Orbifold deconstruction

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Introduction

Common formulations of dynamics (EOM, variational principles, ...) usually require (quasi-)trivial topology $\rightsquigarrow$ realize configuration space as either a submanifold or a quotient of some nice geometric structure: constrained dynamics vs gauge symmetries.

Dynamics formulated on covering space $\rightsquigarrow$ state identifications.

$$\boxed{\text{covering (deck) transformations} = \text{gauge transformations}}$$

New superselection sectors from 'twisted' boundary conditions.

Difficult computations (mostly numerical, e.g. lattice QCD).

Best understood for 2D conformal models (orbifolding).

Basic problems: structure of twisted sectors and fixed point resolution (usually require *ad hoc* techniques).

Under control only in special cases:

- toroidal orbifolds (compactified free bosons, i.e. lattice models);
- holomorphic orbifolds (self-dual models);
- permutation orbifolds (permutation symmetries).

Moore's 'conjecture': all rational conformal models are GKO cosets or orbifolds thereof.

Can one reverse orbifolding?

Orbifolds

2D conformal model characterized by chiral symmetry algebra $\mathbb{V}$ (nice VOA) and L-R coupling (partition functions).

Primary fields (simple $\mathbb{V}$-modules, aka. superselection sectors) characterized by their conformal weight $\mathtt{h}_p$ (lowest eigenvalue of L_0) and chiral character (trace function)

$$\chi_p(q) = \mathrm{Tr}_p\left(q^{L_0 - c/24}\right)$$

describing the spectrum of L_0.

Fusion rules: composition of superselection sectors (tensor products).

Fusion rules related to modular properties *via* Verlinde's formula.

For $G < \mathrm{Aut}(\mathbb{V})$, the chiral algebra of the G-orbifold is the fixed point subalgebra $\mathbb{V}^G = \{v \in \mathbb{V} \mid gv = v \text{ for all } g \in G\}$.

Primaries of the G-orbifold from G-twisted modules of $\mathbb{V}$.

G permutes the G-twisted modules (outer action), with $h \in G$ taking a g-twisted module to a hgh^{-1}-twisted module $\rightsquigarrow$ G-orbits organized into twisted sectors labeled by conjugacy classes.

Twisted modules in the same G-orbit identified with each other ('state identification').

Stabilizer $G_M = \{g \in G \mid gM = M\}$ of the twisted module M represented projectively on M (with associated 2-cocycle ϑ_M) $\rightsquigarrow$ M splits into isotypic components M_ϕ labeled by irreps $\phi \in \mathtt{Irr}(G_M|\vartheta_M)$.

isotypic components $\leftrightsquigarrow$ primaries of the orbifold

G-orbits of (twisted) modules $\leftrightsquigarrow$ blocks of primaries

$$\text{number of primaries} \;=\; \frac{1}{|G|}\sum_{xy=yx}\;\sum_{M\in\mathrm{Fix}(x,y)}\frac{\vartheta_M(x,y)}{\vartheta_M(y,x)}$$

$$\text{number of blocks} \;=\; \sum_{M}\frac{1}{[G:G_M]}$$

Each block $\mathfrak{b}$ characterized by inertia subgroup $\mathtt{I}_\mathfrak{b}$ (stabilizer G_M of any module M in the orbit corresponding to $\mathfrak{b}$) and 2-cocycle $\vartheta_\mathfrak{b}\in Z^2(\mathtt{I}_\mathfrak{b},\mathbb{C})$.

Integrally spaced L_0 spectrum for untwisted modules $\rightsquigarrow$ the conformal weights of primaries from a block in the untwisted sector differ by integers.

Vacuum block $\mathfrak{b}_0$ (untwisted sector) has trivial cocyle $\rightsquigarrow$ all elements of $\mathfrak{b}_0$ have integer conformal weights, and correspond to (ordinary) irreps of G, with matching fusion rules and (quantum) dimensions.

The vacuum block is a twister: a set of primaries with integer conformal weights (and quantum dimensions) closed under fusion.

Question: can we identify the orbifold from its vacuum block?

