Quantum Langlands duality of representations of W-algebras

(joint work with Edward Frenkel)

REPRESENTATION THEORY XVI – Dubrovnik

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Here, ${}^Lk$ is the complex number defined by

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which plays an important role in the classical geometric Langlands program.

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Here π_k is the Heisenberg vertex algebra associated with $\mathfrak{h}$ at level k.

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$$\sum_{i=0}^{n} W_i(z)(\alpha_0 \partial_z)^{n-1}$$
$$=: (\alpha_0\partial_z + b_1(z))(\alpha_0\partial_z + b_2(z)) \ldots (\alpha_0\partial_z + b_n(z)) :,$$

where $\alpha_0 = \sqrt{k+n} + \frac{1}{\sqrt{k+n}}$ and $b_i(z)$ is the generating fields of π_k satisfying the OPE

$$b_i(z)b_j(w) \sim \begin{cases} \frac{1-1/n}{(z-w)^2} & (i = j) \\ \frac{-1/m}{(z-w)^2} & (i \neq j). \end{cases}$$

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It turned out this is an important problem even for an irrational k, which plays an essential role in quantum geometric Langlands program.

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(A similar, but more subtle, equivalence is expected for rational k as well.)

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$$\begin{tikzcd} \mathrm{KL}(\widehat{\mathfrak{g}})_k \arrow[rr, "\mathrm{FLE}_k"] \arrow[d, "\mathrm{Loc}"'] & & \mathrm{Whit}_{^Lk}(\mathrm{Gr}_{^LG}) \arrow[d, "\mathrm{Poinc}"] \\ D\text{-mod}_k(\mathrm{Bun}_G) \arrow[rr] \arrow[d] & & D\text{-mod}_{-^Lk-2^Lh^\vee}(\mathrm{Bun}_{^LG}) \arrow[d, "\Gamma"'] \\ \mathrm{Whit}_{-k-2h^\vee}(\mathrm{Gr}_G) \arrow[rr, "\mathrm{FLE}^{-1}_{-^Lk-2^Lh}"] & & \mathrm{KL}(\widehat{^L\mathfrak{g}})_{-^Lk-2^Lh^{vee}}, \end{tikzcd}$$

where $\mathrm{KL}(\widehat{\mathfrak{g}})_k$ is the Kazhdan-Lusztig parabolic category $\mathbb{O}$ of $\widehat{\mathfrak{g}}$ at level k, $\mathrm{Whit}_k(\mathrm{Gr}_G)$ is the Whittaker category on the affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$ at level k, and FLE_k is the equivalence $\mathrm{KL}(\widehat{\mathfrak{g}})_k \xrightarrow{\sim} \mathrm{Whit}_{^Lk}(\mathrm{Gr}_{^LG})$ of chiral categories (fundamental local equivalence proved by Gatisgory).

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According to Dennis Gaistgory, one can obtain a desired equivalence using the Master chiral algebra, which is a certain vertex algerba. However, to define such a vertex algebra one needs to check a compatibility condition, and this compatibility condition was stated as a conjecture on the isomorphism between certain representations of W-algebras under the Feigin-Frenkel duality.

Family of modules appearing Gaitsgory's conjecture

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We are going to define a family of $\mathscr{W}^k(\mathfrak{g})$-modules

$$T^k_{\lambda,\check{\mu}}, \quad (\lambda, \check{\mu}) \in P_+ \times \check{P}_+.$$

Drinfeld-Sokolov reduction

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$$H^\bullet_{DS}(M) = H^{\infty/2+\bullet}(\mathfrak{n}[t,t^{-1}], M \otimes \mathbb{C}_\Psi),$$

where $\Psi : \mathfrak{n}[t,t^{-1}] \to \mathbb{C}$ is the character defined by

$$\Psi(x_\alpha t^n) = \begin{cases} \delta_{n,-1} & \text{if } \alpha \text{ is simple} \\ 0 & \text{otherwise.} \end{cases}$$

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We have a functor

$$\begin{array}{ccc} \{\text{smooth } \widehat{\mathfrak{g}}\text{-modules of level } k\} & \to & \mathscr{W}^k(\mathfrak{g})\text{-Mod}, \\ M & \mapsto & H^0_{DS}(M). \end{array}$$

Twisted Drinfeld-Sokolov reduction

For $\check{\mu} \in \check{P}_+$, define the character $\Psi_\mu : \mathfrak{n}[t, t^{-1}] \to \mathbb{C}$ by

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and set

$$H^{\bullet}_{DS,\check{\mu}}(M) = H^{\infty/2+\bullet}(\mathfrak{n}[t, t^{-1}], M \otimes \mathbb{C}_{\Psi_{\check{\mu}}}).$$

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It follows that $H^{\bullet}_{DS,\check{\mu}}(M)$ is naturally a $\mathscr{W}^k(\mathfrak{g})$-module.

