Rational Diophantine sextuples

Vinko Petričević
vpetrice@math.hr

University of Zagreb, Faculty of Science, Department of Mathematics, Croatia

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A. Dujella, M. Kazalicki and P., Rational Diophantine sextuples containing two regular quadruples and one regular quintuple

A. Dujella, M. Kazalicki and P., There are infinitely many rational Diophantine sextuples with square denominators

Introduction

A set of m nonzero rationals $\{a_1, a_2, \ldots, a_m\}$ is called *a rational Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. The first example of a rational Diophantine quadruple was the set

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

found by Diophantus. Euler found infinitely many rational Diophantine quintuples, e.g. he was able to extend the integer Diophantine quadruple

$$\{1, 3, 8, 120\}$$

found by Fermat, to the rational quintuple

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}.$$

In 1999, Gibbs found the first example of a rational Diophantine sextuple

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\},$$

and in 2016 he collected over 1000 examples of positive rational Diophantine sextuples with relatively small numerators and denominators, and found 3 almost septuples.
In 2016 Dujella, Kazalicki, Mikić and Szikszai showed that there are infinitely many rational Diophantine triples that can be extended to a Diophantine sextuple in infinitely many ways, while in 2017 Dujella and Kazalicki (inspired by the work of Piezas) described another construction of parametric families of rational Diophantine sextuples.
No example of a rational Diophantine septuple is known.
Recently we (joint work with Dujella and Kazalicki) have extended the search for Diophantine sets with small height and included also examples with mixed signs. We have found many thousands examples of Diophantine sextuples with small heights, and we have found more than 40 almost septuples, and two new infinite families of sextuples.

Regular Diophantine sets

The triple $\{a, b, c\}$ is called regular if

$$(a+b-c)^2 = 4(ab+1).$$

The quadruple $\{a, b, c, d\}$ is called regular if

$$(a+b-c-d)^2 = 4(ab+1)(cd+1).$$

The quintuple $\{a, b, c, d, e\}$ is called regular if

$$(abcde+2abc+a+b+c-d-e)^2 = 4(ab+1)(ac+1)(bc+1)(de+1).$$

The sextuple $\{a, b, c, d, e, f\}$ is called regular if

$$\begin{aligned}(abcde+abcdf+abced-abdef-acdef-bcdef \\ +2abc-2def+a+b+c-d-e-f)^2 \\ = 4(ab+1)(ac+1)(bc+1)(de+1)(df+1)(ef+1).\end{aligned}$$

Theorem 2.1

There are infinitely many rational Diophantine sextuples which contain one regular Diophantine quintuple and two regular Diophantine quadruples.

We use parametrization of Diophantine triples due to Lasić, which is symmetric in the three involved parameters:

$$a_1 = \frac{2t_1(1+t_1t_2(1+t_2t_3))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},$$
$$a_2 = \frac{2t_2(1+t_2t_3(1+t_3t_1))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},$$
$$a_3 = \frac{2t_3(1+t_3t_1(1+t_1t_2))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)}.$$

Proof of Theorem 2.1

Let $\{a_1, a_2, a_3, a_4\}$ and $\{a_1, a_2, a_3, a_5\}$ be regular Diophantine quadruples, i.e. a_4 and a_5 are solutions of the quadratic equation

$$(a_1 + a_2 - a_3 - x)^2 - 4(a_1a_2 + 1)(a_3x + 1) = 0.$$

We obtain that

$$a_4 = \frac{-2(1 - t_3 + t_2t_3)(t_3t_1 + 1 - t_1)(-t_2 + 1 + t_1t_2)(-1 + t_1t_2t_3)}{(1 + t_1t_2t_3)^3},$$

$$a_5 = \frac{2(t_3 + t_2t_3 + 1)(t_3t_1 + t_1 + 1)(1 + t_2 + t_1t_2)(1 + t_1t_2t_3)}{(-1 + t_1t_2t_3)^3}.$$

In order that $\{a_1, a_2, a_3, a_4, a_5\}$ be a rational Diophantine quintuple, it remains to satisfy the condition that $a_4a_5 + 1$ is a perfect square.