Braiding restricted to a twister is involutive $\rightsquigarrow$ elements of the twister $\mathfrak{g}$ are the simple objects of a symmetric monoidal category.

Deligne's theorem: the subring of the fusion ring generated by a twister is isomorphic to the character ring of some finite group.

Fusion matrices

For a primary p define the fusion matrix

$$[\mathtt{N}(p)]_{qr} = N_{pq}^{r}$$

Span a commutative matrix algebra (the Verlinde algebra $\mathcal{V}$)

$$\mathtt{N}(p)\,\mathtt{N}(q) = \sum_r N_{pq}^{r}\mathtt{N}(r)$$

Commuting matrices with non-negative elements $\rightsquigarrow$ common Perron-Frobenius eigenvector (quantum dimension) $\mathtt{d}_p \geq 1$.

$$\sum_r N_{pq}^{r}\mathtt{d}_r = \mathtt{d}_p\mathtt{d}_q$$

Modular S-matrix

$$S_{pq} = \frac{1}{\sqrt{\sum_r \mathtt{d}_r^2}} \sum_r N_{pq}^{r} \mathtt{d}_r \exp\{2\pi \mathtt{i}\,(\mathtt{h}_p + \mathtt{h}_q - \mathtt{h}_r)\}$$

Verlinde's theorem:

$$\boldsymbol{\rho}_p(\mathtt{N}(q)) = \frac{S_{qp}}{S_{\mathtt{0}p}}$$

is an irrep of $\mathcal{V}$ for each primary p, where $\mathtt{0}$ denotes the vacuum primary.

$$\boxed{\text{irreducible representations of } \mathcal{V} \leftrightsquigarrow \text{primary fields}}$$

