Higher level Zhu algebras and indecomposable modules for vertex operator algebras

Katrina Barron — University of Notre Dame

some work is joint with Nathan Vander Werf and Jinwei Yang;
some work is joint with Darlayne Addabbo

Representation XVI,
Dubrovnik 2019

Outline

Preliminaries and Motivation

- Given a VOA V, Frenkel and Zhu in 1992 defined an associative algebra, now called a *Zhu algebra*, denoted $A_0(V)$, and proved that if V is rational, there is a bijection

$$\left\{ \begin{array}{c} \text{Irreducible } \mathbb{N}\text{-gradable} \\ V\text{-modules} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Irreducible} \\ A_0(V)\text{-modules} \end{array} \right\}$$

- In 1998, Dong, Li, and Mason, greatly expanded this notion by defining *higher level Zhu algebras* $A_n(V)$, for $n \in \mathbb{Z}_+$, and established further functorial relations, e.g.

$$\left\{ \begin{array}{c} \text{Irreducible } \mathbb{N}\text{-gradable} \\ V\text{-modules with nonzero} \\ \text{degree } n \text{ subspaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Irreducible} \\ A_n(V)\text{-modules} \end{array} \right\}$$

- Question: What extra information can $A_n(V)$ provide, for $n \in \mathbb{Z}_+$, beyond what $A_0(V)$ gives?

Preliminaries and Motivation

- Given a VOA V, Frenkel and Zhu in 1992 defined an associative algebra, now called a *Zhu algebra*, denoted $A_0(V)$, and proved that if V is rational, there is a bijection

$$\left\{ \begin{array}{c} \text{Irreducible } \mathbb{N}\text{-gradable} \\ V\text{-modules} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Irreducible} \\ A_0(V)\text{-modules} \end{array} \right\}$$

- In 1998, Dong, Li, and Mason, greatly expanded this notion by defining *higher level Zhu algebras* $A_n(V)$, for $n \in \mathbb{Z}_+$, and established further functorial relations, e.g.

$$\left\{ \begin{array}{c} \text{Irreducible } \mathbb{N}\text{-gradable} \\ V\text{-modules with nonzero} \\ \text{degree } n \text{ subspaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Irreducible} \\ A_n(V)\text{-modules} \end{array} \right\}$$

- Question: What extra information can $A_n(V)$ provide, for $n \in \mathbb{Z}_+$, beyond what $A_0(V)$ gives?

Preliminaries and Motivation

- Given a VOA V, Frenkel and Zhu in 1992 defined an associative algebra, now called a *Zhu algebra*, denoted $A_0(V)$, and proved that if V is rational, there is a bijection

$$\left\{ \begin{array}{c} \text{Irreducible } \mathbb{N}\text{-gradable} \\ V\text{-modules} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Irreducible} \\ A_0(V)\text{-modules} \end{array} \right\}$$

- In 1998, Dong, Li, and Mason, greatly expanded this notion by defining *higher level Zhu algebras* $A_n(V)$, for $n \in \mathbb{Z}_+$, and established further functorial relations, e.g.

$$\left\{ \begin{array}{c} \text{Irreducible } \mathbb{N}\text{-gradable} \\ V\text{-modules with nonzero} \\ \text{degree } n \text{ subspaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Irreducible} \\ A_n(V)\text{-modules} \end{array} \right\}$$

- Question: What extra information can $A_n(V)$ provide, for $n \in \mathbb{Z}_+$, beyond what $A_0(V)$ gives?

Motivation

- For VOAs that admit indecomposable non-simple modules, the higher level $A_n(V)$ (i.e. $n > 0$) can be used to construct such modules, in particular if they have indecomposables not generated by their level zero degree space.
- The Zhu algebras $A_n(V)$ can be used to calculate the graded *pseudo-traces* for V-modules, following Miyamoto 2004.
- That is: if V is C_2-cofinite (and thus has a finite number of indecomposables), but is irrational (not of semi-simple representation type), then the space of graded dimensions does not span the space of one-point functions $\mathcal{C}_1(V)$, and is not invariant under the action of $SL_2(\mathbb{Z})$.
- But, if generalizations called *pseudo-traces* are included, then the space of all such q-series does span $\mathcal{C}_1(V)$ and satisfies modular invariance.
- Applications: K.B. and N. Vander Werf, Intersections of screening operators arising from rank two lattice VOAs (2018 & 2019). (See also K.B. and N. Vander Werf, Letters in Math. Phys. 2019.)

Motivation

- For VOAs that admit indecomposable non-simple modules, the higher level $A_n(V)$ (i.e. $n > 0$) can be used to construct such modules, in particular if they have indecomposables not generated by their level zero degree space.
- The Zhu algebras $A_n(V)$ can be used to calculate the graded *pseudo-traces* for V-modules, following Miyamoto 2004.
- That is: if V is C_2-cofinite (and thus has a finite number of indecomposables), but is irrational (not of semi-simple representation type), then the space of graded dimensions does not span the space of one-point functions $\mathcal{C}_1(V)$, and is not invariant under the action of $SL_2(\mathbb{Z})$.
- But, if generalizations called *pseudo-traces* are included, then the space of all such q-series does span $\mathcal{C}_1(V)$ and satisfies modular invariance.
- Applications: K.B. and N. Vander Werf, Intersections of screening operators arising from rank two lattice VOAs (2018 & 2019). (See also K.B. and N. Vander Werf, Letters in Math. Phys. 2019.)

Motivation

- For VOAs that admit indecomposable non-simple modules, the higher level $A_n(V)$ (i.e. $n > 0$) can be used to construct such modules, in particular if they have indecomposables not generated by their level zero degree space.
- The Zhu algebras $A_n(V)$ can be used to calculate the graded *pseudo-traces* for V-modules, following Miyamoto 2004.
- That is: if V is C_2-cofinite (and thus has a finite number of indecomposables), but is irrational (not of semi-simple representation type), then the space of graded dimensions does not span the space of one-point functions $\mathcal{C}_1(V)$, and is not invariant under the action of $SL_2(\mathbb{Z})$.
- But, if generalizations called *pseudo-traces* are included, then the space of all such q-series does span $\mathcal{C}_1(V)$ and satisfies modular invariance.
- Applications: K.B. and N. Vander Werf, Intersections of screening operators arising from rank two lattice VOAs (2018 & 2019). (See also K.B. and N. Vander Werf, Letters in Math. Phys. 2019.)

Motivation

- For VOAs that admit indecomposable non-simple modules, the higher level $A_n(V)$ (i.e. $n > 0$) can be used to construct such modules, in particular if they have indecomposables not generated by their level zero degree space.
- The Zhu algebras $A_n(V)$ can be used to calculate the graded *pseudo-traces* for V-modules, following Miyamoto 2004.
- That is: if V is C_2-cofinite (and thus has a finite number of indecomposables), but is irrational (not of semi-simple representation type), then the space of graded dimensions does not span the space of one-point functions $\mathcal{C}_1(V)$, and is not invariant under the action of $SL_2(\mathbb{Z})$.
- But, if generalizations called *pseudo-traces* are included, then the space of all such q-series does span $\mathcal{C}_1(V)$ and satisfies modular invariance.
- Applications: K.B. and N. Vander Werf, Intersections of screening operators arising from rank two lattice VOAs (2018 & 2019). (See also K.B. and N. Vander Werf, Letters in Math. Phys. 2019.)

Motivation

- For VOAs that admit indecomposable non-simple modules, the higher level $A_n(V)$ (i.e. $n > 0$) can be used to construct such modules, in particular if they have indecomposables not generated by their level zero degree space.
- The Zhu algebras $A_n(V)$ can be used to calculate the graded *pseudo-traces* for V-modules, following Miyamoto 2004.
- That is: if V is C_2-cofinite (and thus has a finite number of indecomposables), but is irrational (not of semi-simple representation type), then the space of graded dimensions does not span the space of one-point functions $\mathcal{C}_1(V)$, and is not invariant under the action of $SL_2(\mathbb{Z})$.
- But, if generalizations called *pseudo-traces* are included, then the space of all such q-series does span $\mathcal{C}_1(V)$ and satisfies modular invariance.
- Applications: K.B. and N. Vander Werf, Intersections of screening operators arising from rank two lattice VOAs (2018 & 2019). (See also K.B. and N. Vander Werf, Letters in Math. Phys. 2019.)

Zhu's Algebras $A_n(V)$

- Let V be a VOA. For $n \in \mathbb{N}$, and $u, v \in V$ homogeneous, let define
$$u \circ_n v = \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\, u+n} Y(u,x)v}{x^{2n+2}}.$$
Let $O_n(V)$ be the subspace of V spanned by elements of the form $u \circ_n v$ and $(L(-1)+L(0))v$ for $u, v \in V$.
- The vector space $A_n(V)$ is defined to be the quotient space $V/O_n(V)$.
- Then $O_n(V)$ is a two-sided ideal of V, and $A_n(V)$ is an associative algebra under the multiplication $*_n$ defined by
$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\, u+n} Y(u,x)v}{x^{n+m+1}},$$
for $u, v \in V$ homogeneous.

Zhu's Algebras $A_n(V)$

- Let V be a VOA. For $n \in \mathbb{N}$, and $u, v \in V$ homogeneous, let define
$$u \circ_n v = \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt} u + n} Y(u,x)v}{x^{2n+2}}.$$
Let $O_n(V)$ be the subspace of V spanned by elements of the form $u \circ_n v$ and $(L(-1) + L(0))v$ for $u, v \in V$.
- The vector space $A_n(V)$ is defined to be the quotient space $V/O_n(V)$.
- Then $O_n(V)$ is a two-sided ideal of V, and $A_n(V)$ is an associative algebra under the multiplication $*_n$ defined by
$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt} u + n} Y(u,x)v}{x^{n+m+1}},$$
for $u, v \in V$ homogeneous.

Zhu's Algebras $A_n(V)$

- Let V be a VOA. For $n \in \mathbb{N}$, and $u, v \in V$ homogeneous, let define
$$u \circ_n v = \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt} u + n} Y(u,x)v}{x^{2n+2}}.$$
Let $O_n(V)$ be the subspace of V spanned by elements of the form $u \circ_n v$ and $(L(-1)+L(0))v$ for $u, v \in V$.
- The vector space $A_n(V)$ is defined to be the quotient space $V/O_n(V)$.
- Then $O_n(V)$ is a two-sided ideal of V, and $A_n(V)$ is an associative algebra under the multiplication $*_n$ defined by
$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt} u + n} Y(u,x)v}{x^{n+m+1}},$$
for $u, v \in V$ homogeneous.

Relationship of $A_n(V)$ to $A_{n-1}(V)$ for $n \in \mathbb{Z}_+$

- $O_n(V) \subset O_{n-1}(V)$ for $n \in \mathbb{Z}_+$.
- Thus there is a natural surjective algebra homomorphism

$$\begin{array}{rrcl} \pi : & A_n(V) & \longrightarrow & A_{n-1}(V) \\ & v + O_n(V) & \mapsto & v + O_{n-1}(V) \end{array} \tag{1}$$

- **Modules for $A_n(V)$:** An $A_n(V)$-module, U, is said to *factor through* $A_{n-1}(V)$ if $\ker \pi = O_{n-1}(V)$ acts trivially on U, giving U a well-defined $A_{n-1}(V)$-module structure.
- **Modules for V:** A weak V-module W is called *admissible* or $\mathbb{N}$-*gradable* if

$$W = \coprod_{k \in \mathbb{N}} W(k),$$

with $v_m W(k) \subset W(k + \mathrm{wt}\, v - m - 1)$ for $v \in V$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$.

- Without loss of generality, we can assume $W(0) \neq 0$.

Relationship of $A_n(V)$ to $A_{n-1}(V)$ for $n \in \mathbb{Z}_+$

- $O_n(V) \subset O_{n-1}(V)$ for $n \in \mathbb{Z}_+$.
- Thus there is a natural surjective algebra homomorphism

$$\begin{aligned} \pi : \quad A_n(V) &\longrightarrow A_{n-1}(V) \\ v + O_n(V) &\mapsto v + O_{n-1}(V) \end{aligned} \tag{1}$$

- **Modules for $A_n(V)$:** An $A_n(V)$-module, U, is said to *factor through* $A_{n-1}(V)$ if $\ker \pi = O_{n-1}(V)$ acts trivially on U, giving U a well-defined $A_{n-1}(V)$-module structure.
- **Modules for V:** A weak V-module W is called *admissible* or $\mathbb{N}$-*gradable* if

$$W = \coprod_{k \in \mathbb{N}} W(k),$$

with $v_m W(k) \subset W(k + \mathrm{wt}\, v - m - 1)$ for $v \in V$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$.
- Without loss of generality, we can assume $W(0) \neq 0$.

