n_{p+1}$ monomials

$$x_{n_{p+1}\alpha_3}(m_2)x_{n_p\alpha_3}(m_1) \quad \text{with} \quad m_1 - 2n_2 < m_2 \leqslant m_1$$

  express as a linear combination of monomials

$$x_{n_{p+1}\alpha_3}(j_2)x_{n_p\alpha_3}(j_1) \quad \text{such that} \quad j_2 \leqslant j_1 - 2n_2$$

  and monomials which contain a quasi-particle of color i and charge $n_p + 1$.

Difference conditions for $i = 3$

$$m_{p,3} \leq -n_{p,3} - 2n_{p,3}(p-1) + \sum_{q=1}^{r_2^{(1)}} \min\left\{2n_{q,2}, n_{p,3}\right\}, \quad 1 \leq p \leq r_3^{(1)}$$

$$m_{p+1,3} \leq m_{p,3} - 2n_{p,3}, \quad n_{p+1,3} = n_{p,3}$$

Difference conditions for $i = 3$

$$m_{p,3} \leq -n_{p,3} - 2n_{p,3}(p-1) + \sum_{q=1}^{r_2^{(1)}} \min\left\{2n_{q,2}, n_{p,3}\right\}, \quad 1 \leq p \leq r_3^{(1)}$$

$$m_{p+1,3} \leq m_{p,3} - 2n_{p,3}, \quad n_{p+1,3} = n_{p,3}$$

- $n_{p,3} = 1$, for $1 \leq p \leq r_3^{(2)} + 1$

$$m_{p,3} \leq -1 - 2(p-1) + r_2^{(1)}$$

- $n_{p,3} = 2$, for $1 \leq p \leq r_3^{(2)}$

$$m_{p,3} \leq -2 - 4(p-1) + 2r_2^{(1)}$$

Difference conditions for $i = 3$

$$m_{p,3} \leq -n_{p,3} - 2n_{p,3}(p-1) + \sum_{q=1}^{r_2^{(1)}} \min\left\{2n_{q,2}, n_{p,3}\right\}, \quad 1 \leq p \leq r_3^{(1)}$$

$$m_{p+1,3} \leq m_{p,3} - 2n_{p,3}, \quad n_{p+1,3} = n_{p,3}$$

- $n_{p,3} = 1$, for $1 \leq p \leq r_3^{(2)} + 1$

$$m_{p,3} \leq -1 - 2(p-1) + r_2^{(1)}$$

- $n_{p,3} = 2$, for $1 \leq p \leq r_3^{(2)}$

$$m_{p,3} \leq -2 - 4(p-1) + 2r_2^{(1)}$$

- contribution to the character:

$$\sum_{r_3^{(1)} \geq r_3^{(2)} \geq 0} \frac{q^{{r_3^{(1)}}^2 + {r_3^{(2)}}^2 - r_2^{(1)}(r_3^{(1)} + r_3^{(2)})}}{(q;q)_{r_3^{(1)} - r_3^{(2)}}(q;q)_{r_3^{(2)}}} y^{r_3^{(1)} + r_3^{(2)}}$$

Difference conditions for $i = 4$

$$x_{n_{r_4^{(1)},4}\alpha_4}(m_{r_4^{(1)},4}) \cdots x_{n_{1,4}\alpha_4}(m_{1,4}) x_{\alpha_3}(m_{r_3^{(1)},3}) \cdots x_{\alpha_3}(m_{2,3}) x_{\alpha_3}(m_{1,3})$$

- $m_{p,4} \leq -n_{p,4} - 2n_{p,4}(p-1), \;\; 1 \leq p \leq r_3^{(1)}$
- $m_{p+1,4} \leq m_{p,4} - 2n_{p,4}, \;\; n_{p+1,4} = n_{p,4}$