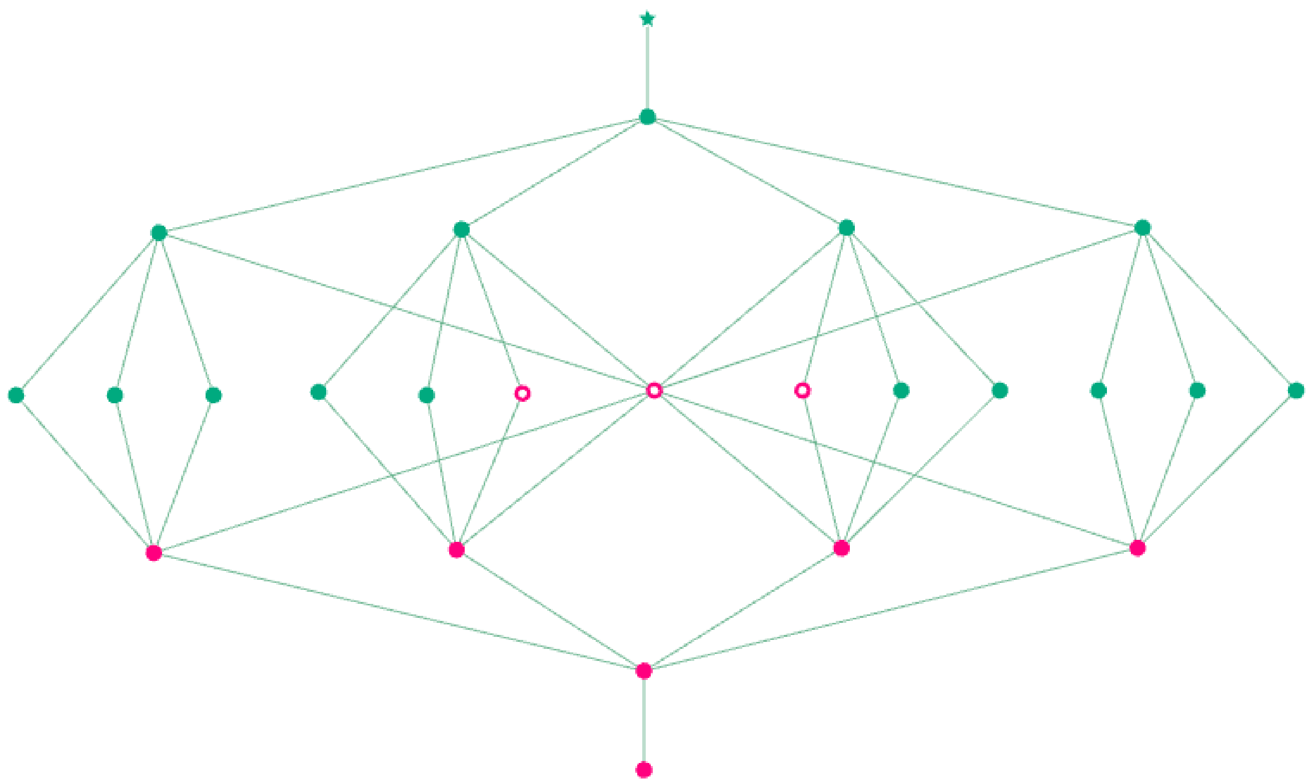
Perron's theorem $\rightsquigarrow$

$$|\,\boldsymbol{\rho}_p(\mathtt{N}(q))\,| \leq \boldsymbol{\rho}_{\mathtt{0}}(\mathtt{N}(q)) = \mathtt{d}_q$$

Twisters

A set $\mathfrak{g}$ of primaries is fusion closed if $N_{pq}^{r} > 0$ for $p, q \in \mathfrak{g}$ implies $r \in \mathfrak{g}$.

Form a modular (even Arguesian) lattice $\mathscr{L}$.



$\widehat{\mathfrak{g}}$: subalgebra of $\mathcal{V}$ spanned by the fusion matrices $\mathtt{N}(\alpha)$ for $\alpha \in \mathfrak{g}$.

Primaries p and q belong to the same twist class if the restrictions to $\widehat{\mathfrak{g}}$ of the irreps $\boldsymbol{\rho}_p$ and $\boldsymbol{\rho}_q$ coincide.

$$\boxed{\text{twist classes} \leftrightsquigarrow \text{irreducible representations of } \widehat{\mathfrak{g}}}$$

$\rightsquigarrow$ the number of twist classes equals the cardinality of $\mathfrak{g}$.

Notation: for a class $\mathtt{C}$ and $\alpha \in \mathfrak{g}$ let $\alpha(\mathtt{C}) = \boldsymbol{\rho}_p(\alpha)$ for any $p \in \mathtt{C}$.

Orthogonality relations: for $\alpha, \beta \in \mathfrak{g}$

$$\sum_{\mathtt{C}} \frac{\alpha(\mathtt{C})\,\overline{\beta(\mathtt{C})}}{\|\mathtt{C}\|} = \begin{cases} 1 & \text{if } \alpha = \beta; \\ 0 & \text{otherwise} \end{cases}$$

where

$$\|\mathtt{C}\| = \frac{1}{\sum\limits_{p \in \mathtt{C}} S_{0p}^2}$$

Second orthogonality: for twist classes $\mathtt{C}_1$ and $\mathtt{C}_2$

$$\sum_{\alpha\in\mathfrak{g}}\alpha(\mathtt{C}_1)\,\overline{\alpha(\mathtt{C}_2)}=\begin{cases}\|\mathtt{C}_1\| & \text{if } \mathtt{C}_1=\mathtt{C}_2;\\ 0 & \text{otherwise.}\end{cases}$$

Trivial class $\mathfrak{g}^\perp$: twist class containing the vacuum primary $\mathtt{0}$.

Remark: $p\in\mathfrak{g}^\perp$ iff $\mathtt{h}_q-\mathtt{h}_p-\mathtt{h}_\alpha\in\mathbb{Z}$ whenever $N_{\alpha p}^{q}>0$ for some $\alpha\in\mathfrak{g}$.

Product rule: If $p\in\mathfrak{g}^\perp$ and $N_{pq}^{r}>0$, then q and r belong to the same twist class.

Consequence: $\mathfrak{g}^\perp\in\mathscr{L}$ and $(\mathfrak{g}^\perp)^\perp=\mathfrak{g}$, hence the lattice $\mathscr{L}$ is self-dual.

Blocks of $\mathfrak{g}$ = twist classes of $\mathfrak{g}^\perp$ (number of blocks = cardinality of $\mathfrak{g}^\perp$).