Relationship of $A_n(V)$ to $A_{n-1}(V)$ for $n \in \mathbb{Z}_+$

- $O_n(V) \subset O_{n-1}(V)$ for $n \in \mathbb{Z}_+$.
- Thus there is a natural surjective algebra homomorphism

$$\begin{aligned} \pi: \quad A_n(V) &\longrightarrow A_{n-1}(V) \\ v + O_n(V) &\mapsto v + O_{n-1}(V) \end{aligned} \tag{1}$$

- **Modules for $A_n(V)$:** An $A_n(V)$-module, U, is said to *factor through* $A_{n-1}(V)$ if $\ker \pi = O_{n-1}(V)$ acts trivially on U, giving U a well-defined $A_{n-1}(V)$-module structure.
- **Modules for V:** A weak V-module W is called *admissible* or $\mathbb{N}$*-gradable* if

$$W = \coprod_{k \in \mathbb{N}} W(k),$$

with $v_m W(k) \subset W(k + \mathrm{wt}\, v - m - 1)$ for $v \in V$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$.
- Without loss of generality, we can assume $W(0) \neq 0$.

Relationship of $A_n(V)$ to $A_{n-1}(V)$ for $n \in \mathbb{Z}_+$

- $O_n(V) \subset O_{n-1}(V)$ for $n \in \mathbb{Z}_+$.
- Thus there is a natural surjective algebra homomorphism

$$\begin{aligned} \pi : \quad A_n(V) &\longrightarrow A_{n-1}(V) \\ v + O_n(V) &\mapsto v + O_{n-1}(V) \end{aligned} \tag{1}$$

- **Modules for $A_n(V)$:** An $A_n(V)$-module, U, is said to *factor through* $A_{n-1}(V)$ if $\ker \pi = O_{n-1}(V)$ acts trivially on U, giving U a well-defined $A_{n-1}(V)$-module structure.
- **Modules for V:** A weak V-module W is called *admissible* or $\mathbb{N}$-*gradable* if

$$W = \coprod_{k \in \mathbb{N}} W(k),$$

with $v_m W(k) \subset W(k + \mathrm{wt}\, v - m - 1)$ for $v \in V$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$.

- Without loss of generality, we can assume $W(0) \neq 0$.

Relationship of $A_n(V)$ to $A_{n-1}(V)$ for $n \in \mathbb{Z}_+$

- $O_n(V) \subset O_{n-1}(V)$ for $n \in \mathbb{Z}_+$.
- Thus there is a natural surjective algebra homomorphism

$$\begin{aligned} \pi : \quad A_n(V) &\longrightarrow A_{n-1}(V) \\ v + O_n(V) &\mapsto v + O_{n-1}(V) \end{aligned} \tag{1}$$

- **Modules for $A_n(V)$:** An $A_n(V)$-module, U, is said to *factor through* $A_{n-1}(V)$ if $\ker \pi = O_{n-1}(V)$ acts trivially on U, giving U a well-defined $A_{n-1}(V)$-module structure.
- **Modules for V:** A weak V-module W is called *admissible* or $\mathbb{N}$-*gradable* if

$$W = \coprod_{k \in \mathbb{N}} W(k),$$

with $v_m W(k) \subset W(k + \mathrm{wt}\, v - m - 1)$ for $v \in V$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$.
- Without loss of generality, we can assume $W(0) \neq 0$.

Functors on the module categories of $A_n(V)$ and of V.

Dong-Li-Mason define two functors

$$\Omega_n \; : \; \begin{matrix}\text{Category of}\\ \text{weak } V\text{-modules}\end{matrix} \longrightarrow \begin{matrix}\text{Category of}\\ A_n(V)\text{-modules}\end{matrix} \tag{2}$$

$$L_n \; : \; \begin{matrix}\text{Category of}\\ A_n(V)\text{-modules}\end{matrix} \longrightarrow \begin{matrix}\text{Category of}\\ \mathbb{N}\text{-gradable } V\text{-modules}\end{matrix} \tag{3}$$

as follows:

Let W be a weak V-module and define

$$\Omega_n(W) = \{w \in W \mid v_i w = 0 \text{ if } \mathrm{wt}\, v_i < -n \text{ for } v \in V \text{ of homogeneous weight}\}. \tag{4}$$

Then $\Omega_n(W)$ is an $A_n(V)$-module by letting

$$[a] = a + O_n(V) \in A_n(V) \quad \text{act as} \quad o(a) = a_{\mathrm{wt}\, a - 1} \tag{5}$$

for $a \in V$.

Functors on the module categories of $A_n(V)$ and of V.

Dong-Li-Mason define two functors

$$\Omega_n \ : \ \begin{array}{c}\text{Category of} \\ \text{weak } V\text{-modules}\end{array} \longrightarrow \begin{array}{c}\text{Category of} \\ A_n(V)\text{-modules}\end{array} \tag{2}$$

$$L_n \ : \ \begin{array}{c}\text{Category of} \\ A_n(V)\text{-modules}\end{array} \longrightarrow \begin{array}{c}\text{Category of} \\ \mathbb{N}\text{-gradable } V\text{-modules}\end{array} \tag{3}$$

as follows:

Let W be a weak V-module and define

$$\Omega_n(W) = \{w \in W \mid v_i w = 0 \text{ if } \mathrm{wt}\, v_i < -n \text{ for } v \in V \text{ of homogeneous weight}\}. \tag{4}$$

Then $\Omega_n(W)$ is an $A_n(V)$-module by letting

$$[a] = a + O_n(V) \in A_n(V) \quad \text{act as} \quad o(a) = a_{\mathrm{wt}\, a - 1} \tag{5}$$

for $a \in V$.

Functors on the module categories of $A_n(V)$ and of V.

Dong-Li-Mason define two functors

$$\Omega_n \ : \ \begin{array}{c}\text{Category of}\\ \text{weak } V\text{-modules}\end{array} \longrightarrow \begin{array}{c}\text{Category of}\\ A_n(V)\text{-modules}\end{array} \tag{2}$$

$$L_n \ : \ \begin{array}{c}\text{Category of}\\ A_n(V)\text{-modules}\end{array} \longrightarrow \begin{array}{c}\text{Category of}\\ \mathbb{N}\text{-gradable } V\text{-modules}\end{array} \tag{3}$$

as follows:

Let W be a weak V-module and define

$$\Omega_n(W) = \{w \in W \mid v_i w = 0 \text{ if } \mathrm{wt}\, v_i < -n \ \text{ for } v \in V \text{ of homogeneous weight}\}. \tag{4}$$

Then $\Omega_n(W)$ is an $A_n(V)$-module by letting

$$[a] = a + O_n(V) \in A_n(V) \quad \text{act as} \quad o(a) = a_{\mathrm{wt}\, a - 1} \tag{5}$$

for $a \in V$.

Defining $L_n : A_n(V)$-module $\mapsto \mathbb{N}$-gradable V-mod

To define L_n, some further notions a necessary:

Let

$$\hat{V} = \mathbb{C}[t, t^{-1}] \otimes V/(D\mathbb{C}[t, t^{-1}] \otimes V), \tag{6}$$

where $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$.

For $v \in V$, denote by $v(m)$ the image of $v \otimes t^m$ in $\hat{V}$. Then $\hat{V}$ is a $\mathbb{Z}$-graded Lie algebra by defining $\deg v(m) = \mathrm{wt}\, v - m - 1$ and

$$[u(j), v(k)] = \sum_{i=0}^{\infty} \binom{j}{i} (u_i v)(j + k - i),$$

for $u, v \in V$, $j, k \in \mathbb{Z}$.

Denote the homogeneous subspace of degree m by $\hat{V}(m)$. Then $\hat{V}(0)$, is a Lie subalgebra of $\hat{V}$.

Defining $L_n : A_n(V)$-module $\mapsto \mathbb{N}$-gradable V-mod

To define L_n, some further notions a necessary:
Let

$$\hat{V} = \mathbb{C}[t, t^{-1}] \otimes V/(D\mathbb{C}[t, t^{-1}] \otimes V), \tag{6}$$

where $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$.

For $v \in V$, denote by $v(m)$ the image of $v \otimes t^m$ in $\hat{V}$. Then $\hat{V}$ is a $\mathbb{Z}$-graded Lie algebra by defining $\deg v(m) = \mathrm{wt}\, v - m - 1$ and

$$[u(j), v(k)] = \sum_{i=0}^{\infty} \binom{j}{i} (u_i v)(j + k - i),$$

for $u, v \in V$, $j, k \in \mathbb{Z}$.

Denote the homogeneous subspace of degree m by $\hat{V}(m)$. Then $\hat{V}(0)$, is a Lie subalgebra of $\hat{V}$.

Defining $L_n : A_n(V)$-module $\mapsto$ $\mathbb{N}$-gradable V-mod

To define L_n, some further notions a necessary:
Let

$$\hat{V} = \mathbb{C}[t, t^{-1}] \otimes V/(D\mathbb{C}[t, t^{-1}] \otimes V), \tag{6}$$

where $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$.

For $v \in V$, denote by $v(m)$ the image of $v \otimes t^m$ in $\hat{V}$. Then $\hat{V}$ is a $\mathbb{Z}$-graded Lie algebra by defining $\deg v(m) = \mathrm{wt}\, v - m - 1$ and

$$[u(j), v(k)] = \sum_{i=0}^{\infty} \binom{j}{i} (u_i v)(j + k - i),$$

for $u, v \in V$, $j, k \in \mathbb{Z}$.

Denote the homogeneous subspace of degree m by $\hat{V}(m)$. Then $\hat{V}(0)$, is a Lie subalgebra of $\hat{V}$.

Defining $L_n : A_n(V)$-module $\mapsto \mathbb{N}$-gradable V-mod

To define L_n, some further notions a necessary:
Let

$$\hat{V} = \mathbb{C}[t, t^{-1}] \otimes V / (D\mathbb{C}[t, t^{-1}] \otimes V), \tag{6}$$

where $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$.

For $v \in V$, denote by $v(m)$ the image of $v \otimes t^m$ in $\hat{V}$. Then $\hat{V}$ is a $\mathbb{Z}$-graded Lie algebra by defining $\deg v(m) = \mathrm{wt}\, v - m - 1$ and

$$[u(j), v(k)] = \sum_{i=0}^{\infty} \binom{j}{i} (u_i v)(j + k - i),$$

for $u, v \in V$, $j, k \in \mathbb{Z}$.

Denote the homogeneous subspace of degree m by $\hat{V}(m)$. Then $\hat{V}(0)$, is a Lie subalgebra of $\hat{V}$.

Defining L_n : $A_n(V)$-module $\mapsto$ $\mathbb{N}$-gradable V-mod

$A_n(V)$ is a Lie algebra via $[u, v] = u *_n v - v *_n u$, and then

$$\begin{array}{ccc} \hat{V}(0) & \longrightarrow & A_n(V) \\ v(\mathrm{wt}\, v - 1) & \mapsto & v + O_n(V) \end{array} \tag{7}$$

is a well-defined Lie algebra epimorphism.

Thus, for U an $A_n(V)$-module, we can lift U to a module for the Lie algebra $\hat{V}(0)$, and then to one for $P_n = \bigoplus_{p<-n} \hat{V}(p) \oplus \hat{V}(0)$ by letting $\hat{V}(-p)$ act trivially for $p \neq 0$.

Define

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = \mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(P_n)} U.$$

$M_n(U)$ is $\mathbb{N}$-graded as follows: Let U be degree n. Denote by $\mathcal{U}(\hat{V})_k$ the degree k space of $\mathcal{U}(\hat{V})$ induced from $\hat{V}$. Define the i-th degree subspace of $M_n(U)$ to be $M_n(U)(i) = \mathcal{U}(\hat{V})_{i-n}U$.

Defining L_n : $A_n(V)$-module $\mapsto$ $\mathbb{N}$-gradable V-mod

$A_n(V)$ is a Lie algebra via $[u, v] = u *_n v - v *_n u$, and then

$$\begin{array}{ccc} \hat{V}(0) & \longrightarrow & A_n(V) \\ v(\mathrm{wt}\, v - 1) & \mapsto & v + O_n(V) \end{array} \tag{7}$$

is a well-defined Lie algebra epimorphism.

Thus, for U an $A_n(V)$-module, we can lift U to a module for the Lie algebra $\hat{V}(0)$, and then to one for $P_n = \bigoplus_{p<-n} \hat{V}(p) \oplus \hat{V}(0)$ by letting $\hat{V}(-p)$ act trivially for $p \neq 0$.

Define

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = \mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(P_n)} U.$$

$M_n(U)$ is $\mathbb{N}$-graded as follows: Let U be degree n. Denote by $\mathcal{U}(\hat{V})_k$ the degree k space of $\mathcal{U}(\hat{V})$ induced from $\hat{V}$. Define the i-th degree subspace of $M_n(U)$ to be $M_n(U)(i) = \mathcal{U}(\hat{V})_{i-n}U$.