We obtain the condition that

$$\begin{aligned} p(t_1,t_2,t_3) &= (-8t_2^3t_3^3-8t_3^2t_2^2-3t_3^4t_2^4+4t_2^2+4t_2^2t_3^4+4t_2^4t_3^2+8t_2^3t_3)t_1^4 \\ &+(8t_2^2t_3-16t_2t_3^2-8t_2^3t_3^2+8t_2-8t_2^3t_3^4-8t_2^4t_3^3-8t_2^2t_3^3+8t_3^4t_2)t_1^3 \\ &+(-8t_3^2-8t_2^2-8t_2t_3-8t_2^3t_3^3-8t_2^2t_3^4+8t_3^3t_2+4t_3^4+4-18t_3^2t_2^2 \\ &\quad +4t_3^4t_2^4-8t_2^4t_3^2-16t_2^3t_3)t_1^2 \\ &+(8t_2^4t_3^3-8t_2^2t_3-16t_2^2t_3^3-8t_2t_3^2-8t_2+8t_3^3+8t_2^3t_3^2-8t_3)t_1 \\ &-3-8t_2t_3+4t_2^4t_3^2-8t_3^2t_2^2+4t_3^2+4t_2^2+8t_2^3t_3 \end{aligned} \tag{2.1}$$

is a perfect square. We compute the discriminant of the quartic polynomial p with the respect to t_1 and factorize it. One of the factors is

$$p_1(t_2,t_3)=3+10t_2t_3-3t_3^2+3t_3^2t_2^2$$

(other factors either have much larger degree or correspond to quintuples with an element equal to 0). The condition $p_1(t_2,t_3)=0$ (which ensures that the polynomial p with the respect of t_1 has a double root) leads to $9t_3^2+16$ be a perfect square, say $9t_3^2+16=(3t_3+u)^2$. We get

$$t_3=\frac{16-u^2}{6u},\quad t_2=\frac{u^2+10u+16}{(u-4)(u+4)}.$$

Inserting this in (2.1), we obtain that a_4a_5+1 is a perfect square. Thus, we obtained a two-parametric family (in parameters t_1 and u) of rational Diophantine quintuples which contain two regular quadruples.
Now we extend the nonregular quadruple $\{a_1, a_3, a_4, a_5\}$ (analogously we could choose any of other two nonregular quadruples contained in the quintuple $\{a_1, a_2, a_3, a_4, a_5\}$) to regular quintuples $\{a_1, a_3, a_4, a_5, a_6\}$ and $\{a_1, a_3, a_4, a_5, a_7\}$, i.e. a_6 and a_7 are solutions of the quadratic equation

$$(a_1a_3a_4a_5x+2a_1a_3a_4+a_1+a_3+a_4-a_5-x)^2-4(a_1a_3+1)(a_1a_4+1)(a_3a_4+1)(a_5x+1)=0.$$

We will not use a_7 is our construction, so we give here only the value of a_6:

$$\begin{aligned}a_6 &= 6(u+4)(u+8)(u+2)(u-4)(2t_1u^2+3u^2+20t_1u+12u+32t_1)\\ &\times(t_1u^2+10t_1u+16t_1-6u)(t_1u^2+10t_1u+16t_1+6u)(t_1u^2+10t_1u+16t_1-24-6u)\\ &\times(4096t_1^2+15360t_1^2u+15168t_1^2u^2+5920t_1^2u^3+948t_1^2u^4+60t_1^2u^5+t_1^2u^6-12288t_1u\\ &\quad-7680t_1u^2+480t_1u^4+48t_1u^5-5184u^2-2592u^3-324u^4)^{-2}.\end{aligned}$$