Blocks partition the set of primaries: p and q belong to the same block iff $N_{\alpha p}^{q}>0$ for some $\alpha\in\mathfrak{g}$ ($\mathfrak{g}$ is the trivial block).

Restriction of fusion matrices to the elements of a block $\mathfrak{b}$ provides an integral representation $\mathtt{N}_{\mathfrak{b}}$ of $\widehat{\mathfrak{g}}$.

$$\boxed{\left.\begin{array}{l}\text{classes: irreducible}\\ \text{blocks: integral}\end{array}\right\}\ \text{representations of } \widehat{\mathfrak{g}}}$$

Overlap $\langle \mathfrak{b}, \mathtt{C} \rangle$: multiplicity of irrep $\boldsymbol{\rho}_{\mathtt{C}}$ in $\mathtt{N}_{\mathfrak{b}}$ (non-negative integer)

$$\langle \mathfrak{b}, \mathtt{C} \rangle = \sum_{p \in \mathfrak{b}} \sum_{q \in \mathtt{C}} |S_{pq}|^2$$

$|\mathfrak{b}| = \sum_{\mathtt{C}} \langle \mathfrak{b}, \mathtt{C} \rangle$ and $|\mathtt{C}| = \sum_{\mathfrak{b}} \langle \mathfrak{b}, \mathtt{C} \rangle$.

Note: $\langle \mathfrak{b}, \mathtt{C} \rangle = 0$ implies $S_{pq} = 0$ for all $p \in \mathfrak{b}$ and $q \in \mathtt{C}$.

$\langle \mathfrak{g}, \mathtt{C} \rangle = 1$ for every class $\mathtt{C}$ and $\langle \mathfrak{b}, \mathfrak{g}^{\perp} \rangle = 1$ for every block $\mathfrak{b}$.

Integrality theorem: $\mathfrak{g}\subseteq\mathfrak{g}^\perp$ implies $\mathtt{d}_\alpha\in\mathbb{Z}$ and $\mathtt{h}_\alpha\in\frac{1}{2}\mathbb{Z}$ for all $\alpha\in\mathfrak{g}$.

$\mathfrak{g}\subseteq\mathfrak{g}^\perp$ **iff** $N_{\alpha\beta}^{\gamma}>0$ for $\alpha,\beta,\gamma\in\mathfrak{g}$ implies $\mathtt{h}_\gamma-\mathtt{h}_\alpha-\mathtt{h}_\beta\in\mathbb{Z}$ **iff** every class is a union of blocks.

$\mathfrak{g}\in\mathscr{L}$ is a twister if $\mathtt{h}_\alpha\in\mathbb{Z}$ for all $\alpha\in\mathfrak{g}$ (consequently $\mathfrak{g}\subseteq\mathfrak{g}^\perp$).

Every $\mathfrak{g}\subseteq\mathfrak{g}^\perp$ is a twister or a $\mathbb{Z}_2$-extension thereof ('swister'?).

If $\mathfrak{g}\in\mathscr{L}$ is a twister, then

1. a block is contained in the trivial class $\mathfrak{g}^\perp$ iff the conformal weights of its elements differ by integers;

2. to each block $\mathfrak{b}$ corrresponds an algebraic integer $\mathcal{D}_\mathfrak{b}$ ('block dimension') such that $\dfrac{\mathtt{d}_p}{\mathcal{D}_\mathfrak{b}}\in\mathbb{Z}_+$ for all $p\in\mathfrak{b}$.

A twister determines a Tannakian subcategory of the modular tensor category associated to the conformal model $\rightsquigarrow$

$$\boxed{\text{the ring } \widehat{\mathfrak{g}} \text{ associated to a twister is isomorphic to the representation ring of some finite group } G \text{ (Deligne's theorem).}}$$

Problem: there may exist several nonisomorphic groups with isomorphic representation rings (e.g. $\mathbb{D}_8$, the dihedral group of order 8, and Q, the group of unit quaternions). How to distinguish between the different possibilities?

Solution: use the λ-ring structure (exterior powers) of the representation ring, i.e. the Adams operations.