Defining L_n : $A_n(V)$-module $\mapsto$ $\mathbb{N}$-gradable V-mod

$A_n(V)$ is a Lie algebra via $[u, v] = u *_n v - v *_n u$, and then

$$\begin{array}{rcl} \hat{V}(0) & \longrightarrow & A_n(V) \\ v(\mathrm{wt}\, v - 1) & \mapsto & v + O_n(V) \end{array} \tag{7}$$

is a well-defined Lie algebra epimorphism.

Thus, for U an $A_n(V)$-module, we can lift U to a module for the Lie algebra $\hat{V}(0)$, and then to one for $P_n = \bigoplus_{p<-n} \hat{V}(p) \oplus \hat{V}(0)$ by letting $\hat{V}(-p)$ act trivially for $p \neq 0$.

Define

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = \mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(P_n)} U.$$

$M_n(U)$ is $\mathbb{N}$-graded as follows: Let U be degree n. Denote by $\mathcal{U}(\hat{V})_k$ the degree k space of $\mathcal{U}(\hat{V})$ induced from $\hat{V}$. Define the i-th degree subspace of $M_n(U)$ to be $M_n(U)(i) = \mathcal{U}(\hat{V})_{i-n}U$.

Defining $L_n : A_n(V)$-module $\mapsto \mathbb{N}$-gradable V-mod

$A_n(V)$ is a Lie algebra via $[u, v] = u *_n v - v *_n u$, and then

$$\begin{array}{rcl} \hat{V}(0) & \longrightarrow & A_n(V) \\ v(\mathrm{wt}\, v - 1) & \mapsto & v + O_n(V) \end{array} \tag{7}$$

is a well-defined Lie algebra epimorphism.

Thus, for U an $A_n(V)$-module, we can lift U to a module for the Lie algebra $\hat{V}(0)$, and then to one for $P_n = \bigoplus_{p<-n} \hat{V}(p) \oplus \hat{V}(0)$ by letting $\hat{V}(-p)$ act trivially for $p \neq 0$.

Define

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = \mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(P_n)} U.$$

$M_n(U)$ is $\mathbb{N}$-graded as follows: Let U be degree n. Denote by $\mathcal{U}(\hat{V})_k$ the degree k space of $\mathcal{U}(\hat{V})$ induced from $\hat{V}$. Define the i-th degree subspace of $M_n(U)$ to be $M_n(U)(i) = \mathcal{U}(\hat{V})_{i-n}U$.

Defining L_n : $A_n(V)$-module $\mapsto$ admissible V-mod

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = \mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(P_n)} U.$$

would be an $\mathbb{N}$-gradable V-module, except that it does not in general satisfy associativity. Also, there is another subspace that is in general nontrivial that we need to quotient out by. It so happens that the associativity relations are contained in this second subspace!

Let $U^* = \mathrm{Hom}(U, \mathbb{C})$. Extend U^* to $M_n(U)$ first by an induction to $M_n(U)(n)$ and then by letting U^* annihilate $\bigoplus_{i \neq n} M_n(U)(i)$.

Set

$$J = \{v \in M_n(U) \,|\, \langle u', xv\rangle = 0 \text{ for all } u' \in U^*, x \in \mathcal{U}(\hat{V})\}$$

and then let

$$L_n(U) = M_n(U)/J.$$

Defining L_n : $A_n(V)$-module $\mapsto$ admissible V-mod

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = \mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(P_n)} U.$$

would be an $\mathbb{N}$-gradable V-module, except that it does not in general satisfy associativity. Also, there is another subspace that is in general nontrivial that we need to quotient out by. It so happens that the associativity relations are contained in this second subspace!

Let $U^* = \mathrm{Hom}(U, \mathbb{C})$. Extend U^* to $M_n(U)$ first by an induction to $M_n(U)(n)$ and then by letting U^* annihilate $\bigoplus_{i \neq n} M_n(U)(i)$.

Set

$$J = \{v \in M_n(U) \,|\, \langle u', xv \rangle = 0 \text{ for all } u' \in U^*, x \in \mathcal{U}(\hat{V})\}$$

and then let

$$L_n(U) = M_n(U)/J.$$

Defining L_n : $A_n(V)$-module $\mapsto$ admissible V-mod

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = \mathcal{U}(\hat{V}) \otimes_{\mathcal{U}(P_n)} U.$$

would be an $\mathbb{N}$-gradable V-module, except that it does not in general satisfy associativity. Also, there is another subspace that is in general nontrivial that we need to quotient out by. It so happens that the associativity relations are contained in this second subspace!

Let $U^* = \mathrm{Hom}(U, \mathbb{C})$. Extend U^* to $M_n(U)$ first by an induction to $M_n(U)(n)$ and then by letting U^* annihilate $\bigoplus_{i \neq n} M_n(U)(i)$.

Set

$$J = \{v \in M_n(U) \,|\, \langle u', xv \rangle = 0 \text{ for all } u' \in U^*, x \in \mathcal{U}(\hat{V})\}$$

and then let

$$L_n(U) = M_n(U)/J.$$

A modification of a theorem of Dong, Li, and Mason

The following is a modification of a theorem of Dong, Li ,and Mason:

Theorem [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

Let U be a nonzero $A_n(V)$-module, for $n \in \mathbb{N}$.

- $L_n(U)$ is an $\mathbb{N}$-gradable V-module.
- If we assume further, for $n > 0$, that **there is no nonzero submodule of U that can factor through $A_{n-1}(V)$**, then

$$L_n(U)(0) \neq 0, \quad \text{and} \quad \Omega_n/\Omega_{n-1}(L_n(U)) \cong U.$$

Note that the added assumption in bold is in general necessary as we shall see in an example below.

A modification of a theorem of Dong, Li, and Mason

The following is a modification of a theorem of Dong, Li ,and Mason:

Theorem [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

Let U be a nonzero $A_n(V)$-module, for $n \in \mathbb{N}$.

- $L_n(U)$ is an $\mathbb{N}$-gradable V-module.
- If we assume further, for $n > 0$, that **there is no nonzero submodule of U that can factor through $A_{n-1}(V)$**, then

$$L_n(U)(0) \neq 0, \quad \text{and} \quad \Omega_n/\Omega_{n-1}(L_n(U)) \cong U.$$

Note that the added assumption in bold is in general necessary as we shall see in an example below.

Consequences of Main Theorem

We also have

Corollary [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

Suppose that for some fixed $n \in \mathbb{Z}_+$, $A_n(V)$ has a direct sum decomposition

$$A_n(V) \cong A_{n-1}(V) \oplus A_n'(V),$$

for $A_n'(V)$ a direct sum complement to $A_{n-1}(V)$, and let U be an $A_n(V)$ module. If U is trivial as an $A_{n-1}(V)$-module, then

$$\Omega_n/\Omega_{n-1}(L_n(U)) \cong U.$$

This will also be illustrated in comparing two examples below, after which we will give some more results about the structure of $L_n(U)$ for U indecomposable.

Consequences of Main Theorem

We also have

Corollary [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

Suppose that for some fixed $n \in \mathbb{Z}_+$, $A_n(V)$ has a direct sum decomposition

$$A_n(V) \cong A_{n-1}(V) \oplus A'_n(V),$$

for $A'_n(V)$ a direct sum complement to $A_{n-1}(V)$, and let U be an $A_n(V)$ module. If U is trivial as an $A_{n-1}(V)$-module, then

$$\Omega_n/\Omega_{n-1}(L_n(U)) \cong U.$$

This will also be illustrated in comparing two examples below, after which we will give some more results about the structure of $L_n(U)$ for U indecomposable.

Example 1: Heisenberg VOA

Let $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\alpha$ with a bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle \alpha, \alpha \rangle = 1$. Let

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$$

be the affinization of $\mathfrak{h}$ with bracket relations

$$[a(m), b(n)] = m\langle a, b\rangle \delta_{m+n,0}\mathbf{k}, \quad a, b \in \mathfrak{h},$$

$$[\mathbf{k}, a(m)] = 0,$$

where we define $a(m) = a \otimes t^m$ for $m \in \mathbb{Z}$ and $a \in \mathfrak{h}$.
Set

$$\hat{\mathfrak{h}}^+ = \mathfrak{h} \otimes t\mathbb{C}[t] \qquad \text{and} \qquad \hat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}].$$

Consider the induced $\hat{\mathfrak{h}}$-module given by

$$\begin{aligned} V &= M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t]\otimes\mathfrak{h}\oplus\mathbb{C}\mathbf{k})} \mathbb{C}\mathbf{1} \simeq S(\hat{\mathfrak{h}}^-) \qquad \text{(linearly)} \\ &= \mathbb{C}[\alpha(-1)\mathbf{1}, \alpha(-2)\mathbf{1}, \alpha(-3)\mathbf{1}, \ldots]. \end{aligned}$$

Example 1: Heisenberg VOA

$V = M_a(1)$ with $\omega_a = \frac{1}{2}\alpha(-1)^2\mathbf{1} + a\alpha(-2)\mathbf{1}$, for $a \in \mathbb{C}$, is a VOA with $c = 1 - 12a^2$, and

$$A_0(M_a(1)) = \mathbb{C}[x,y]/(x^2 - y) \cong \mathbb{C}[x],$$

where $x = \alpha(-1)\mathbf{1} + O_0(V)$ and $y = \alpha(-1)^2\mathbf{1} + O_0(V)$.

Proposition [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

For $V = M_a(1)$,

$$A_1(V) = \mathbb{C}[x,y]/(x^2 - y)(x^2 - y + 2),$$

where $x = \alpha(-1)\mathbf{1} + O_1(V)$ and $y = \alpha(-1)^2\mathbf{1} + O_1(V)$.

Furthermore, since $I_0 = (x^2 - y)$ and $I_1 = (x^2 - y + 2)$ are relatively prime, i.e. $I_0 \cap I_1 = 0$, and $I_0 + I_1 = \mathbb{C}[x,y]$, we have

$$A_1(V) \cong \mathbb{C}[x,y]/(x^2 - y) \oplus \mathbb{C}[x,y]/(x^2 - y + 2) \cong \mathbb{C}[x] \oplus \mathbb{C}[x].$$

Example 1: Heisenberg VOA

$V = M_a(1)$ with $\omega_a = \frac{1}{2}\alpha(-1)^2\mathbf{1} + a\alpha(-2)\mathbf{1}$, for $a \in \mathbb{C}$, is a VOA with $c = 1 - 12a^2$, and

$$A_0(M_a(1)) = \mathbb{C}[x, y]/(x^2 - y) \cong \mathbb{C}[x],$$

where $x = \alpha(-1)\mathbf{1} + O_0(V)$ and $y = \alpha(-1)^2\mathbf{1} + O_0(V)$.

Proposition [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

For $V = M_a(1)$,

$$A_1(V) = \mathbb{C}[x, y]/(x^2 - y)(x^2 - y + 2),$$

where $x = \alpha(-1)\mathbf{1} + O_1(V)$ and $y = \alpha(-1)^2\mathbf{1} + O_1(V)$.

Furthermore, since $I_0 = (x^2 - y)$ and $I_1 = (x^2 - y + 2)$ are relatively prime, i.e. $I_0 \cap I_1 = 0$, and $I_0 + I_1 = \mathbb{C}[x, y]$, we have

$$A_1(V) \cong \mathbb{C}[x, y]/(x^2 - y) \oplus \mathbb{C}[x, y]/(x^2 - y + 2) \cong \mathbb{C}[x] \oplus \mathbb{C}[x].$$

Example 1: Heisenberg VOA

That is for $V = M_a(1)$, we have

$$A_1(V) \cong A_0(V) \oplus A_0'(V)$$

and thus if U is an indecomposable $A_1(V)$-module that is itself not an $A_0(V)$-module then

$$\Omega_1/\Omega_0(L_1(U)) \cong U.$$

In particular, the original statement of Dong, Li and Mason holds for $n = 1$ and $V = M_a(1)$.

More concretely, the indecomposable modules for

$$A_0(M_a(1)) \cong \mathbb{C}[x, y]/(y - x^2) \cong \mathbb{C}[x]$$

are given by

$$U_0(\lambda, k) = \mathbb{C}[x, y]/((y - x^2), (x - \lambda)^{k+1}) \cong \mathbb{C}[x]/(x - \lambda)^{k+1} \quad (8)$$

for $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, and

$$L_0(U_0(\lambda, k)) \cong M_a(1) \otimes_{\mathbb{C}} \Omega(\lambda, k),$$

Example 1: Heisenberg VOA

That is for $V = M_a(1)$, we have

$$A_1(V) \cong A_0(V) \oplus A_0'(V)$$

and thus if U is an indecomposable $A_1(V)$-module that is itself not an $A_0(V)$-module then

$$\Omega_1/\Omega_0(L_1(U)) \cong U.$$

In particular, the original statement of Dong, Li and Mason holds for $n = 1$ and $V = M_a(1)$.