The only missing condition in order that $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a rational Diophantine sextuple is that a_2a_6+1 is a perfect square. This condition leads to the quartic in t_1 over $\mathbb{Q}(u)$:

$$(u^{12}+120u^{11}+5496u^{10}+125600u^9+1639440u^8+13075200u^7+65656320u^6$$
$$+209203200u^5+419696640u^4+514457600u^3+360185856u^2+125829120u+16777216)t_1^4$$
$$+(24u^{12}++1296u^{11}+32256u^{10}+446208u^9+3461760u^8+13047552u^7-208760832u^5$$
$$-886210560u^4-1827667968u^3-2113929216u^2-1358954496u-402653184)t_1^3$$
$$+(36u^{12}+1296u^{11}+18072u^{10}+48096u^9-1681632u^8-22516992u^7-127051776u^6$$
$$-360271872u^5-430497792u^4+197001216u^3+1184366592u^2+1358954496u+603979776)t_1^2$$
$$+(-432u^{11}-15552u^{10}-259200u^9-2267136u^8-9116928u^7+145870848u^5$$
$$+580386816u^4+1061683200u^3+1019215872u^2+452984832u)t_1$$
$$+1296u^{10}+41472u^9+31643136u^6+670032u^8+6054912u^7+96878592u^5+171528192u^4$$
$$+169869312u^3+84934656u^2=z^2.$$

Since this quartic has a $\mathbb{Q}(u)$-rational point at infinity, it can be transformed by birational transformations into an elliptic curve over $\mathbb{Q}(u)$ (the singular point at infinity on the quartic corresponds to the point at infinity and an additional point P_1 on the elliptic curve).

The quartic has another $\mathbb{Q}(u)$-rational point corresponding to $t_1 = \frac{-3(u+4)u}{2(u^2+10u+16)}$. It gives $a_6 = 0$, so it does not yield a rational Diophantine sextuples. However, if we denote the corresponding point on the elliptic curve by P_2, then the point $2P_2$ on the elliptic curve corresponds to the point with

$$t_1 = \frac{3(3u^4 + 40u^3 + 368u^2 + 1280u + 1024)}{4(u^2 + 10u + 16)(u + 20)u}$$

on the quartic, and by inserting this value, we obtain the parametric family of rational Diophantine sextuples which satisfies the properties from Theorem 2.1.

$$\left\{\frac{-12u(u+4)(3u^4+8u^3+224u^2+576u+512)(3u^3+28u^2+256u+256)}{(u+8)(u+2)(u-4)(3u^3+8u^2+144u+128)(3u^4+48u^3+528u^2+1280u+1024)},\right.$$
$$\frac{8u(u+20)(3u^5+8u^4+64u^3-640u^2-2304u-2048)(u+8)(u+2)}{3(u+4)(u-4)(3u^3+8u^2+144u+128)(3u^4+48u^3+528u^2+1280u+1024)},$$
$$\frac{2(u+4)(u-4)(39u^7+776u^6+8096u^5+48640u^4+226048u^3+587776u^2+770048u+393216)}{3(u+8)(u+2)(3u^3+8u^2+144u+128)(3u^4+48u^3+528u^2+1280u+1024)},$$
$$\frac{-8(u^2+4u+32)(3u^3+14u^2-40u-64)(9u^3+8u^2+112u+384)(3u^4+48u^3+528u^2+1280u+1024)}{3(u+8)(u+4)(u+2)(u-4)(3u^3+8u^2+144u+128)^3},$$
$$\frac{4u(u+2)(17u^2+48u+48)(3u^5+8u^4-176u^3-2944u^2-9216u-8192)(3u^3+8u^2+144u+128)(u+8)^2}{3(u+4)(u-4)(3u^4+48u^3+528u^2+1280u+1024)^3},$$
$$\left.\frac{12(u+2)(u-4)(5u+8)(u+4)(3u^2+8u+64)(3u^3+8u^2+144u+128)(3u^4+48u^3+528u^2+1280u+1024)}{(u+8)(16384+69632u+64768u^2+22272u^3+3680u^4+576u^5+9u^6)^2}\right\}.$$

E.g. for $u=-1$ we get the rational Diophantine sextuple

$$\left\{\frac{27900}{17479}, \frac{471352}{112365}, \frac{261770}{17479}, \frac{185535272}{419265}, \frac{63737828}{526368735}, \frac{79554420}{408480247}\right\}.$$

By taking other linear combinations of the points P_1 and P_2 we can also obtain (more complicated) families of rational Diophantine sextuples.