Adams operations

For $n \in \mathbb{Z}$ and elements $\alpha, \beta \in \mathfrak{g}$ of a twister $\mathfrak{g}$ let

$$\mathtt{Z}_n(\alpha,\beta) = \sum_{p,q} N_{\alpha p}^{q} S_{\beta p} S_{0q} \mathsf{e}^{2\pi \mathtt{i} n(\mathtt{h}_\alpha + \mathtt{h}_p - \mathtt{h}_q)}$$

There exists an endomorphism $\Psi^n \in \mathrm{End}(\widehat{\mathfrak{g}})$ such that

- $\Psi^n(\alpha) = \sum_{\beta \in \mathfrak{g}} \mathtt{Z}_n(\alpha,\beta)\beta$ for $\alpha \in \mathfrak{g}$;
- $\Psi^n(\alpha) = \alpha^n$ for invertible elements $\alpha \in \widehat{\mathfrak{g}}$;
- Ψ^n is functorial;
- $\Psi^n \circ \Psi^m = \Psi^{nm}$.

$\Psi^n \in \mathrm{End}(\widehat{\mathfrak{g}})$ is the n^{th} Adams operation on $\widehat{\mathfrak{g}}$, endowing the latter with the structure of a λ-ring, able to distinguish (but for Brauer pairs) non-isomorphic groups with identical fusion rules, e.g. $\mathbb{D}_8$ and Q.

Ψ^n permutes the characteristic functions of the twist classes $\rightsquigarrow$ existence of power maps $\mathtt{C} \mapsto \mathtt{C}^n$ for each $n \in \mathbb{Z}$.

Order of the twist class $\mathtt{C}$ = least positive integer n such that $\mathtt{C}^n = \mathfrak{g}^{\perp}$.

If a block $\mathfrak{b}$ is contained in a twist class of order n, then the conformal weights inside $\mathfrak{b}$ can only differ by integer multiples of $\frac{1}{n}$.

Orbifold deconstruction

vacuum block	twister
irrep of twist group	element of twister
twisted sectors	twist classes
orbits of twisted modules	blocks
stabilizer of orbit	inertia subgroup

Step 1: select a twister.

Step 2: determine the corresponding twist classes and blocks.

Step 3: compute the character table and the power maps.

Step 4: identify the twist group G (up to possible Brauer pairs).

Step 5: for each block $\mathfrak{b}$ contained in the trivial class determine the inertia subgroup $\mathtt{I}_\mathfrak{b}$ and associated 2-cocycle $\vartheta_\mathfrak{b} \in Z^2(\mathtt{I}_\mathfrak{b}, \mathbb{C}^\times)$.

Remark. For most applications it is enough to know the order $\mathtt{e}_\mathfrak{b}$ of (the cohomology class of) $\vartheta_\mathfrak{b}$ and the index $[G : \mathtt{I}_\mathfrak{b}]$ of the inertia subgroup.

Step 6: compute the spectrum of the deconstructed model.

Remark. To each block $\mathfrak{b} \subseteq \mathfrak{g}^\perp$ correspond $[G : \mathtt{I}_\mathfrak{b}]$ different primaries of respective conformal weights $\mathtt{h}_\mathfrak{b} = \min\{\mathtt{h}_p \,|\, p \in \mathfrak{b}\}$, quantum dimensions

$$\mathtt{d}_\mathfrak{b} = \frac{\mathcal{D}_\mathfrak{b}}{\mathtt{e}_\mathfrak{b}\,[G : \mathtt{I}_\mathfrak{b}]}$$

and chiral characters

$$\boldsymbol{\chi}_{\mathfrak{b}}(\tau) = \frac{\mathtt{e}_{\mathfrak{b}}}{\mathcal{D}_{\mathfrak{b}}} \sum_{p\in\mathfrak{b}} \mathtt{d}_p \chi_p(\tau)$$