Difference conditions for $i = 4$

Relation among quasi-particles of different colors

$$\prod_{p=1}^{r_4^{(1)}} \prod_{q=1}^{r_3^{(1)}} \left(1 - \frac{z_{q,3}}{z_{p,4}}\right)^{\min\left\{n_{q,3}, n_{p,4}\right\}} x_{\alpha_4}(z_{r_4^{(1)},4}) \cdots x_{\alpha_4}(z_{1,4})$$

$$x_{\alpha_3}(z_{r_3^{(1)},3}) \cdots x_{\alpha_3}(z_{1,3}) v_L$$

$$\in \prod_{p=1}^{r_4^{(1)}} z_{p,4}^{-\sum_{q=1}^{r_3^{(1)}} \min\left\{n_{q,3}, n_{p,4}\right\}} W_{L(\Lambda_0)} \left[\left[z_{1,3}, \ldots, z_{r_4^{(1)},4}\right]\right]$$

Difference conditions for $i = 4$

$$m_{p,4} \leq -n_{p,4} - 2n_{p,4}(p-1) + \sum_{q=1}^{r_3^{(1)}} \min\left\{n_{q,3}, n_{p,4}\right\}, \quad 1 \leq p \leq r_4^{(1)}$$

$$m_{p+1,4} \leq m_{p,4} - 2n_{p,4}, \quad n_{p+1,4} = n_{p,4}$$

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$$m_{p,4} \leq -n_{p,4} - 2n_{p,4}(p-1) + \sum_{q=1}^{r_3^{(1)}} \min\{n_{q,3}, n_{p,4}\}, \quad 1 \leq p \leq r_4^{(1)}$$

$$m_{p+1,4} \leq m_{p,4} - 2n_{p,4}, \quad n_{p+1,4} = n_{p,4}$$

- $n_{p,4} = 1$, for $1 \leq p \leq r_4^{(2)} + 1$

$$m_{p,4} \leq -1 - 2(p-1) + r_3^{(1)}$$

- $n_{p,4} = 2$, for $1 \leq p \leq r_4^{(2)}$

$$m_{p,4} \leq -2 - 4(p-1) + r_3^{(1)} + r_3^{(2)}$$

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$$m_{p+1,4} \le m_{p,4} - 2n_{p,4}, \quad n_{p+1,4} = n_{p,4}$$

- $n_{p,4} = 1$, for $1 \le p \le r_4^{(2)} + 1$

$$m_{p,4} \le -1 - 2(p-1) + r_3^{(1)}$$

- $n_{p,4} = 2$, for $1 \le p \le r_4^{(2)}$

$$m_{p,4} \le -2 - 4(p-1) + r_3^{(1)} + r_3^{(2)}$$

- contribution to the character:

$$\sum_{r_4^{(1)} \ge r_4^{(2)} \ge 0} \frac{q^{{r_4^{(1)}}^2 + {r_4^{(2)}}^2 - r_3^{(1)} r_4^{(1)} - r_3^{(2)} r_4^{(2)}}}{(q;q)_{r_4^{(1)} - r_4^{(2)}} (q;q)_{r_4^{(2)}}} y^{r_4^{(1)} + r_4^{(2)}}$$

Principal subspaces of $F_4^{(1)}$

Basis for the principal subspace W_V

$$\bigcup_{n_{r_i^{(1)},i}\leq\ldots\leq n_{1,i}} \{b = b(\alpha_4)\cdots b(\alpha_1)v$$

$$= x_{n_{r_4^{(1)},4}\alpha_l}(m_{r_4^{(1)},4})\cdots x_{n_{1,4}\alpha_4}(m_{1,4})\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})v :$$

$$\left| \begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i-1}^{(1)}} min\left\{k_i n_{q,i-1}, n_{p,i}\right\} - 2(p-1)n_{p,i}, 1 \leq p \leq r_i^{(1)}, \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1 \end{array} \right\}$$

- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i} \leq 1,\ 1 \leq i \leq 2$
- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i} \leq 2,\ 3 \leq i \leq 4$

$$\text{ch } W_{L(\Lambda_0)} = \sum_{\substack{r_1^{(1)}\geqslant 0\\ r_2^{(1)}\geqslant 0}} \frac{q^{\sum_{i=1}^{2} r_i^{(t)^2} - r_1^{(t)} r_2^{(t)}}}{(q)_{r^{(1)}}(q)_{r_2^{(1)}}} \prod_{i=1}^{2} y_i^{n_i}$$

$$\times \sum_{\substack{r_3^{(1)}\geqslant r_3^{(2)}\geqslant 0\\ r_4^{(1)}\geqslant r_4^{(2)}\geqslant 0}} \frac{q^{\sum_{i=3}^{4}\sum_{t=1}^{2} r_i^{(t)^2} - \sum_{t=1}^{2} r_3^{(t)} r_4^{(t)} - r_2^{(t)}(r_3^{(1)} + r_3^{(2)})}}{\prod_{i=3}^{4}(q)_{r_i^{(1)} - r_i^{(2)}}\cdots(q)_{r_i^{(2k)}}} \prod_{i=3}^{4} y_i^{n_i},$$

Basis of principal subspace $W_{L(k\Lambda_0)}$ of $F_4^{(1)}$

Basis for the principal subspace W_V

$$\bigcup_{n_{r_i^{(1)},i}\leq\ldots\leq n_{1,i}} \{b = b(\alpha_4)\cdots b(\alpha_1)v$$

$$= x_{n_{r_4^{(1)},4}\alpha_l}(m_{r_4^{(1)},4})\cdots x_{n_{1,4}\alpha_4}(m_{1,4})\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})v :$$

$$\left| \begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i-1}^{(1)}} min\left\{k_i n_{q,i-1}, n_{p,i}\right\} - 2(p-1)n_{p,i}, 1 \leq p \leq r_i^{(1)}, \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1 \end{array} \right\}$$

- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i} \leq k,\ 1 \leq i \leq 2$
- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i} \leq 2k,\ 3 \leq i \leq 4$

$$\mathrm{ch}\ W_{L(k\Lambda_0)} = \sum_{\substack{r_1^{(1)}\geqslant\cdots\geqslant r_1^{(k)}\geqslant 0 \\ r_2^{(1)}\geqslant\cdots\geqslant r_2^{(k)}\geqslant 0}} \frac{q^{\sum_{i=1}^{2}\sum_{t=1}^{k} r_i^{(t)^2} - \sum_{t=1}^{k} r_1^{(t)} r_2^{(t)}}}{\prod_{i=1}^{2}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots(q)_{r_i^{(k)}}} \prod_{i=1}^{2} y_i^{n_i}$$

$$\times \sum_{\substack{r_3^{(1)}\geqslant\cdots\geqslant r_3^{(2k)}\geqslant 0 \\ r_4^{(1)}\geqslant\ldots\geqslant r_4^{(2k)}\geqslant 0}} \frac{q^{\sum_{i=3}^{4}\sum_{t=1}^{2k} r_i^{(t)^2} - \sum_{t=1}^{2k} r_3^{(t)} r_4^{(t)} - \sum_{t=1}^{k} r_2^{(t)}(r_3^{(2t-1)} + r_3^{(2t)})}}{\prod_{i=3}^{4}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots(q)_{r_i^{(2k)}}} \prod_{i=3}^{4} y_i^{n_i},$$

Basis of principal subspace $W_{N(k\Lambda_0)}$ of $F_4^{(1)}$

Basis for the principal subspace W_V

$$\bigcup_{n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i}} \{b = b(\alpha_4) \cdots b(\alpha_1) v$$

$$= x_{n_{r_4^{(1)},4}\alpha_l}(m_{r_4^{(1)},4}) \cdots x_{n_{1,4}\alpha_4}(m_{1,4}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(m_{1,1}) v :$$

$$\left| \begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i-1}^{(1)}} min\left\{k_i n_{q,i-1}, n_{p,i}\right\} - 2(p-1)n_{p,i}, 1 \leq p \leq r_i^{(1)}, \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1 \end{array} \right\}$$

- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i},\ 1 \leq i \leq 2$
- $n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i},\ 3 \leq i \leq 4$