More concretely, the indecomposable modules for

$$A_0(M_a(1)) \cong \mathbb{C}[x, y]/(y - x^2) \cong \mathbb{C}[x]$$

are given by

$$U_0(\lambda, k) = \mathbb{C}[x, y]/((y - x^2), (x - \lambda)^{k+1}) \cong \mathbb{C}[x]/(x - \lambda)^{k+1} \tag{8}$$

for $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, and

$$L_0(U_0(\lambda, k)) \cong M_a(1) \otimes_{\mathbb{C}} \Omega(\lambda, k),$$

Example 1: Heisenberg VOA

That is for $V = M_a(1)$, we have

$$A_1(V) \cong A_0(V) \oplus A_0'(V)$$

and thus if U is an indecomposable $A_1(V)$-module that is itself not an $A_0(V)$-module then

$$\Omega_1/\Omega_0(L_1(U)) \cong U.$$

In particular, the original statement of Dong, Li and Mason holds for $n = 1$ and $V = M_a(1)$.

More concretely, the indecomposable modules for

$$A_0(M_a(1)) \cong \mathbb{C}[x,y]/(y - x^2) \cong \mathbb{C}[x]$$

are given by

$$U_0(\lambda, k) = \mathbb{C}[x,y]/((y - x^2), (x - \lambda)^{k+1}) \cong \mathbb{C}[x]/(x - \lambda)^{k+1} \quad (8)$$

for $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, and

$$L_0(U_0(\lambda, k)) \cong M_a(1) \otimes_{\mathbb{C}} \Omega(\lambda, k),$$

where $\Omega(\lambda, k)$ is a $(k+1)$-dimensional vacuum space such that $\alpha(0)$ acts with Jordan form given by

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}. \tag{9}$$

The zero mode of ω_a which is given by

$$L(0) = \sum_{m \in \mathbb{Z}_+} \alpha(-m)\alpha(m) + \frac{1}{2}\alpha(0)^2 - a\alpha(0) \tag{10}$$

acts on $\Omega(\lambda, k)$ such that the only eigenvalue is $\frac{1}{2}\lambda^2 - a\lambda$, which is the lowest conformal weight of $M_a(1) \otimes \Omega(\lambda, k)$.

where $\Omega(\lambda, k)$ is a $(k+1)$-dimensional vacuum space such that $\alpha(0)$ acts with Jordan form given by

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}. \tag{9}$$

The zero mode of ω_a which is given by

$$L(0) = \sum_{m \in \mathbb{Z}_+} \alpha(-m)\alpha(m) + \frac{1}{2}\alpha(0)^2 - a\alpha(0) \tag{10}$$

acts on $\Omega(\lambda, k)$ such that the only eigenvalue is $\frac{1}{2}\lambda^2 - a\lambda$, which is the lowest conformal weight of $M_a(1) \otimes \Omega(\lambda, k)$.

$L(0) - (\frac{1}{2}\lambda^2 - a\lambda)Id_k$ with respect to a Jordan basis for $\alpha(0)$ acting on $\Omega(\lambda, k)$ is given by

$$\begin{bmatrix} 0 & \lambda - a & \frac{1}{2} - a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda - a & \frac{1}{2} - a & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda - a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \lambda - a & \frac{1}{2} - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} . \qquad (11)$$

Thus $L(0)$ is diagonalizable if and only if:
(i) $k = 0$, which corresponds to the case when $M_a(1) \otimes \Omega(\lambda, k)$ is irreducible;
(ii) $k = 1$ and $\lambda = a$;
or
(iii) $k > 1$ and $\lambda = a = \frac{1}{2}$.
These $M_a(1) \otimes \Omega(\lambda, k)$ exhaust all the indecomposable generalized $M_a(1)$-modules.

$L(0) - (\frac{1}{2}\lambda^2 - a\lambda)Id_k$ with respect to a Jordan basis for $\alpha(0)$ acting on $\Omega(\lambda, k)$ is given by

$$\begin{bmatrix} 0 & \lambda - a & \frac{1}{2} - a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda - a & \frac{1}{2} - a & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda - a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \lambda - a & \frac{1}{2} - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} . \tag{11}$$

Thus $L(0)$ is diagonalizable if and only if:

(i) $k = 0$, which corresponds to the case when $M_a(1) \otimes \Omega(\lambda, k)$ is irreducible;

(ii) $k = 1$ and $\lambda = a$;

or

(iii) $k > 1$ and $\lambda = a = \frac{1}{2}$.

These $M_a(1) \otimes \Omega(\lambda, k)$ exhaust all the indecomposable generalized $M_a(1)$-modules.

$L(0) - (\frac{1}{2}\lambda^2 - a\lambda)Id_k$ with respect to a Jordan basis for $\alpha(0)$ acting on $\Omega(\lambda, k)$ is given by

$$\left[\begin{array}{ccccccc} 0 & \lambda - a & \frac{1}{2} - a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda - a & \frac{1}{2} - a & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda - a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \lambda - a & \frac{1}{2} - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]. \tag{11}$$

Thus $L(0)$ is diagonalizable if and only if:
(i) $k = 0$, which corresponds to the case when $M_a(1) \otimes \Omega(\lambda, k)$ is irreducible;
(ii) $k = 1$ and $\lambda = a$;
or
(iii) $k > 1$ and $\lambda = a = \frac{1}{2}$.
These $M_a(1) \otimes \Omega(\lambda, k)$ exhaust all the indecomposable generalized $M_a(1)$-modules.

$L(0) - (\frac{1}{2}\lambda^2 - a\lambda)Id_k$ with respect to a Jordan basis for $\alpha(0)$ acting on $\Omega(\lambda, k)$ is given by

$$\begin{bmatrix} 0 & \lambda - a & \frac{1}{2} - a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda - a & \frac{1}{2} - a & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda - a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \lambda - a & \frac{1}{2} - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{11}$$

Thus $L(0)$ is diagonalizable if and only if:
(i) $k = 0$, which corresponds to the case when $M_a(1) \otimes \Omega(\lambda, k)$ is irreducible;
(ii) $k = 1$ and $\lambda = a$;
or
(iii) $k > 1$ and $\lambda = a = \frac{1}{2}$.
These $M_a(1) \otimes \Omega(\lambda, k)$ exhaust all the indecomposable generalized $M_a(1)$-modules.

$L(0) - (\frac{1}{2}\lambda^2 - a\lambda)Id_k$ with respect to a Jordan basis for $\alpha(0)$ acting on $\Omega(\lambda, k)$ is given by

$$\left[\begin{array}{ccccccc} 0 & \lambda - a & \frac{1}{2} - a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda - a & \frac{1}{2} - a & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda - a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \lambda - a & \frac{1}{2} - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]. \qquad (11)$$

Thus $L(0)$ is diagonalizable if and only if:
(i) $k = 0$, which corresponds to the case when $M_a(1) \otimes \Omega(\lambda, k)$ is irreducible;
(ii) $k = 1$ and $\lambda = a$;
or
(iii) $k > 1$ and $\lambda = a = \frac{1}{2}$.
These $M_a(1) \otimes \Omega(\lambda, k)$ exhaust all the indecomposable generalized $M_a(1)$-modules.

$L(0) - (\frac{1}{2}\lambda^2 - a\lambda)Id_k$ with respect to a Jordan basis for $\alpha(0)$ acting on $\Omega(\lambda, k)$ is given by

$$\begin{bmatrix} 0 & \lambda - a & \frac{1}{2} - a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda - a & \frac{1}{2} - a & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda - a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \lambda - a & \frac{1}{2} - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{11}$$

Thus $L(0)$ is diagonalizable if and only if:
(i) $k = 0$, which corresponds to the case when $M_a(1) \otimes \Omega(\lambda, k)$ is irreducible;
(ii) $k = 1$ and $\lambda = a$;
or
(iii) $k > 1$ and $\lambda = a = \frac{1}{2}$.
These $M_a(1) \otimes \Omega(\lambda, k)$ exhaust all the indecomposable generalized $M_a(1)$-modules.

The $\mathbb{N}$-grading of $M_a(1) \otimes \Omega(\lambda, k)$ is explicitly given by

$$M_a(1) \otimes \Omega(\lambda, k) = \coprod_{m \in \mathbb{N}} M_a(1)_m \otimes \Omega(\lambda, k)$$

where $M_a(1)_m$ is the weight m space of the vertex operator algebra $M_a(1)$ and thus $M_a(1)_m \otimes \Omega(\lambda, k)$ is the space of generalized eigenvectors of weight $m + \frac{1}{2}\lambda^2 - a\lambda$ with respect to $L(0)$.

The indecomposable modules for

$$A_1(M_a(1)) \cong \mathbb{C}[x, y]/((y - x^2)(y - x^2 - 2)) \cong A_0(M_a(1)) \oplus \mathbb{C}[x]$$

are given by the indecomposable modules $U_0(\lambda, k)$ for $A_0(M_a(1)) \cong \mathbb{C}[x]$ as before or by

$$U_1(\lambda, k) = \mathbb{C}[x, y]/((y - x^2 - 2), (x - \lambda)^{k+1}) \cong \mathbb{C}[x]/(x - \lambda)^{k+1} \quad (12)$$

for $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, in which case,

$$U_1(\lambda, k) \cong \alpha(-1)\mathbf{1} \otimes \Omega(\lambda, k) \quad \text{and} \quad L_1(U_1(\lambda, k)) \cong M_a(1) \otimes \Omega(\lambda, k).$$

The $\mathbb{N}$-grading of $M_a(1) \otimes \Omega(\lambda, k)$ is explicitly given by

$$M_a(1) \otimes \Omega(\lambda, k) = \coprod_{m \in \mathbb{N}} M_a(1)_m \otimes \Omega(\lambda, k)$$

where $M_a(1)_m$ is the weight m space of the vertex operator algebra $M_a(1)$ and thus $M_a(1)_m \otimes \Omega(\lambda, k)$ is the space of generalized eigenvectors of weight $m + \frac{1}{2}\lambda^2 - a\lambda$ with respect to $L(0)$.

The indecomposable modules for

$$A_1(M_a(1)) \cong \mathbb{C}[x, y]/((y - x^2)(y - x^2 - 2)) \cong A_0(M_a(1)) \oplus \mathbb{C}[x]$$

are given by the indecomposable modules $U_0(\lambda, k)$ for $A_0(M_a(1)) \cong \mathbb{C}[x]$ as before or by

$$U_1(\lambda, k) = \mathbb{C}[x, y]/((y - x^2 - 2), (x - \lambda)^{k+1}) \cong \mathbb{C}[x]/(x - \lambda)^{k+1} \quad (12)$$

for $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, in which case,

$$U_1(\lambda, k) \cong \alpha(-1)\mathbf{1} \otimes \Omega(\lambda, k) \quad \text{and} \quad L_1(U_1(\lambda, k)) \cong M_a(1) \otimes \Omega(\lambda, k).$$

Thus the possible cases for $\Omega_1/\Omega_0(L_1(U))$ for U an indecomposable $A_1(M_a(1))$-module are

$$U = U_0(\lambda, k), \quad L_1(U)(0) = 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U_1(\lambda, k) \ncong U$$

or

$$U = U_1(\lambda, k) \quad L_1(U)(0) = \Omega(\lambda, k) \neq 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U.$$

Note however that in the case of $U = U_0(\lambda, k)$, the $M_a(1)$-module $L_1(U)$ is in fact $M_a(1) \otimes \Omega(\lambda, k)$ but the grading as an $\mathbb{N}$-gradable module is shifted up one.

By regrading to obtain an $\mathbb{N}$-gradable $M_a(1)$-module such that the first nonzero degree is 0, this module is again just $M_a(1) \otimes \Omega(\lambda, k)$.

Thus $A_1(M_a(1))$ gives no new information about the indecomposable $M_a(1)$-modules not already given by $A_0(M_a(1))$.

Thus the possible cases for $\Omega_1/\Omega_0(L_1(U))$ for U an indecomposable $A_1(M_a(1))$-module are

$$U = U_0(\lambda, k), \quad L_1(U)(0) = 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U_1(\lambda, k) \not\cong U$$

or

$$U = U_1(\lambda, k) \quad L_1(U)(0) = \Omega(\lambda, k) \neq 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U.$$

Note however that in the case of $U = U_0(\lambda, k)$, the $M_a(1)$-module $L_1(U)$ is in fact $M_a(1) \otimes \Omega(\lambda, k)$ but the grading as an $\mathbb{N}$-gradable module is shifted up one.