Step 7: if the above did not single out the deconstructed model, compute the block-fusion coefficients

$$\mathcal{N}_{\mathfrak{a}\mathfrak{b}}^{\mathfrak{c}} = \frac{\mathtt{e}_{\mathfrak{a}}\mathtt{e}_{\mathfrak{b}}\mathtt{e}_{\mathfrak{c}}}{|\mathtt{I}_{\mathfrak{a}}||\mathtt{I}_{\mathfrak{b}}|} \sum_{p\in\mathfrak{a}} \sum_{q\in\mathfrak{b}} \sum_{r\in\mathfrak{c}} N_{pq}^{r} \frac{\mathtt{d}_p\mathtt{d}_q\mathtt{d}_r}{\mathcal{D}_{\mathfrak{a}}\mathcal{D}_{\mathfrak{b}}\mathcal{D}_{\mathfrak{c}}} \tag{1}$$

that characterize the fusion rules of the deconstructed model (but for fixed-point resolution).

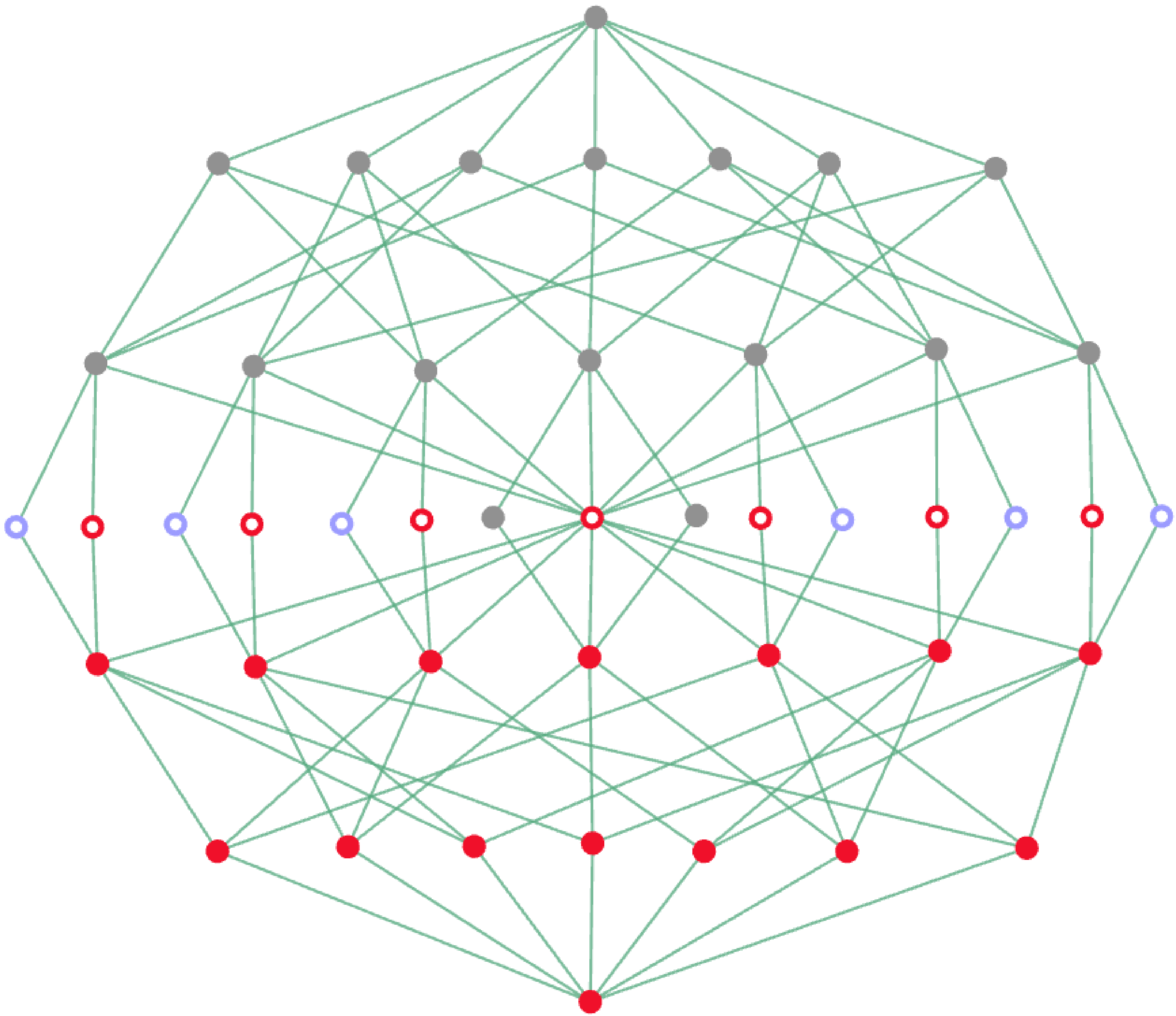
A permutation orbifold example

$\mathbf{D}$ = transitive permutation group of degree 4 generated by the cyclic permutations $(1,2,3,4)$ and $(2,4)$ (isomorphic to the dihedral group of order 8, the symmetry group of a square).

Permutation orbifold $\mathbb{V}^\natural \wr \mathbf{D}$: conformal model of central charge $c = 96$, with 22 primaries of known conformal weights, chiral characters, etc.

Moonshine module is self-dual $\rightsquigarrow$ fusion rules and modular properties described by the (untwisted) double of $\mathbf{D}$.

45 fusion closed sets (of which 22 twisters), with Hasse-diagram



7 maximal twisters, all self-dual $\rightsquigarrow$ corresponding deconstructions are self-dual too (with only one primary).

twist group	trace function
$\mathbb{D}_8$	$J(q)^4$
$\mathbb{D}_8$	$J(q)^4 - 590652J(q)^2 - 64481280J(q) + 55552950252$
$\mathbb{D}_8$	$J(q)^4 - 590652J(q)^2 - 64481280J(q) + 55552359600$
$\mathbb{D}_8$	$J(q)^4 - 590652J(q)^2 - 64481277J(q) + 55552950252$
$\mathbb{D}_8$	$J(q)^4 - 393768J(q)^2 + 38763309456 = \left(J(q)^2 - 196884\right)^2$
$\mathbb{D}_8$	$J(q)^4 - 393768J(q)^2 - 42987519J(q) - 1728009288$