Character of $W_{N(k\Lambda_0)}$

$$\frac{1}{\prod_{\alpha\in R_+}(\alpha;q)_\infty} =$$

$$= \sum_{\substack{r_1^{(1)}\geqslant r_1^{(2)}\geqslant\cdots\geqslant 0\\ r_2^{(1)}\geqslant r_2^{(2)}\geqslant\cdots\geqslant 0}} \frac{q^{\sum_{i=1}^{2}\sum_{t\geq 1} r_i^{(t)^2} - \sum_{t\geq 1} r_1^{(t)} r_2^{(t)}}}{\prod_{i=1}^{2}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots} \prod_{i=1}^{2} y_i^{n_i}$$

$$\times \sum_{\substack{r_3^{(1)}\geqslant r_3^{(2)}\geqslant\cdots\geqslant 0\\ r_4^{(1)}\geqslant r_4^{(2)}\geqslant\cdots\geqslant 0}} \frac{q^{\sum_{i=3}^{4}\sum_{t\geq 1} r_i^{(t)^2} - \sum_{t\geq 1} r_3^{(t)} r_4^{(t)} - \sum_{t\geq 1} r_2^{(t)}(r_3^{(2t-1)}+r_3^{(2t)})}}{\prod_{i=3}^{4}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots} \prod_{i=3}^{4} y_i^{n_i}$$

Proof of linear independence

- Sketch for the level $k = 1$:

$$\sum_a c_a b_a v_{\Lambda_0} = 0$$

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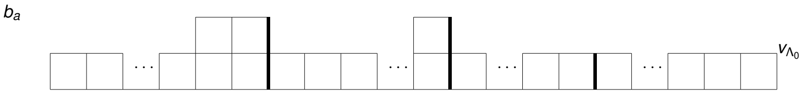

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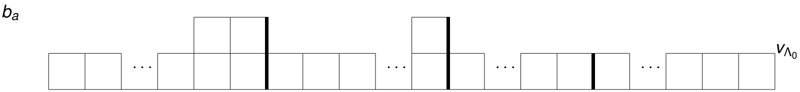
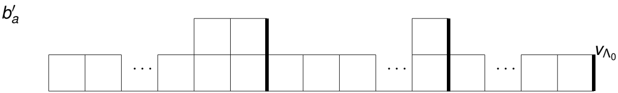


- Idea:

$$\sum_a c_a b'_a v_{\Lambda_0} = 0$$



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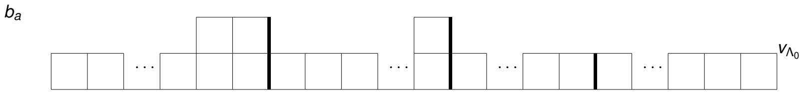

- Idea:

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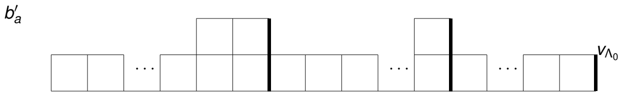

- $b'_a v_{\Lambda_0}$ are elements of principal subspace of level 1 standard $C_3^{(1)}$-module with a highest weight Λ_0.

Proof of linear independence

In the proof of linear independence we employ

- operator A_θ

$$A_\theta = \mathrm{Res}_z z^{-1}\, x_\theta(z) = x_\theta(-1)$$

  - $[A_\theta, x_\alpha(m)] = 0$
  - $A_\theta v_{\Lambda_0} = x_\theta(-1) v_{\Lambda_0}$

$$A_\theta \sum_a c_a b_a v_{\Lambda_0} = \sum_a c_a b_a x_\theta(-1) v_{\Lambda_0} = 0$$

Proof of linear independence

In the proof of linear independence of we employ

- the "Weyl group translation" operators e_θ and e_{α_1}

$$e_\alpha = \exp\ x_{-\alpha}(1) \exp\left(-x_\alpha(-1)\right) \exp\ x_{-\alpha}(1)$$

$$\exp\ x_\alpha(0) \exp\left(-x_{-\alpha}(0)\right) \exp\ x_\alpha(0), \quad \alpha = \theta, \alpha_1$$

  - $e_\alpha v_{\Lambda_0} = -x_\alpha(-1) v_{\Lambda_0}$, for α -long root
  - $x_\beta(j) e_\alpha = e_\alpha x_\beta(j + \beta(\alpha^\vee))$,