By regrading to obtain an $\mathbb{N}$-gradable $M_a(1)$-module such that the first nonzero degree is 0, this module is again just $M_a(1) \otimes \Omega(\lambda, k)$.

Thus $A_1(M_a(1))$ gives no new information about the indecomposable $M_a(1)$-modules not already given by $A_0(M_a(1))$.

Thus the possible cases for $\Omega_1/\Omega_0(L_1(U))$ for U an indecomposable $A_1(M_a(1))$-module are

$$U = U_0(\lambda, k), \quad L_1(U)(0) = 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U_1(\lambda, k) \not\cong U$$

or

$$U = U_1(\lambda, k) \quad L_1(U)(0) = \Omega(\lambda, k) \neq 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U.$$

Note however that in the case of $U = U_0(\lambda, k)$, the $M_a(1)$-module $L_1(U)$ is in fact $M_a(1) \otimes \Omega(\lambda, k)$ but the grading as an $\mathbb{N}$-gradable module is shifted up one.

By regrading to obtain an $\mathbb{N}$-gradable $M_a(1)$-module such that the first nonzero degree is 0, this module is again just $M_a(1) \otimes \Omega(\lambda, k)$.

Thus $A_1(M_a(1))$ gives no new information about the indecomposable $M_a(1)$-modules not already given by $A_0(M_a(1))$.

Thus the possible cases for $\Omega_1/\Omega_0(L_1(U))$ for U an indecomposable $A_1(M_a(1))$-module are

$$U = U_0(\lambda, k), \quad L_1(U)(0) = 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U_1(\lambda, k) \not\cong U$$

or

$$U = U_1(\lambda, k) \quad L_1(U)(0) = \Omega(\lambda, k) \neq 0, \text{ and } \Omega_1/\Omega_0(L_1(U)) \cong U.$$

Note however that in the case of $U = U_0(\lambda, k)$, the $M_a(1)$-module $L_1(U)$ is in fact $M_a(1) \otimes \Omega(\lambda, k)$ but the grading as an $\mathbb{N}$-gradable module is shifted up one.

By regrading to obtain an $\mathbb{N}$-gradable $M_a(1)$-module such that the first nonzero degree is 0, this module is again just $M_a(1) \otimes \Omega(\lambda, k)$.

Thus $A_1(M_a(1))$ gives no new information about the indecomposable $M_a(1)$-modules not already given by $A_0(M_a(1))$.

Example 2: Virasoro VOA

- Let $\mathcal{L} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \in \mathbb{Z}\}$ be the Virasoro algebra with central charge $\mathbf{c}$.
- Let $\mathcal{L}^{\geq 0} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \geq 0\}$.
- Let $\mathbb{C}_{c,h}$ be the 1-dimensional $\mathcal{L}^{\geq 0}$-module where $\mathbf{c}$ acts as c for some $c \in \mathbb{C}$, L_0 acts as h for some $h \in \mathbb{C}$, and L_n acts trivially for $n \geq 1$.
- Form the induced $\mathcal{L}$-module

$$M(c,h) = U(\mathcal{L}) \otimes_{\mathcal{L}^{\geq 0}} \mathbb{C}_{c,h}.$$

- Write $L(n)$ for the operator on a Virasoro module corresponding to L_n, and $\mathbf{1}_{c,h} = 1 \in \mathbb{C}_{c,h}$.
- Then

$$V_{Vir} = M(c,0)/\langle L(-1)\mathbf{1}_{c,0}\rangle$$

has a natural VOA structure with vacuum vector $1 = \mathbf{1}_{c,0}$, and $\omega = L(-2)\mathbf{1}_{c,0}$.

Example 2: Virasoro VOA

- Let $\mathcal{L} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \in \mathbb{Z}\}$ be the Virasoro algebra with central charge $\mathbf{c}$.
- Let $\mathcal{L}^{\geq 0} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \geq 0\}$.
- Let $\mathbb{C}_{c,h}$ be the 1-dimensional $\mathcal{L}^{\geq 0}$-module where $\mathbf{c}$ acts as c for some $c \in \mathbb{C}$, L_0 acts as h for some $h \in \mathbb{C}$, and L_n acts trivially for $n \geq 1$.
- Form the induced $\mathcal{L}$-module

$$M(c,h) = U(\mathcal{L}) \otimes_{\mathcal{L}^{\geq 0}} \mathbb{C}_{c,h}.$$

- Write $L(n)$ for the operator on a Virasoro module corresponding to L_n, and $\mathbf{1}_{c,h} = 1 \in \mathbb{C}_{c,h}$.
- Then

$$V_{Vir} = M(c,0)/\langle L(-1)\mathbf{1}_{c,0}\rangle$$

has a natural VOA structure with vacuum vector $\mathbf{1} = \mathbf{1}_{c,0}$, and $\omega = L(-2)\mathbf{1}_{c,0}$.

Example 2: Virasoro VOA

- Let $\mathcal{L} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \in \mathbb{Z}\}$ be the Virasoro algebra with central charge $\mathbf{c}$.
- Let $\mathcal{L}^{\geq 0} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \geq 0\}$.
- Let $\mathbb{C}_{c,h}$ be the 1-dimensional $\mathcal{L}^{\geq 0}$-module where $\mathbf{c}$ acts as c for some $c \in \mathbb{C}$, L_0 acts as h for some $h \in \mathbb{C}$, and L_n acts trivially for $n \geq 1$.
- Form the induced $\mathcal{L}$-module

$$M(c,h) = U(\mathcal{L}) \otimes_{\mathcal{L}^{\geq 0}} \mathbb{C}_{c,h}.$$

- Write $L(n)$ for the operator on a Virasoro module corresponding to L_n, and $\mathbf{1}_{c,h} = 1 \in \mathbb{C}_{c,h}$.
- Then

$$V_{Vir} = M(c,0)/\langle L(-1)\mathbf{1}_{c,0}\rangle$$

has a natural VOA structure with vacuum vector $1 = \mathbf{1}_{c,0}$, and $\omega = L(-2)\mathbf{1}_{c,0}$.

Example 2: Virasoro VOA

- Let $\mathcal{L} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \in \mathbb{Z}\}$ be the Virasoro algebra with central charge $\mathbf{c}$.
- Let $\mathcal{L}^{\geq 0} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \geq 0\}$.
- Let $\mathbb{C}_{c,h}$ be the 1-dimensional $\mathcal{L}^{\geq 0}$-module where $\mathbf{c}$ acts as c for some $c \in \mathbb{C}$, L_0 acts as h for some $h \in \mathbb{C}$, and L_n acts trivially for $n \geq 1$.
- Form the induced $\mathcal{L}$-module

$$M(c,h) = U(\mathcal{L}) \otimes_{\mathcal{L}^{\geq 0}} \mathbb{C}_{c,h}.$$

- Write $L(n)$ for the operator on a Virasoro module corresponding to L_n, and $\mathbf{1}_{c,h} = 1 \in \mathbb{C}_{c,h}$.
- Then

$$V_{Vir} = M(c,0)/\langle L(-1)\mathbf{1}_{c,0}\rangle$$

has a natural VOA structure with vacuum vector $1 = \mathbf{1}_{c,0}$, and $\omega = L(-2)\mathbf{1}_{c,0}$.

Example 2: Virasoro VOA

- Let $\mathcal{L} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \in \mathbb{Z}\}$ be the Virasoro algebra with central charge $\mathbf{c}$.
- Let $\mathcal{L}^{\geq 0} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \geq 0\}$.
- Let $\mathbb{C}_{c,h}$ be the 1-dimensional $\mathcal{L}^{\geq 0}$-module where $\mathbf{c}$ acts as c for some $c \in \mathbb{C}$, L_0 acts as h for some $h \in \mathbb{C}$, and L_n acts trivially for $n \geq 1$.
- Form the induced $\mathcal{L}$-module

$$M(c, h) = U(\mathcal{L}) \otimes_{\mathcal{L}^{\geq 0}} \mathbb{C}_{c,h}.$$

- Write $L(n)$ for the operator on a Virasoro module corresponding to L_n, and $\mathbf{1}_{c,h} = 1 \in \mathbb{C}_{c,h}$.
- Then

$$V_{Vir} = M(c, 0)/\langle L(-1)\mathbf{1}_{c,0}\rangle$$

has a natural VOA structure with vacuum vector $1 = \mathbf{1}_{c,0}$, and $\omega = L(-2)\mathbf{1}_{c,0}$.

Example 2: Virasoro VOA

- Let $\mathcal{L} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \in \mathbb{Z}\}$ be the Virasoro algebra with central charge $\mathbf{c}$.
- Let $\mathcal{L}^{\geq 0} = \mathrm{span}_{\mathbb{C}}\{L_n, \mathbf{c} \mid n \geq 0\}$.
- Let $\mathbb{C}_{c,h}$ be the 1-dimensional $\mathcal{L}^{\geq 0}$-module where $\mathbf{c}$ acts as c for some $c \in \mathbb{C}$, L_0 acts as h for some $h \in \mathbb{C}$, and L_n acts trivially for $n \geq 1$.
- Form the induced $\mathcal{L}$-module

$$M(c,h) = U(\mathcal{L}) \otimes_{\mathcal{L}^{\geq 0}} \mathbb{C}_{c,h}.$$

- Write $L(n)$ for the operator on a Virasoro module corresponding to L_n, and $\mathbf{1}_{c,h} = 1 \in \mathbb{C}_{c,h}$.
- Then

$$V_{Vir} = M(c,0)/\langle L(-1)\mathbf{1}_{c,0}\rangle$$

has a natural VOA structure with vacuum vector $1 = \mathbf{1}_{c,0}$, and $\omega = L(-2)\mathbf{1}_{c,0}$.

Example 2: Virasoro VOA

Proposition [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

$$\begin{aligned} A_0(V_{Vir}) &\cong \mathbb{C}[x,y]/(y - x^2 - 2x) \\ &\cong \mathbb{C}[x], \end{aligned}$$

where $x = L(-2)\mathbf{1} + O_0(V_{Vir})$ and $y = L(-2)^2\mathbf{1} + O_0(V_{Vir})$, and

$$\begin{aligned} A_1(V_{Vir}) &\cong \mathbb{C}[x,y]/(y - x^2 - 2x)(y - x^2 - 6x + 4) \\ &\cong \mathbb{C}[x',y']/(x'y'), \end{aligned}$$

where $x = L(-2)\mathbf{1} + O_1(V_{Vir})$ and $y = L(-2)^2\mathbf{1} + O_1(V_{Vir})$.

Modules for rings such as $\mathbb{C}[x',y']/(x'y')$ were classified in 1969 by Nazarova and Roiter, showing a rich array of types of modules arising from the fact that

$$A_1(V) = \mathbb{C}[x,y]/I_0 I_1 \quad \text{with} \quad I_0 \cap I_1 \neq 0.$$

This nontrivial intersection of ideals gives interesting examples.

Example 2: Virasoro VOA

Proposition [K.B., N. Vander Werf, and J. Yang, Jour. Pure Appl. Alg. 2019]:

$$\begin{aligned} A_0(V_{Vir}) &\cong \mathbb{C}[x,y]/(y - x^2 - 2x) \\ &\cong \mathbb{C}[x], \end{aligned}$$

where $x = L(-2)\mathbf{1} + O_0(V_{Vir})$ and $y = L(-2)^2\mathbf{1} + O_0(V_{Vir})$, and

$$\begin{aligned} A_1(V_{Vir}) &\cong \mathbb{C}[x,y]/(y - x^2 - 2x)(y - x^2 - 6x + 4) \\ &\cong \mathbb{C}[x', y']/(x'y'), \end{aligned}$$

where $x = L(-2)\mathbf{1} + O_1(V_{Vir})$ and $y = L(-2)^2\mathbf{1} + O_1(V_{Vir})$.

Modules for rings such as $\mathbb{C}[x', y']/(x'y')$ were classified in 1969 by Nazarova and Roiter, showing a rich array of types of modules arising from the fact that

$$A_1(V) = \mathbb{C}[x,y]/I_0 I_1 \quad \text{with} \quad I_0 \cap I_1 \neq 0.$$

This nontrivial intersection of ideals gives interesting examples.

An illustrative family of V_{Vir}-modules

In particular, illustrative examples for:

1. The necessity of the extra condition we imposed on the $A_n(V)$-module U for $\Omega_n/\Omega_{n-1} \circ L_n(U) \cong U$ to hold;

2. When $A_{n-1}(V)$ is not naturally isomorphic to a direct summand of $A_n(V)$, how the structure of indecomposable V-modules is affected.