Theorem 3.1

There are infinitely many rational Diophantine sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect squares.

We can describe one such family in the following way. Let $C : t^4 - 1 = -6s^2$ be a genus one curve defined over $\mathbb{Q}$. It is birationally equivalent to the elliptic curve $E' : y^2 = x^3 + 9x$. Denote by $T = [4, 10]$ the point of infinite order in the Mordell-Weil group $E'(\mathbb{Q})$. For a positive integer k denote by t_k the t-coordinate of the point on C that corresponds to the point $[k]T$ on $E'(\mathbb{Q})$ under birational equivalence. Let $\mathcal{F}(t)$ be the family of Diophantine sextuples:

$$\left\{-\frac{9\left(t^2+1\right)}{8(t-1)(t+1)}, \frac{\left(t^2-7\right)\left(t^2+1\right)\left(7t^2-1\right)}{8(t-1)^3(t+1)^3}, \frac{8(t-1)^3(t+1)^3}{9(t^2+1)^3}, -\frac{2\left(t^2+5\right)\left(5t^2+1\right)}{9(t-1)(t+1)(t^2+1)},\right.$$
$$-\frac{2t\left(t^2-4t-3\right)\left(3t^2-4t-1\right)\left(t^3+8t^2+5t+4\right)\left(4t^3-5t^2+8t-1\right)}{(t-1)(t+1)(t^2+1)(t^4+34t^2+1)^2},$$
$$\left.\frac{2t\left(t^2+4t-3\right)\left(3t^2+4t-1\right)\left(t^3-8t^2+5t-4\right)\left(4t^3+5t^2+8t+1\right)}{(t-1)(t+1)(t^2+1)(t^4+34t^2+1)^2}\right\}.$$

If k is a positive integer such that $k \equiv 1, 2 \pmod 3$, then $\mathcal{F}(t_k)$ is a rational Diophantine sextuple such that denominators of all the elements in the sextuple are perfect squares.

Numerical search

We started with brute-force search for all Diophantine sets with numerators and denominatos in the range between -2^{15} and 2^{15}.

We used graph theory. We construct a graph, connecting the numbers k and l with an edge provided they satisfy $k \cdot l = x^2 - 1$, for some $x \in \mathbb{Q}$. So we were searching for cliques (completely connected subgraphs) in that graph.

For example, that graph have about $2 \cdot 10^9$ nodes and more than 10^{11} edges (almost 1TB of disk space, or about 500GB with better *representation*). We write program in C++. Search for all cliques took about one week on 12-cores computer with 250GB of memory (with usual hard drive).
Next, we tried to extend each Diophantine set with well known regular elements.

(We also tried to extend sextuples using Stoll's program `ratpoints`).

Our starting point is an example of Diophantine sextuple with square denominators

$$I = \left\{ \frac{75}{8^2}, -\frac{3325}{64^2}, -\frac{12288}{125^2}, \frac{123}{10^2}, \frac{3498523}{2260^2}, \frac{698523}{2260^2} \right\}$$

which we have discovered (together with seventeen other examples) by a numerical search.
Let $\{a, b, c, d\}$ be a rational Diophantine quadruple with elements in $\mathbb{Q}(t)$ such that

$$(abcd - 3)^2 = 4(ab + cd + 3), \tag{3.1}$$

and let x_1 and x_2 be the roots of

$$(abcdx + 2abc + a + b + c - d - x)^2 = 4(ab + 1)(ac + 1)(bc + 1)(dx + 1).$$

If $x_1 x_2 \neq 0$ then Proposition 1 from Dujella and Kazlicki paper from 2017. states that $\{a, b, c, d, x_1, x_2\}$ is a Diophantine sextuple.