Remark

Spectral flows do not preserve the category $\mathcal{O}$.

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One can define the $\mathscr{W}^{{}^L k}({}^L\mathfrak{g})$-module $\check{T}^{\check{k}}_{\check{\mu},\lambda} = H^0_{DS,\lambda}(\mathbb{V}_\mu^{{}^L k})$ as well.

Main Results

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i). *We have* $H^{i \neq 0}_{DS,\check{\mu}}(\mathbb{V}^k_\lambda) = 0$.

ii). *Let* k *be irrational. Then* $T^k_{\lambda,\mu}$ *is an irreducible highest weight representation of* $\mathscr{W}^k(\mathfrak{g})$*, and we have*

$$T^k_{\lambda,\check{\mu}} \cong \check{T}^{\check{k}}_{\check{\mu},\lambda}$$

under the Feigin-Frenkel duality isomorphism $\mathscr{W}^k(\mathfrak{g}) \cong \mathscr{W}^{\check{k}}({}^L\mathfrak{g})$.

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$$\mathbb{V}^k_\mu \otimes \mathbb{L}^1_\nu \cong \bigoplus_{\substack{\lambda \in P_+ \\ \lambda - \mu - \nu \in Q_+}} \mathbb{V}^{k+1}_\lambda \otimes T^\ell_{\mu,\lambda},$$

where $\ell = \frac{k+h^\vee}{k+h^\vee+1}$ ([A.-Creutzig-Linshaw 2019]).

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Remark

$T^k_{\lambda,\mu}$ also appear in the higher rank triplet algebras ([Shoma Sugimoto]).

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given by

$$M \otimes N \mapsto H^0_{DS}(M \star FLE_{^L k}(N)),$$
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where $\star$ is the convolution product defined by Frenkel-Gaitsgory.

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We have $H^0_{DS}(\mathbb{V}^k_\lambda \star FLE_{^L k}(\mathbb{V}^{^L k}_{\check{\mu}})) = T^k_{\lambda,\check{\mu}}$ and
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where $\star$ is the convolution product defined by Frenkel-Gaitsgory.

We have $H^0_{DS}(\mathbb{V}^k_\lambda \star FLE_{^Lk}(\mathbb{V}^{^Lk}_{\check{\mu}})) = T^k_{\lambda,\check{\mu}}$ and $H^0_{DS}(FLE_k(\mathbb{V}^k_\lambda) \star \mathbb{V}^{^Lk}_{\check{\mu}})) = T^{^Lk}_{\check{\mu},\lambda}$.

So the previous theorem says that these two functors coincide.

Back to Master chiral algebra

Using FLE, we have to functors

$$\mathrm{KL}_k(\widehat{\mathfrak{g}}) \otimes \mathrm{KL}_{{}^L k}(\widehat{{}^L\mathfrak{g}}) \to \mathscr{W}^k(\mathfrak{g})\text{-Mod}$$

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Let P denote this functor.

Back to Master chiral algebra

The Master chiral algebra is defined as

$$\mathbb{M} = (\mathsf{id} \otimes P \otimes \mathsf{id})(\mathcal{D}^{ch}_{G,k} \otimes \mathcal{D}^{ch}_{{}^L G, -{}^L k - 2{}^L h^\vee}),$$

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where $\mathcal{D}^{ch}_{G,k} = \bigoplus_{\lambda \in P_+} \mathbb{V}^k_\lambda \otimes \mathbb{V}^{-k-2h^\vee}_{\lambda^*}$ is the chiral differential operators on G defined by Malikov-Schechtman-Vaintrob and Beilinson-Drinfeld, where $\lambda^* = -w_0(\lambda)$.

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According to Gaisgory, the Ran space version of our theorem can be used to prove the quantum geometric Langlands correspondence.

Thank you!