For instance: For $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} U &= \mathbb{C}[x,y]/((y - x^2 - 2x)^{k+1}, (y - x^2 - 6x + 4)) \\ &\cong \mathbb{C}[x]/(x-1)^{k+1} \end{aligned}$$

are indecomposable $A_1(V_{Vir})$-modules that do not factor through $A_0(V_{Vir})$, but do have a nontrivial submodule that factors through $A_0(V_{Vir})$.

An illustrative family of V_{Vir}-modules

In particular, illustrative examples for:

1. The necessity of the extra condition we imposed on the $A_n(V)$-module U for $\Omega_n/\Omega_{n-1} \circ L_n(U) \cong U$ to hold;

2. When $A_{n-1}(V)$ is not naturally isomorphic to a direct summand of $A_n(V)$, how the structure of indecomposable V-modules is affected.

For instance: For $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} U &= \mathbb{C}[x,y]/((y - x^2 - 2x)^{k+1}, (y - x^2 - 6x + 4)) \\ &\cong \mathbb{C}[x]/(x-1)^{k+1} \end{aligned}$$

are indecomposable $A_1(V_{Vir})$-modules that do not factor through $A_0(V_{Vir})$, but do have a nontrivial submodule that factors through $A_0(V_{Vir})$.

An illustrative family of V_{Vir}-modules

In particular, illustrative examples for:

1. The necessity of the extra condition we imposed on the $A_n(V)$-module U for $\Omega_n/\Omega_{n-1} \circ L_n(U) \cong U$ to hold;

2. When $A_{n-1}(V)$ is not naturally isomorphic to a direct summand of $A_n(V)$, how the structure of indecomposable V-modules is affected.

For instance: For $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} U &= \mathbb{C}[x,y]/((y-x^2-2x)^{k+1}, (y-x^2-6x+4)) \\ &\cong \mathbb{C}[x]/(x-1)^{k+1} \end{aligned}$$

are indecomposable $A_1(V_{Vir})$-modules that do not factor through $A_0(V_{Vir})$, but do have a nontrivial submodule that factors through $A_0(V_{Vir})$.

An illustrative family of V_{Vir}-modules

In particular, illustrative examples for:

1. The necessity of the extra condition we imposed on the $A_n(V)$-module U for $\Omega_n/\Omega_{n-1} \circ L_n(U) \cong U$ to hold;

2. When $A_{n-1}(V)$ is not naturally isomorphic to a direct summand of $A_n(V)$, how the structure of indecomposable V-modules is affected.

For instance: For $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} U &= \mathbb{C}[x,y]/((y - x^2 - 2x)^{k+1}, (y - x^2 - 6x + 4)) \\ &\cong \mathbb{C}[x]/(x-1)^{k+1} \end{aligned}$$

are indecomposable $A_1(V_{Vir})$-modules that do not factor through $A_0(V_{Vir})$, but do have a nontrivial submodule that factors through $A_0(V_{Vir})$.

An illustrative family of V_{Vir}-modules

More concretely, let w be a lowest weight vector for the Virasoro algebra such that

$$L(0)^k w \neq 0; \quad L(0)^{k+1} w = 0.$$

Set

$$U' = \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^i w \mid i = 0, \dots, k\}.$$

Lemma [K.B., N. Vander Werf, and J. Yang, JPAA 2019]:

As $A_1(V_{Vir})$-modules,

$$U = \mathbb{C}[x]/(x-1)^{k+1} \cong U'$$

under the isomorphism

$$\phi : U \longrightarrow U', \qquad \overline{(x-1)^i} \mapsto L(-1)L(0)^i w,$$

where $\overline{(x-1)^i}$ is the image of $(x-1)^i$ in U under the canonical projection and $i = 0, \dots, k$.

An illustrative family of V_{Vir}-modules

More concretely, let w be a lowest weight vector for the Virasoro algebra such that

$$L(0)^k w \neq 0; \quad L(0)^{k+1} w = 0.$$

Set

$$U' = \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^i w \mid i = 0, \ldots, k\}.$$

Lemma [K.B., N. Vander Werf, and J. Yang, JPAA 2019]:

As $A_1(V_{Vir})$-modules,

$$U = \mathbb{C}[x]/(x-1)^{k+1} \cong U'$$

under the isomorphism

$$\phi : U \longrightarrow U', \qquad \overline{(x-1)^i} \mapsto L(-1)L(0)^i w,$$

where $\overline{(x-1)^i}$ is the image of $(x-1)^i$ in U under the canonical projection and $i = 0, \ldots, k$.

An illustrative family of V_{Vir}-modules

So $U \cong \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^i w \mid i = 0, \ldots, k\} \neq 0$ and U has the nontrivial submodule corresponding to

$$\mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w\},$$

that factors through $A_0(V_{Vir})$, although U does not.

We have

$$\begin{aligned} L_1(U) &= M_1(U)/J = (\mathcal{U}(\hat{V}_{Vir}) \otimes_{\mathcal{U}(P_1)} U)/J \\ &= \mathrm{span}_{\mathbb{C}}\{L(-n_1)L(-n_2)\cdots L(-n_r)L(0)^i w \mid i = 0, \ldots, k, \\ &\qquad\qquad r \in \mathbb{N},\ n_1 \geq n_2 \geq \cdots \geq n_r \geq 1\} \, mod\, J \end{aligned}$$

and

$$\Omega_0(L_1(U)) = L_1(U)(0) \oplus \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w + J\}$$

$$\Omega_1(L_1(U)) = L_1(U)(0) \oplus L_1(U)(1) \oplus \text{higher degree terms}$$

An illustrative family of V_{Vir}-modules

So $U \cong \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^i w \mid i = 0, \ldots, k\} \neq 0$ and U has the nontrivial submodule corresponding to

$$\mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w\},$$

that factors through $A_0(V_{Vir})$, although U does not.

We have

$$\begin{aligned} L_1(U) &= M_1(U)/J = (\mathcal{U}(\hat{V}_{Vir}) \otimes_{\mathcal{U}(P_1)} U)/J \\ &= \mathrm{span}_{\mathbb{C}}\{L(-n_1)L(-n_2)\cdots L(-n_r)L(0)^i w \mid i = 0, \ldots, k, \\ &\qquad\qquad r \in \mathbb{N},\ n_1 \geq n_2 \geq \cdots \geq n_r \geq 1\} mod\, J \end{aligned}$$

and

$$\begin{aligned} \Omega_0(L_1(U)) &= L_1(U)(0) \oplus \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w + J\} \\ \Omega_1(L_1(U)) &= L_1(U)(0) \oplus L_1(U)(1) \oplus \text{higher degree terms} \end{aligned}$$

An illustrative family of V_{Vir}-modules

So $U \cong \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^i w \mid i = 0, \ldots, k\} \neq 0$ and U has the nontrivial submodule corresponding to

$$\mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w\},$$

that factors through $A_0(V_{Vir})$, although U does not.

We have

$$\begin{aligned} L_1(U) &= M_1(U)/J = (\mathcal{U}(\hat{V}_{Vir}) \otimes_{\mathcal{U}(P_1)} U)/J \\ &= \mathrm{span}_{\mathbb{C}}\{L(-n_1)L(-n_2)\cdots L(-n_r)L(0)^i w \mid i = 0, \ldots, k, \\ &\qquad\qquad r \in \mathbb{N},\ n_1 \geq n_2 \geq \cdots \geq n_r \geq 1\} mod\, J \end{aligned}$$

and

$$\Omega_0(L_1(U)) = L_1(U)(0) \oplus \mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w + J\}$$

$$\Omega_1(L_1(U)) = L_1(U)(0) \oplus L_1(U)(1) \oplus \text{higher degree terms}$$

A illustrative family of V_{Vir}-modules

And thus

$$\begin{aligned} \Omega_1/\Omega_0(L_1(U)) &\cong U/\mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w\} \\ &\qquad \oplus \text{higher degree terms} \\ &\ncong U, \end{aligned}$$

illustrating the necessity of the extra condition in our Main Theorem (modifying a theorem of Dong, Li and Mason).

Observe that this is due to the fact that $A_1(V)$ does not have $A_0(V)$ as an isomorphic direct sum component for $V = V_{Vir}$ (in comparison to $M_a(1)$ which does have $A_0(M_a(1))$ isomorphic to a direct sum component of $A_1(M_a(1))$.

And that this is due to the fact that $A_1(V_{Vir}) \cong \mathbb{C}[x, y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have nontrivial intersection, i.e., are not relatively prime, where as $A_1(M(1)) \cong \mathbb{C}[x, y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have trivial intersection, i.e. are relatively prime.

A illustrative family of V_{Vir}-modules

And thus

$$\begin{aligned} \Omega_1/\Omega_0(L_1(U)) &\cong U/\mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w\} \\ &\qquad \oplus \text{higher degree terms} \\ &\ncong U, \end{aligned}$$

illustrating the necessity of the extra condition in our Main Theorem (modifying a theorem of Dong, Li and Mason).

Observe that this is due to the fact that $A_1(V)$ does not have $A_0(V)$ as an isomorphic direct sum component for $V = V_{Vir}$ (in comparison to $M_a(1)$ which does have $A_0(M_a(1))$ isomorphic to a direct sum component of $A_1(M_a(1))$.

And that this is due to the fact that $A_1(V_{Vir}) \cong \mathbb{C}[x, y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have nontrivial intersection, i.e., are not relatively prime, where as $A_1(M(1)) \cong \mathbb{C}[x, y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have trivial intersection, i.e. are relatively prime.

A illustrative family of V_{Vir}-modules

And thus

$$\begin{aligned} \Omega_1/\Omega_0(L_1(U)) &\cong U/\mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w\} \\ &\qquad \oplus \text{higher degree terms} \\ &\ncong U, \end{aligned}$$

illustrating the necessity of the extra condition in our Main Theorem (modifying a theorem of Dong, Li and Mason).

Observe that this is due to the fact that $A_1(V)$ does not have $A_0(V)$ as an isomorphic direct sum component for $V = V_{Vir}$ (in comparison to $M_a(1)$ which does have $A_0(M_a(1))$ isomorphic to a direct sum component of $A_1(M_a(1))$.

And that this is due to the fact that $A_1(V_{Vir}) \cong \mathbb{C}[x,y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have nontrivial intersection, i.e., are not relatively prime, where as $A_1(M(1)) \cong \mathbb{C}[x,y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have trivial intersection, i.e. are relatively prime.

A illustrative family of V_{Vir}-modules

And thus

$$\begin{aligned} \Omega_1/\Omega_0(L_1(U)) &\cong U/\mathrm{span}_{\mathbb{C}}\{L(-1)L(0)^k w\} \\ &\qquad\qquad \oplus \text{higher degree terms} \\ &\ncong U, \end{aligned}$$

illustrating the necessity of the extra condition in our Main Theorem (modifying a theorem of Dong, Li and Mason).

Observe that this is due to the fact that $A_1(V)$ does not have $A_0(V)$ as an isomorphic direct sum component for $V = V_{Vir}$ (in comparison to $M_a(1)$ which does have $A_0(M_a(1))$ isomorphic to a direct sum component of $A_1(M_a(1))$.

And that this is due to the fact that $A_1(V_{Vir}) \cong \mathbb{C}[x, y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have nontrivial intersection, i.e., are not relatively prime, where as $A_1(M(1)) \cong \mathbb{C}[x, y]/(p(x)q(x))$ where $p(x)$ and $q(x)$ have trivial intersection, i.e. are relatively prime.

Other illustrative examples for V_{Vir}

Virasoro Example 2: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad W \neq \langle W(n) \rangle$$

even if W is simple: Let $c \neq 0$, and

- $L(c, 0)$ = the unique simple minimal VOA with central charge c, up to isomorphism, i.e.,

$$L(c, 0) \cong V_{Vir}(c, 0)/T(c, 0)$$

for $T(c, 0)$ the largest proper ideal.

- Let $V = W = L(c, 0)$. Then $W(0) = \mathbb{C}\mathbf{1}$, with $\mathbf{1} = \mathbf{1}_{(c,0)}$, and $W(1) = 0$. Thus in this case

$$\Omega_0(W) = \mathbb{C}\mathbf{1} = W(0) = W(0) \oplus W(1) = \Omega_1(W).$$

Therefore

$$L_1(\Omega_1/\Omega_0(W)) = 0 \ncong W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 2: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad W \neq \langle W(n) \rangle$$

even if W is simple: Let $c \neq 0$, and

- $L(c,0)$ = the unique simple minimal VOA with central charge c, up to isomorphism, i.e.,

  $$L(c,0) \cong V_{Vir}(c,0)/T(c,0)$$

  for $T(c,0)$ the largest proper ideal.