Since the quadruple (a, b, c, d) satisfies

$$ab+1=t_{12}^2 \quad ac+1=t_{13}^2 \quad ad+1=t_{14}^2$$
$$bc+1=t_{23}^2 \quad bd+1=t_{24}^2 \quad cd+1=t_{34}^2,$$

where t_{ij} are in $\mathbb{Q}(t)$, it follows that $(t_{12}, t_{34}, t_{13}, t_{24}, t_{14}, t_{23}, m'=abcd)$ defines a rational point on an algebraic variety $\mathcal{C}$ defined by the following equations:

$$(t_{12}^2-1)(t_{34}^2-1)=m'$$
$$(t_{13}^2-1)(t_{24}^2-1)=m'$$
$$(t_{14}^2-1)(t_{23}^2-1)=m'.$$

Conversely, the points $(\pm t_{12}, \pm t_{34}, \pm t_{13}, \pm t_{24}, \pm t_{14}, \pm t_{23}, m')$ on $\mathcal{C}$ determine two rational Diophantine quadruples $\pm(a, b, c, d)$ (for example $a^2=(t_{12}^2-1)(t_{13}^2-1)/(t_{23}^2-1)$) provided that the elements a, b, c and d are rational, distinct and non-zero. (Note that if one element is rational, then all the elements are rational.)

The projection $(t_{12}, t_{34}, t_{13}, t_{24}, t_{14}, t_{23}, m') \mapsto m'$ defines a fibration of $\mathcal{C}$ over the affine line, and a generic fiber is the product of three genus one curves $\mathcal{D} : (x^2-1)(y^2-1) = m'$, hence any point on $\mathcal{C}$ corresponds to the three points $Q_1 = (t_{12}, t_{34})$, $Q_2 = (t_{13}, t_{24})$ and $Q_3 = (t_{14}, t_{23})$ on $\mathcal{D}$. The condition (3.1) is equivalent to $t_{12}t_{34} = \pm t_{12} \pm t_{34}$, or $t_{34} = \pm t_{12}/(t_{12} \pm 1)$. Hence, if we set $t_{12} = t$, $t_{34} = t/(t-1)$ and $m' = (t^2-1)(\frac{t^2}{(t-1)^2} - 1) = \frac{2t^2+t-1}{t-1}$, the condition (3.1) is automatically satisfied.
The curve $\mathcal{D}$ over $\mathbb{Q}(t)$

$$\mathcal{D} : (x^2-1)(y^2-1) = \frac{2t^2+t-1}{t-1}$$

is birationally equivalent to the elliptic curve

$$E : S^2 = T^3 - 2 \cdot \frac{2t^2-t+1}{t-1} T^2 + \frac{(2t-1)^2(t+1)^2}{(t-1)^2} T.$$

The map is given by $T = 2(x^2-1)y + 2x^2 - (2-m')$, and $S = 2Tx$, where $m' = \frac{2t^2+t-1}{t-1}$.

Denote by $P = \left[\frac{(2t-1)^2(t+1)}{t-1}, \frac{2t(2t-1)^2(t+1)}{t-1}\right] \in E(\mathbb{Q}(t))$ a point of infinite order on E, and by $R = \left[\frac{(t+1)(2t-1)}{t-1}, \frac{2(t+1)(2t-1)}{t-1}\right]$ a point of order 4. The point $(t_{12}, t_{34}) \in \mathcal{D}(\mathbb{Q}(t))$ corresponds to the point $P \in E(\mathbb{Q}(t))$. The points P and R generate Mordell-Weil group $E(\mathbb{Q}(t))$.
Therefore, to specify the family of Diophantine quadruples which satisfies (3.1) and whose product is $\frac{2t^2+t-1}{t-1}$, we need to specify two points Q_2 and Q_3 in $E(\mathbb{Q}(t))$.
If we go back to our example, we can observe that the first four elements of I satisfy condition (3.1), and that their product is equal to $\frac{2t_0^2+t_0-1}{t_0-1}$ where $t_0 = 113/625$. Inspired by the fact that $\frac{1-t_0}{2} = 16/25$ is a square, we restrict to the subfamily $t := 1 - 2u^2$, i.e. we consider base change of $E/\mathbb{Q}(t)$ to $E/\mathbb{Q}\left(\sqrt{\frac{1-t}{2}}\right)$.