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- $x_\beta(j) e_\alpha = e_\alpha x_\beta(j + \beta(\alpha^\vee))$,

$$\sum_a c_a b_a x_\theta(-1) v_{\Lambda_0} = \sum_a c_a b_a e_\theta v_{\Lambda_0} = e_\theta \sum_a c_a b'_a v_{\Lambda_0} = 0$$

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$$\sum_a c_a b_a x_\theta(-1) v_{\Lambda_0} = \sum_a c_a b_a e_\theta v_{\Lambda_0} = e_\theta \sum_a c_a b'_a v_{\Lambda_0} = 0$$

- repeat until $\sum_a c_a b'_a x_\alpha(-1) v_{\Lambda_0} = 0$

$$\sum_a c_a b'_a x_\alpha(-1) v_{\Lambda_0} = \sum_a c_a b'_a e_\alpha v_{\Lambda_0} = \sum_a c_a b''_a v_{\Lambda_0} = 0$$

Basis of the principal subspace $W_{N(k'\Lambda_0)}$

In the proof of linear independence in the case of $W_{N(k'\Lambda_0)}$ we use:

- $f_{k'} : W_{N(k'\Lambda_0)} \to W_{L(k'\Lambda_0)}$

$$f_{k'}(x_{(k'+1)\alpha_i}(z)) = 0, \quad i = 1, 2$$

$$f_k(x_{(2k'+1)\alpha_2}(z)) = 0, \quad i = 3, 4$$

Principal subspaces in the case of $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$

- $\widetilde{\mathfrak{g}}$ - of type $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$
- $L(\Lambda_0)$, $L(\Lambda_1)$, $L(\Lambda_6)$ the standard $\widetilde{\mathfrak{g}}$ - modules of level 1 in the case of $E_6^{(1)}$
- $L(\Lambda_0)$, $L(\Lambda_1)$ the standard $\widetilde{\mathfrak{g}}$ - modules of level 1 in the case of $E_7^{(1)}$
- $L(\Lambda_0)$ the standard $\widetilde{\mathfrak{g}}$ - modules of level 1 in the case of $E_8^{(1)}$

Principal subspaces in the case of $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$

- $\tilde{\mathfrak{g}}$ - of type $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$
- $L(\Lambda_0)$, $L(\Lambda_1)$, $L(\Lambda_6)$ the standard $\tilde{\mathfrak{g}}$ - modules of level 1 in the case of $E_6^{(1)}$
- $L(\Lambda_0)$, $L(\Lambda_1)$ the standard $\tilde{\mathfrak{g}}$ - modules of level 1 in the case of $E_7^{(1)}$
- $L(\Lambda_0)$ the standard $\tilde{\mathfrak{g}}$ - modules of level 1 in the case of $E_8^{(1)}$
- let $k \geq 1$
- set

$$\Lambda_k = k_0\Lambda_0 + k_j\Lambda_j$$

$$k = k_0 + k_k$$

- $L(\Lambda_k)$ the level k standard $\tilde{\mathfrak{g}}$ - module
- v_{Λ_k} highest weight vector

Principal subspaces in the case of $E_6^{(1)}$, $E_6^{(1)}$ and $E_8^{(1)}$

- principal subspaces are

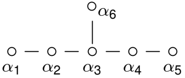
$$W(\Lambda_k) = U(\widetilde{\mathfrak{n}}_+) v_{\Lambda_k}$$

- $W(\Lambda_k) \subset W(\Lambda_j)^{\otimes k_j} \otimes W(\Lambda_0)^{\otimes k_0} \subset L(\Lambda_j)^{\otimes k_j} \otimes L(\Lambda_0)^{\otimes k_0}$

- $v_{\Lambda_k} = \underbrace{v_{\Lambda_j} \otimes v_{\Lambda_j} \otimes \cdots \otimes v_{\Lambda_j}}_{k_j - factors} \otimes \underbrace{v_{\Lambda_0} \otimes v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 - factors}$

Principal subspaces in the case of $E_6^{(1)}$

- $\mathfrak{g}$ is of type E_6



Basis for the principal subspace W_V

$$\bigcup_{n_{r_i^{(1)},i}\leq\ldots\leq n_{1,i}} \{b = b(\alpha_1)\cdots b(\alpha_6)v$$

$$= x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})\cdots x_{n_{r_6^{(1)},6}\alpha_6}(m_{r_6^{(1)},6})\cdots x_{n_{1,6}\alpha_6}(m_{1,6})v :$$