- Let $V = W = L(c,0)$. Then $W(0) = \mathbb{C}\mathbf{1}$, with $\mathbf{1} = \mathbf{1}_{(c,0)}$, and $W(1) = 0$. Thus in this case

  $$\Omega_0(W) = \mathbb{C}\mathbf{1} = W(0) = W(0) \oplus W(1) = \Omega_1(W).$$

  Therefore

  $$L_1(\Omega_1/\Omega_0(W)) = 0 \ncong W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 2: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \not\cong W \quad \textbf{since} \quad W \neq \langle W(n) \rangle$$

even if W is simple: Let $c \neq 0$, and

- $L(c,0)$ = the unique simple minimal VOA with central charge c, up to isomorphism, i.e.,

  $$L(c,0) \cong V_{Vir}(c,0)/T(c,0)$$

  for $T(c,0)$ the largest proper ideal.
- Let $V = W = L(c,0)$. Then $W(0) = \mathbb{C}\mathbf{1}$, with $\mathbf{1} = \mathbf{1}_{(c,0)}$, and $W(1) = 0$. Thus in this case

  $$\Omega_0(W) = \mathbb{C}\mathbf{1} = W(0) = W(0) \oplus W(1) = \Omega_1(W).$$

  Therefore

  $$L_1(\Omega_1/\Omega_0(W)) = 0 \not\cong W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 2: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad W \neq \langle W(n) \rangle$$

even if W is simple: Let $c \neq 0$, and

- $L(c,0)$ = the unique simple minimal VOA with central charge c, up to isomorphism, i.e.,

  $$L(c,0) \cong V_{Vir}(c,0)/T(c,0)$$

  for $T(c,0)$ the largest proper ideal.
- Let $V = W = L(c,0)$. Then $W(0) = \mathbb{C}\mathbf{1}$, with $\mathbf{1} = \mathbf{1}_{(c,0)}$, and $W(1) = 0$. Thus in this case

  $$\Omega_0(W) = \mathbb{C}\mathbf{1} = W(0) = W(0) \oplus W(1) = \Omega_1(W).$$

  Therefore

  $$L_1(\Omega_1/\Omega_0(W)) = 0 \ncong W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 2: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad W \neq \langle W(n) \rangle$$

even if W is simple: Let $c \neq 0$, and

- $L(c,0)$ = the unique simple minimal VOA with central charge c, up to isomorphism, i.e.,

  $$L(c,0) \cong V_{Vir}(c,0)/T(c,0)$$

  for $T(c,0)$ the largest proper ideal.
- Let $V = W = L(c,0)$. Then $W(0) = \mathbb{C}\mathbf{1}$, with $\mathbf{1} = \mathbf{1}_{(c,0)}$, and $W(1) = 0$. Thus in this case

  $$\Omega_0(W) = \mathbb{C}\mathbf{1} = W(0) = W(0) \oplus W(1) = \Omega_1(W).$$

  Therefore

  $$L_1(\Omega_1/\Omega_0(W)) = 0 \ncong W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 2: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad W \neq \langle W(n) \rangle$$

even if W is simple: Let $c \neq 0$, and

- $L(c,0)$ = the unique simple minimal VOA with central charge c, up to isomorphism, i.e.,

  $$L(c,0) \cong V_{Vir}(c,0)/T(c,0)$$

  for $T(c,0)$ the largest proper ideal.
- Let $V = W = L(c,0)$. Then $W(0) = \mathbb{C}\mathbf{1}$, with $\mathbf{1} = \mathbf{1}_{(c,0)}$, and $W(1) = 0$. Thus in this case

  $$\Omega_0(W) = \mathbb{C}\mathbf{1} = W(0) = W(0) \oplus W(1) = \Omega_1(W).$$

  Therefore

  $$L_1(\Omega_1/\Omega_0(W)) = 0 \ncong W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 3: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Let $V = L(c,0)$ and $c \neq c_{p,q}$ for $c_{p,q}$ the minimal series.
- Then $V = L(c,0) = V_{Vir}(c,0)$ and V is a simple VOA.
- Let $W = M(c,0)$ which is not a simple V-module but is indecomposable.
- Since $M(c,0)/\langle L(-1)\rangle$ is simple, we have

$$\begin{aligned}
\Omega_0(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}\} \\
&= W(0) \oplus W(1) \\
\Omega_1(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}, L(-1)^2\mathbf{1}\} \\
&= W(0) \oplus W(1) \oplus \mathbb{C}L(-1)^2\mathbf{1}.
\end{aligned}$$

- Thus

$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$

Other illustrative examples for V_{Vir}

Virasoro Example 3: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Let $V = L(c,0)$ and $c \neq c_{p,q}$ for $c_{p,q}$ the minimal series.
- Then $V = L(c,0) = V_{Vir}(c,0)$ and V is a simple VOA.
- Let $W = M(c,0)$ which is not a simple V-module but is indecomposable.
- Since $M(c,0)/\langle L(-1)\rangle$ is simple, we have

$$\begin{aligned}
\Omega_0(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}\} \\
&= W(0) \oplus W(1) \\
\Omega_1(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}, L(-1)^2\mathbf{1}\} \\
&= W(0) \oplus W(1) \oplus \mathbb{C}L(-1)^2\mathbf{1}.
\end{aligned}$$

- Thus

$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$

Other illustrative examples for V_{Vir}

Virasoro Example 3: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Let $V = L(c,0)$ and $c \neq c_{p,q}$ for $c_{p,q}$ the minimal series.
- Then $V = L(c,0) = V_{Vir}(c,0)$ and V is a simple VOA.
- Let $W = M(c,0)$ which is not a simple V-module but is indecomposable.
- Since $M(c,0)/\langle L(-1)\rangle$ is simple, we have

$$\begin{aligned}
\Omega_0(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}\}\\
&= W(0) \oplus W(1)\\
\Omega_1(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}, L(-1)^2\mathbf{1}\}\\
&= W(0) \oplus W(1) \oplus \mathbb{C}L(-1)^2\mathbf{1}.
\end{aligned}$$

- Thus

$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$

Other illustrative examples for V_{Vir}

Virasoro Example 3: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Let $V = L(c,0)$ and $c \neq c_{p,q}$ for $c_{p,q}$ the minimal series.
- Then $V = L(c,0) = V_{Vir}(c,0)$ and V is a simple VOA.
- Let $W = M(c,0)$ which is not a simple V-module but is indecomposable.
- Since $M(c,0)/\langle L(-1)\rangle$ is simple, we have

$$\begin{aligned}\Omega_0(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}\}\\ &= W(0) \oplus W(1)\\ \Omega_1(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}, L(-1)^2\mathbf{1}\}\\ &= W(0) \oplus W(1) \oplus \mathbb{C}L(-1)^2\mathbf{1}.\end{aligned}$$

- Thus

$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$

Other illustrative examples for V_{Vir}

Virasoro Example 3: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \not\cong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Let $V = L(c,0)$ and $c \neq c_{p,q}$ for $c_{p,q}$ the minimal series.
- Then $V = L(c,0) = V_{Vir}(c,0)$ and V is a simple VOA.
- Let $W = M(c,0)$ which is not a simple V-module but is indecomposable.
- Since $M(c,0)/\langle L(-1)\rangle$ is simple, we have

$$\begin{aligned} \Omega_0(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}\} \\ &= W(0) \oplus W(1) \\ \Omega_1(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}, L(-1)^2\mathbf{1}\} \\ &= W(0) \oplus W(1) \oplus \mathbb{C}L(-1)^2\mathbf{1}. \end{aligned}$$

- Thus

$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$

Other illustrative examples for V_{Vir}

Virasoro Example 3: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Let $V = L(c,0)$ and $c \neq c_{p,q}$ for $c_{p,q}$ the minimal series.
- Then $V = L(c,0) = V_{Vir}(c,0)$ and V is a simple VOA.
- Let $W = M(c,0)$ which is not a simple V-module but is indecomposable.
- Since $M(c,0)/\langle L(-1)\rangle$ is simple, we have

$$\begin{aligned}
\Omega_0(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}\}\\
&= W(0) \oplus W(1)\\
\Omega_1(W) &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, L(-1)\mathbf{1}, L(-1)^2\mathbf{1}\}\\
&= W(0) \oplus W(1) \oplus \mathbb{C}L(-1)^2\mathbf{1}.
\end{aligned}$$

- Thus

$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$

Other illustrative examples for V_{Vir}

Virasoro Example 3, continued: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \ncong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Thus
$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$
which does not factor through $A_0(V)$ since $y - x^2 - 2x$ acts as $2L(-1)L(1)$ on $\mathbb{C}L(-1)^2\mathbf{1}$, which is nontrivial.
- Then $L_1(\mathbb{C}L(-1)^2\mathbf{1}) \cong \langle L(-1)\mathbf{1}\rangle$ since $L(-1)\mathbf{1} = \frac{1}{2}L(1)L(-1)^2\mathbf{1}$ and this spans the zero degree space of $L_1(\mathbb{C}L(-1)^2\mathbf{1})$.
- Therefore we have

$$L_1(\Omega_1/\Omega_0(W)) = L_1(\mathbb{C}L(-1)^2\mathbf{1}) \cong \langle L(-1)\mathbf{1}\rangle \ncong M(c,0) = W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 3, continued: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \not\cong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Thus
$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$
which does not factor through $A_0(V)$ since $y - x^2 - 2x$ acts as $2L(-1)L(1)$ on $\mathbb{C}L(-1)^2\mathbf{1}$, which is nontrivial.
- Then $L_1(\mathbb{C}L(-1)^2\mathbf{1}) \cong \langle L(-1)\mathbf{1}\rangle$ since $L(-1)\mathbf{1} = \frac{1}{2}L(1)L(-1)^2\mathbf{1}$ and this spans the zero degree space of $L_1(\mathbb{C}L(-1)^2\mathbf{1})$.
- Therefore we have

$$L_1(\Omega_1/\Omega_0(W)) = L_1(\mathbb{C}L(-1)^2\mathbf{1}) \cong \langle L(-1)\mathbf{1}\rangle \not\cong M(c,0) = W.$$

Other illustrative examples for V_{Vir}

Virasoro Example 3, continued: An example where

$$L_n(\Omega_n/\Omega_{n-1}(W)) \not\cong W \quad \textbf{since} \quad \Omega_n/\Omega_{n-1}(W) \neq W(n),$$

for W indecomposable but nonsimple, and V simple:

- Thus
$$\Omega_1/\Omega_0(W) \cong \mathbb{C}L(-1)^2\mathbf{1},$$
which does not factor through $A_0(V)$ since $y - x^2 - 2x$ acts as $2L(-1)L(1)$ on $\mathbb{C}L(-1)^2\mathbf{1}$, which is nontrivial.
- Then $L_1(\mathbb{C}L(-1)^2\mathbf{1}) \cong \langle L(-1)\mathbf{1}\rangle$ since $L(-1)\mathbf{1} = \frac{1}{2}L(1)L(-1)^2\mathbf{1}$ and this spans the zero degree space of $L_1(\mathbb{C}L(-1)^2\mathbf{1})$.
- Therefore we have

$$L_1(\Omega_1/\Omega_0(W)) = L_1(\mathbb{C}L(-1)^2\mathbf{1}) \cong \langle L(-1)\mathbf{1}\rangle \not\cong M(c,0) = W.$$

Categorical aspects of simple $A_n(V)$ and V-modules

Theorem [DLM]:

- $\Omega_n(W) \supseteq \bigoplus_{k=0}^{n} W(k)$ with equality if W is simple.
- If U is a simple $A_n(V)$-module, then $L_n(U)$ is a simple $\mathbb{N}$-gradable V-module.
- L_n and Ω_n/Ω_{n-1} induce mutually inverse bijections on the isomorphism classes of simple objects in the category of $A_n(V)$-modules which cannot factor through $A_{n-1}(V)$ and simple objects in the category of $\mathbb{N}$-gradable V-modules.

What can we say about indecomposable modules in the corresponding categories?

In general, if W is not simple, we can have $\Omega_n(W) \supsetneq \bigoplus_{k=0}^{n} W(k)$, as our Virasoro VOA family of modules illustrates.

Theorem [DLM]:

- $\Omega_n(W) \supseteq \bigoplus_{k=0}^{n} W(k)$ with equality if W is simple.
- If U is a simple $A_n(V)$-module, then $L_n(U)$ is a simple $\mathbb{N}$-gradable V-module.
- L_n and Ω_n/Ω_{n-1} induce mutually inverse bijections on the isomorphism classes of simple objects in the category of $A_n(V)$-modules which cannot factor through $A_{n-1}(V)$ and simple objects in the category of $\mathbb{N}$-gradable V-modules.

What can we say about indecomposable modules in the corresponding categories?

In general, if W is not simple, we can have $\Omega_n(W) \supsetneq \bigoplus_{k=0}^{n} W(k)$, as our Virasoro VOA family of modules illustrates.