If we further specialize $u := \frac{3v^2+3}{4v^2-4}$ and denote the resulting elliptic curve also by E, then we obtain another point of infinite order in $E(\mathbb{Q}(v))$

$$Q = \left[-\frac{\left(v^2+5\right)^2\left(5v^2+1\right)^2}{36(v-1)^2(v+1)^2\left(v^2+1\right)^2}, \frac{v\left(v^2+5\right)^2\left(5v^2+1\right)^2}{9(v-1)^2(v+1)^2\left(v^2+1\right)^3} \right].$$

It is easy to check that the family $\mathcal{F}$ corresponds to the triple $(P, Q, P+R) \in E(\mathbb{Q}(v))^3$ (with the substitution $v := t$), and we obtain I if we specialize to $v = -1/7$, i.e. $I = \mathcal{F}(-1/7)$.

For a prime p and $q \in \mathbb{Q}$, we denote by $v_p(q)$ the p-adic valuation of q normalized such that $v_p(p) = 1$. The elliptic curve $E' : y^2 = x^3 + 9x$ is birationally equivalent to C via the map $h : E' \to C$, $h(x, y) = \left(\frac{3-x}{3+x}, \frac{-2y}{(x+3)^2}\right)$. Let $T = [4, 10]$; this generates $E'(\mathbb{Q})$ modulo the torsion subgroup. Denote by $(t_k, s_k) := h([k]T) \in C$.

Lemma 3.2

Let k be a positive integer. If $k \equiv 0 \pmod 3$ then $v_3(s_k) > 0$ and $v_3(t_k - 2) \geq 1$. If $k \equiv 1, 2 \pmod 3$ then $v_3(t_k - 5) \geq 2$ and $v_3(s_k) = 0$.

Proof.

Define $(x_k, y_k) := [k]T$. Since $y_k^2 = x_k^3 + 9x_k$, it follows that either $v_3(x_k) < 0$ or x_k is a square mod 9. The first case occurs when $k \equiv 0 \pmod 3$ (since $[3]T$ is in the kernel of the mod 3 reduction map on E'), and then $t_k = \frac{3-x_k}{3+x_k}$ implies that $v_3(t_k + 1) \geq 2$. In the second case one finds that $v_3(t_k - 5) \geq 2$ (since the squares mod 9 are $1, 4$ and 7). In the first case, since $t_k^4 - 1 = -6s_k^2$ it follows $v_3(s_k) > 0$, while in the second case $v_3(s_k) = 0$. □

Proof of Theorem 3.1

We analyze the denominators of all the elements in $\mathcal{F} = \{a_1(t), a_2(t), \ldots, a_6(t)\}$ separately.

i) Since $a_1(t) = -\frac{9\left(t^2+1\right)}{8(t-1)(t+1)} = -\frac{9\left(t^2+1\right)^2}{8\left(t^4-1\right)}$, we have that $a_1(t_k) = \frac{3\left(t_k^2+1\right)^2}{16s_k^2}$.