$\mathbb{Z}_2^3$	$J(q)^4 - 393768J(q)^2 - 42987519J(q) + 37035300168$

Table 1. Maximal deconstructions of $\mathbb{V}^\natural \wr \mathbf{D}$

Maximal deconstructions have different trace functions $\rightsquigarrow$ they all differ from each other.

Lessons:

1. one and the same conformal model can have several different maximal deconstructions, possibly with non-isomorphic twist groups;

2. one needs to know the power maps in order to identify the twist group;

3. maximal deconstructions of a permutation orbifold related to replication identities, e.g. (from deconstruction no. 7)

$$J(2\tau)^2 + J\left(\frac{\tau}{2}\right)^2 + J\left(\frac{\tau+1}{2}\right)^2$$
$$= J(\tau)^4 - 787536J(\tau)^2 - 85975038J(\tau) + 74070600336$$

Addendum: $L\left(\frac{1}{2},0\right)\wr\mathbb{S}_3$ (aka. $\mathcal{F}(3)^{\mathbb{S}_3}$, cf. Penn's talk) has 49 primaries, but only 3 (non-trivial) twisters. The corresponding decompositions are

twist group	deconstructed model
$\mathbb{Z}_2$	$L\left(\frac{1}{2},0\right)\wr\mathbb{Z}_3$
$\mathbb{S}_3$	$L\left(\frac{1}{2},0\right)^{\otimes 3}$
$\mathbb{S}_4$	$\mathtt{SU}(2)_2$

Each deconstruction is $N=1$ superconformal, with $L\left(\frac{1}{2},0\right)\wr\mathbb{Z}_3$ and (the trivial) $L\left(\frac{1}{2},0\right)\wr\mathbb{S}_3$ isolated in the moduli space of $c=\frac{3}{2}$ superconformal models (Cappelli and d'Appollonio, JHEP0208:039, 2002).

Summary

- effective procedure for orbifold deconstruction
- lattice structure of fusion closed sets
- Adams-operations for twisters
- integrality of quantum dimensions
- structure of twisted modules

and open questions

- characterization of primitive models?
- Moore's conjecture?
- relation between different maximal deconstructions?
- generalized deconstruction using swisters?
- Brauer characters?