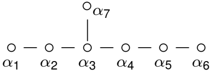
$$\left|\begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i'}^{(1)}} min\left\{n_{q,i'}, n_{p,i}\right\} - 2(p-1)n_{p,i} - \sum_{t=1}^{n_{p,i}} \delta_{i,j_t}, 1 \leq p \leq r_i^{(1)}, \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1 \end{array}\right\},$$

where $i' = 4, 6$ *if* $i = 3$ *and* $i' = i + 1$ *if* $i = 1, 2, 4$, *and*

$$j_t = \begin{cases} 0, & \text{if } 1 \leqslant t \leqslant k_0, \\ j, & \text{if } k_0 < t \leqslant k_0 + k_j. \end{cases}$$

Principal subspaces in the case of $E_7^{(1)}$

- $\mathfrak{g}$ is of type E_7



Basis for the principal subspace W_V

$$\bigcup_{n_{r_i^{(1)},i}\leq\ldots\leq n_{1,i}} \{b = b(\alpha_7)b(\alpha_2)b(\alpha_3)b(\alpha_4)b(\alpha_5)b(\alpha_6)b(\alpha_1)v$$

$$= x_{n_{r_7^{(1)},7}\alpha_7}(m_{r_7^{(1)},7})\cdots x_{n_{1,7}\alpha_7}(m_{1,7})\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})v :$$

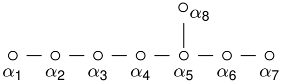
$$\left|\begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i'}^{(1)}} min\left\{n_{q,i'}, n_{p,i}\right\} - 2(p-1)n_{p,i} - \sum_{t=1}^{n_{p,i}} \delta_{i,j_t}, 1 \leq p \leq r_i^{(1)}, \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1 \end{array}\right\},$$

where $i' = 1, 3$ *if* $i = 2$, $i' = i + 1$ *if* $i = 3, 4, 5$, *and* $i' = 5$ *if* $i = 7$ *and*

$$j_t = \begin{cases} 0, & if \quad 1 \leqslant t \leqslant k_0, \\ j, & if \ \ k_0 < t \leqslant k_0 + k_j. \end{cases}$$

Principal subspaces in the case of $E_8^{(1)}$

- $\mathfrak{g}$ is of type E_8



Basis for the principal subspace W_V

$$\bigcup_{n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i}} \{ b = b(\alpha_8)b(\alpha_6)b(\alpha_5)b(\alpha_4)b(\alpha_3)b(\alpha_2)b(\alpha_7)b(\alpha_1)v$$

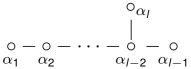
$$= x_{n_{r_8^{(1)},8}\alpha_8}(m_{r_8^{(1)},8}) \cdots x_{n_{1,8}\alpha_8}(m_{1,8}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(m_{1,1})v :$$

$$\left| \begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i'}^{(1)}} min\left\{ n_{q,i'}, n_{p,i} \right\} - 2(p-1)n_{p,i}, 1 \leq p \leq r_i^{(1)}, \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1 \end{array} \right\},$$

where $i' = 5, 7$ *if* $i = 6$, $i' = i - 1$ *if* $i = 2, 3, 4, 5$, *and* $i' = 5, 7$ *if* $i = 8$.

Principal subspaces in the case of $D_l^{(1)}$

- $\mathfrak{g}$ is of type D_l



Basis for the principal subspace W_V

$$\bigcup_{n_{r_i^{(1)},i}\leq\ldots\leq n_{1,i}} \{b = b(\alpha_l)\cdots b(\alpha_1)v$$

$$= x_{n_{r_8^{(1)},8}\alpha_8}(m_{r_8^{(1)},8})\cdots x_{n_{1,8}\alpha_8}(m_{1,8})\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})v :$$

$$\left| \begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i'}^{(1)}} min\left\{n_{q,i'}, n_{p,i}\right\} - 2(p-1)n_{p,i} - \sum_{t=1}^{n_{p,i}} \delta_{i,j_t}, 1 \leq p \leq r_i^{(1)}, \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1 \end{array} \right\},$$

where $i' = l-2$ *if* $i = l$, $i' = i-1$ *otherwise and*

$$j_t = \begin{cases} 0, & \text{if } \ 1 \leqslant t \leqslant k_0, \\ j, & \text{if } \ k_0 < t \leqslant k_0 + k_j. \end{cases},$$

where $j = 0, 1, l-1, l$.