Categorical aspects of simple $A_n(V)$ and V-modules

Theorem [DLM]:

- $\Omega_n(W) \supseteq \bigoplus_{k=0}^{n} W(k)$ with equality if W is simple.
- If U is a simple $A_n(V)$-module, then $L_n(U)$ is a simple $\mathbb{N}$-gradable V-module.
- L_n and Ω_n/Ω_{n-1} induce mutually inverse bijections on the isomorphism classes of simple objects in the category of $A_n(V)$-modules which cannot factor through $A_{n-1}(V)$ and simple objects in the category of $\mathbb{N}$-gradable V-modules.

What can we say about indecomposable modules in the corresponding categories?

In general, if W is not simple, we can have $\Omega_n(W) \supsetneq \bigoplus_{k=0}^{n} W(k)$, as our Virasoro VOA family of modules illustrates.

Categorical aspects for indecomposable modules

Proposition [K.B., N. Vander Werf, and J. Yang]:

If U is an indecomposable $A_n(V)$-module, then $L_n(U)$ is generated by U and thus is an indecomposable $\mathbb{N}$-gradable V-module.

Furthermore, if U is finite-dimensional, then $L_n(U)$ is an indecomposable $\mathbb{N}$-gradable generalized V-module.

Here an $\mathbb{N}$-*gradable generalized* V-*module* is an $\mathbb{N}$-gradable V-module that admits a decomposition

$$W = \coprod_{\lambda \in \mathbb{C}} W_\lambda$$

where

$$W_\lambda = \{w \in W \mid (L(0) - \lambda Id_W)^j w = 0 \text{ for some } j \in \mathbb{Z}_+\},$$

and $W_{n+\lambda} = 0$ for fixed λ and for all sufficiently small integers n.

Categorical aspects for indecomposable modules

Proposition [K.B., N. Vander Werf, and J. Yang]:

If U is an indecomposable $A_n(V)$-module, then $L_n(U)$ is generated by U and thus is an indecomposable $\mathbb{N}$-gradable V-module.

Furthermore, if U is finite-dimensional, then $L_n(U)$ is an indecomposable $\mathbb{N}$-gradable generalized V-module.

Here an $\mathbb{N}$-*gradable generalized* V*-module* is an $\mathbb{N}$-gradable V-module that admits a decomposition

$$W = \coprod_{\lambda \in \mathbb{C}} W_\lambda$$

where

$$W_\lambda = \{w \in W \mid (L(0) - \lambda Id_W)^j w = 0 \text{ for some } j \in \mathbb{Z}_+\},$$

and $W_{n+\lambda} = 0$ for fixed λ and for all sufficiently small integers n.

Categorical questions for indecomposable modules

When is $L_n(\Omega_n/\Omega_{n-1}(W)) \cong W$?

Theorem [K.B., N. Vander Werf, and J. Yang]:

- If W is a $\mathbb{N}$-gradable V-module that is generated by $W(n)$ such that $\Omega_j(W) = \bigoplus_{k=0}^{j} W(k)$, for $j = n$ and $n-1$, then

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong W/W_J$$

for some submodule W_J of W.

- Furthermore, suppose W satisfies the following property:

$$w = 0 \text{ for } w \in W \iff V.w \cap W(n) = 0.$$

Then

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong W.$$

Categorical questions for indecomposable modules

When is $L_n(\Omega_n/\Omega_{n-1}(W)) \cong W$?

Theorem [K.B., N. Vander Werf, and J. Yang]:

- If W is a $\mathbb{N}$-gradable V-module that is generated by $W(n)$ such that $\Omega_j(W) = \bigoplus_{k=0}^{j} W(k)$, for $j = n$ and $n-1$, then

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong W/W_J$$

for some submodule W_J of W.

- Furthermore, suppose W satisfies the following property:

$$w = 0 \text{ for } w \in W \iff V.w \cap W(n) = 0.$$

Then

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong W.$$

Categorical questions for indecomposable modules

When is $L_n(\Omega_n/\Omega_{n-1}(W)) \cong W$?

Theorem [K.B., N. Vander Werf, and J. Yang]:

- If W is a $\mathbb{N}$-gradable V-module that is generated by $W(n)$ such that $\Omega_j(W) = \bigoplus_{k=0}^{j} W(k)$, for $j = n$ and $n-1$, then

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong W/W_J$$

for some submodule W_J of W.

- Furthermore, suppose W satisfies the following property:

$$w = 0 \text{ for } w \in W \iff V.w \cap W(n) = 0.$$

Then

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong W.$$

Structure of $W = L_n(U)$

Theorem [K.B., N. Vander Werf, J. Yang]:

Let U be an $A_n(V)$-module and $W = L_n(U)$. Then

$$\bigoplus_{j=0}^{n} W(j) \subset \Omega_n(W) \subset \bigoplus_{j=0}^{2n+1} W(j),$$

and so all singular vectors, $\Omega_0(W)$, must be contained in $\bigoplus_{j=0}^{n} W(j)$.

Corollary [K.B., N. Vander Werf, J. Yang]:

If $W \cong L_n(W(n))$, then W must satisfy the following:
(i) W is generated by $W(n)$;
(ii) For $w \in W$, we have $Vw \cap W(n) = 0$ if and only if $w = 0$;
(iii) $\Omega_n(W) \subset \bigoplus_{j=0}^{2n+1} W(j)$ and $\Omega_0(W) \subset \bigoplus_{j=0}^{n} W(j)$.

Structure of $W = L_n(U)$

Theorem [K.B., N. Vander Werf, J. Yang]:

Let U be an $A_n(V)$-module and $W = L_n(U)$. Then

$$\bigoplus_{j=0}^{n} W(j) \subset \Omega_n(W) \subset \bigoplus_{j=0}^{2n+1} W(j),$$

and so all singular vectors, $\Omega_0(W)$, must be contained in $\bigoplus_{j=0}^{n} W(j)$.

Corollary [K.B., N. Vander Werf, J. Yang]:

If $W \cong L_n(W(n))$, then W must satisfy the following:
(i) W is generated by $W(n)$;
(ii) For $w \in W$, we have $Vw \cap W(n) = 0$ if and only if $w = 0$;
(iii) $\Omega_n(W) \subset \bigoplus_{j=0}^{2n+1} W(j)$ and $\Omega_0(W) \subset \bigoplus_{j=0}^{n} W(j)$.

Construction techniques for higher levels

First note that in the two papers:

- K.B., N. Vander Werf, and J. Yang, The level one Zhu algebra for the Heisenberg vertex operator algebra, to appear in *Proceedings of the Conference on Affine, Vertex and W-algebras, Istituto Nazionale di Alta Matematica, Rome, Italy, 11–15 Dec. 2017*, ed. D. Adamovic and P. Papi, Springer INdAM Series.
- K.B., N. Vander Werf, and J. Yang, The level one Zhu algebra for the Virasoro vertex operator algebra, to appear in *Proceedings of the International Conference on Vertex Operator Algebras, Number Theory and Related Topics, 11-15 Jun. 2018*, ed. M. Krauel, M. Tuite, and G. Yamskulna, Contemp. Math., Amer. Math. Soc.

we construct $A_1(M(1))$ and $A_1(V_{Vir})$, respectively, using only the VOA itself and some minimal information about the irreducible modules of the VOA.

Level n for Heisenberg VOA

K.B. and D. Addabbo: Level 2 and higher level for Heisenberg & Virasoro VOAs, as well as other constructions .... to appear soon.

For example:

Proposition [K.B. and D. Addabbo]:

For $V = M_a(1)$, there exists a surjection

$$\varphi : \mathbb{C}[x, y][z, w] \longrightarrow A_2(V)$$

where $\mathbb{C}[x, y][z, w]$ is the associative polynomial algebra in commuting variables

$$x \mapsto \alpha(-1)\mathbf{1} + O_2(V) \quad \text{and} \quad y \mapsto \alpha(-1)^2\mathbf{1} + O_2(V)$$

and non-commuting variables

$$z \mapsto \alpha(-1)^3\mathbf{1} + O_2(V) \quad \text{and} \quad w \mapsto \alpha(-1)^4\mathbf{1} + O_2(V).$$

Moreover..........

Level n for Heisenberg VOA

K.B. and D. Addabbo: Level 2 and higher level for Heisenberg & Virasoro VOAs, as well as other constructions .... to appear soon. For example:

Proposition [K.B. and D. Addabbo]:

For $V = M_a(1)$, there exists a surjection

$$\varphi : \mathbb{C}[x, y][z, w] \longrightarrow A_2(V)$$

where $\mathbb{C}[x, y][z, w]$ is the associative polynomial algebra in commuting variables

$$x \mapsto \alpha(-1)\mathbf{1} + O_2(V) \quad \text{and} \quad y \mapsto \alpha(-1)^2\mathbf{1} + O_2(V)$$

and non-commuting variables

$$z \mapsto \alpha(-1)^3\mathbf{1} + O_2(V) \quad \text{and} \quad w \mapsto \alpha(-1)^4\mathbf{1} + O_2(V).$$

Moreover..........

Example of level two for Heisenberg VOA

Proposition [K.B. and D. Addabbo]:

Moreover for $\varphi : \mathbb{C}[x, y][z, w] \longrightarrow A_2(V)$ the following polynomials are contained in Ker φ:

$$\begin{aligned}
p_y(x, y) &= (x^2 - y)(x^2 - y + 2)(x^2 - y + 4)\\
p_z(x, y, z) &= (x^3 - z)(x^3 - z + 12x)((x^3 - z + 12x)^2 - 36)\\
p_w(x, y, z, w) &= (x^4 - w)(x^4 - w + 12x^2)((w + 3x^4 + 24x^2\\
&\qquad - 4zx)^2 - 12(w + 3x^4 + 24x^2 - 4zx))
\end{aligned}$$

and in fact

$$\begin{aligned}
A_2(V) \cong\ & \mathbb{C}[x, y, z, w]/((x^2 - y), (x^3 - z), (x^4 - w))\\
&\oplus \mathbb{C}[x, y, z, w]/((x^2 - y + 2), (x^3 - z + 12x), (x^4 - w + 12x^2)\\
&\oplus \mathbb{C}[x, y, z, w]/((x^2 - y + 4), (\tilde{z}^2 - 1), (\tilde{w}^2 - \tilde{w}), (x\tilde{z} - \tilde{z}x),\\
&\qquad (x\tilde{w} - \tilde{w}x), ((\tilde{z}\tilde{w} - \tilde{w}\tilde{z})^4 - 1))
\end{aligned}$$

Example of level two for Heisenberg VOA

Proposition [K.B. and D. Addabbo]:

Moreover for $\varphi : \mathbb{C}[x,y][z,w] \longrightarrow A_2(V)$ the following polynomials are contained in Ker φ:

$$\begin{aligned}
p_y(x,y) &= (x^2-y)(x^2-y+2)(x^2-y+4)\\
p_z(x,y,z) &= (x^3-z)(x^3-z+12x)((x^3-z+12x)^2-36)\\
p_w(x,y,z,w) &= (x^4-w)(x^4-w+12x^2)((w+3x^4+24x^2\\
&\qquad -4zx)^2-12(w+3x^4+24x^2-4zx))
\end{aligned}$$

and in fact

$$\begin{aligned}
A_2(V) \cong\ & \mathbb{C}[x,y,z,w]/((x^2-y),(x^3-z),(x^4-w))\\
&\oplus \mathbb{C}[x,y,z,w]/((x^2-y+2),(x^3-z+12x),(x^4-w+12x^2))\\
&\oplus \mathbb{C}[x,y,z,w]/((x^2-y+4),(\tilde{z}^2-1),(\tilde{w}^2-\tilde{w}),(x\tilde{z}-\tilde{z}x),\\
&\qquad (x\tilde{w}-\tilde{w}x),((\tilde{z}\tilde{w}-\tilde{w}\tilde{z})^4-1))
\end{aligned}$$

Example of level two for Heisenberg VOA

Proposition [K.B. and D. Addabbo]:

Or equivalently

$$
\begin{aligned}
A_2(V) &\cong \mathbb{C}[x] \oplus \mathbb{C}[x] \oplus (M_2(\mathbb{C}) \otimes \mathbb{C}[x]) \\
&\cong A_1(V) \oplus (M_2(\mathbb{C}) \otimes \mathbb{C}[x]) .
\end{aligned}
$$

Example of level two for Heisenberg VOA

Proposition [K.B. and D. Addabbo]:

Or equivalently

$$\begin{aligned} A_2(V) &\cong \mathbb{C}[x] \oplus \mathbb{C}[x] \oplus (M_2(\mathbb{C}) \otimes \mathbb{C}[x]) \\ &\cong A_1(V) \oplus (M_2(\mathbb{C}) \otimes \mathbb{C}[x]) . \end{aligned}$$

THANK YOU!