Characters of W_Λ

Theorem

Set $n_i = r_i^{(1)} + \cdots + r_i^{(k)}$ *for* $i = 1, \ldots, l$. *For any rectangular weight* $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ *of level* $k = k_0 + k_j$ *we have*

$$chW_{L(\Lambda)} = \sum_{\substack{r_1^{(1)} \geqslant \cdots \geqslant r_1^{(k)} \geqslant 0 \\ \vdots \\ r_l^{(1)} \geqslant \cdots \geqslant r_l^{(k)} \geqslant 0}} \frac{q^{\sum_{i=1}^{l}\sum_{t=1}^{k} {r_i^{(t)}}^2 - \sum_{i=2}^{l}\sum_{t=1}^{k} r_{i'}^{(t)} r_i^{(t)} + \sum_{i=1}^{l}\sum_{t=1}^{k} r_i^{(t)} \delta_{ij_t}}}{\prod_{i=1}^{l} (q)_{r_i^{(1)} - r_i^{(2)}} \cdots (q)_{r_i^{(k)}}} \prod_{i=1}^{l} y_i^{n_i}.$$

Further directions

- extend these approach to principal subspaces of affine Lie algebra of type $F_4^{(1)}$ ($B_l^{(1)}$, $C_l^{(1)}$ and $G_2^{(1)}$) this is ongoing program with S. Kožić and M. Primc

- $L(\Lambda_0)$, $L(\Lambda_4)$ the standard $\widetilde{\mathfrak{g}}$ - modules of level 1 in the case of $F_4^{(1)}$
- let $k \geq 1$
- set

$$\Lambda_k = k_0\Lambda_0 + k_j\Lambda_j$$
$$k = k_0 + k_k$$

- $L(\Lambda_k)$ the level k standard $\widetilde{\mathfrak{g}}$ - module
- v_{Λ_k} highest weight vector
- principal subspaces are

$$W(\Lambda_k) = U(\widetilde{\mathfrak{n}}_+)v_{\Lambda_k}$$

Further directions

- we expect:

For any rectangular weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ of level $k = k_0 + k_j$ for $F_4^{(1)}$ we have

$$\mathrm{ch}\; W_{L(k\Lambda_0)} = \sum_{\substack{r_1^{(1)}\geqslant\cdots\geqslant r_1^{(k)}\geqslant 0\\ r_2^{(1)}\geqslant\cdots\geqslant r_2^{(k)}\geqslant 0}} \frac{q^{\sum_{i=1}^{2}\sum_{t=1}^{k} r_i^{(t)^2} - \sum_{t=1}^{k} r_1^{(t)} r_2^{(t)}}}{\prod_{i=1}^{2}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots(q)_{r_i^{(k)}}}\prod_{i=1}^{2} y_i^{n_i}$$

$$\times \sum_{\substack{r_3^{(1)}\geqslant\cdots\geqslant r_3^{(2k)}\geqslant 0\\ r_4^{(1)}\geqslant\ldots\geqslant r_4^{(2k)}\geqslant 0}} \frac{q^{\sum_{i=3}^{4}\sum_{t=1}^{2k} r_i^{(t)^2} - \sum_{t=1}^{2k} r_3^{(t)} r_4^{(t)} - \sum_{t=1}^{k} r_2^{(t)}(r_3^{(2t-1)} + r_3^{(2t)}) + \sum_{t=1}^{k_4} r_4^{t}\delta_{4,j_t}}}{\prod_{i=3}^{4}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots(q)_{r_i^{(2k)}}}\prod_{i=3}^{4} y_i^{n_i}$$

Thank you!