Similarly, $a_3(t) = \frac{8(t-1)^3(t+1)^3}{9(t^2+1)^3} = \frac{8\left(t^4-1\right)^3}{9(t^2+1)^6}$, and $a_3(t_k) = \frac{-3\cdot 2^6 s_k^6}{(t_k^2+1)^6}$. The claim follows from Lemma 3.2 since it implies that $v_3(s_k) = 0$ if $k \equiv 1, 2 \pmod 3$.

ii) We have that $a_2(t) = \frac{(t^2-7)(t^2+1)(7t^2-1)}{8(t-1)^3(t+1)^3} = \frac{(t^2-7)(t^2+1)^4(7t^2-1)}{8(t^4-1)^3}$. The only primes that can divide the denominator of $a_2(t_k) = \frac{-(t_k^2-7)(t_k^2+1)^4(7t_k^2-1)}{3\cdot 24^2 s_k^6}$ are the primes that divide s_k or $1/t_k$. Assume that p is a prime different than 2 and 3. Since $t_k^4 - 1 = -6s_k^2$, if $v_p(t_k) < 0$, then $v_p(s_k) = 2v_p(t_k)$. Hence for such p, $v_p(a_2(t_k)) = 1$ if $p = 7$ and 0 otherwise. If $v_p(s_k) > 0$, then $v_p(a_2(t_k))$ is even unless $v_p((t_k^2 - 7)(7t_k^2 - 1)) > 0$. Since the resultant of the polynomials $t^4 - 1$ and $(t^2 - 7)(7t^2 - 1)$ is equal to $2^{16}3^4$ (and is divisible only by 2 and 3) this cannot happen. To rule out $p = 3$ case, we note that $v_3\left((t_k^2 - 7)(7t_k^2 - 1)\right) \geq 3$ since Lemma 3.2 implies that $v_3(t_k - 5) \geq 2$ if $k \equiv 1, 2 \pmod 3$. Hence $v_3(a_2(t_k)) \geq 0$. If $v_2(s_k) > 0$, then $v_2(t_k) = 0$ and $v_2((t_k^2 - 7)(7t_k^2 - 1)) = 2$ is even number. Finally, if $v_2(t_k) < 0$, then $4v_2(t_k) = 2v_2(s_k) + 1$ which is not possible.

iii) Since $a_4(t) = -\frac{2(t^2+5)(5t^2+1)}{9(t^4-1)}$ then $a_4(t_k) = \frac{(t_k^2+5)(5t_k^2+1)}{3\cdot 3^2 s_k^2}$. Let p be a prime different than 2 and 3. Similarly as in part ii), if $v_p(t_k) < 0$ then $v_p(a_4(t_k)) = 1$ if $p = 5$ and 0 otherwise. If $v_p(s_k) > 0$, then $v_p(a_4(t_k))$ is even unless $v_p\left(\left(t_k^2+5\right)\left(5t_k^2+1\right)\right) > 0$. Since the resultant of the polynomials t^4-1 and $\left(t^2+5\right)\left(5t^2+1\right)$ is $2^{12}3^4$ this cannot happen. Note that $v_3\left(\left(t_k^2+5\right)\left(5t_k^2+1\right)\right) \geq 3$ (Lemma 3.2), hence if $k \equiv 1,2 \pmod 3$ then $v_3(a_4(t_k) \geq 0$. As earlier, $v_2(t_k) < 0$ is not possible, and if $v_2(s_k) > 0$ then $v_2(t_k) = 0$ and $v_2\left(\left(t_k^2+5\right)\left(5t_k^2+1\right)\right) = 2$.

iv) We have that

$$a_5(t_k) = \frac{t_k\left(t_k^2-4t_k-3\right)\left(3t_k^2-4t_k-1\right)\left(t_k^3+8t_k^2+5t_k+4\right)\left(4t_k^3-5t_k^2+8t_k-1\right)}{3s_k^2\left(t_k^4+34t_k^2+1\right)^2}.$$

Let p be a prime different than 2 and 3. If $v_p(t_k) < 0$ then $v_p(a_5(t_k)) = 11v_p(t_k) - 8v_p(t_k) - 2v_p(s_k) = -v_p(t_k) > 0$. If $v_p(s_k) > 0$, then $v_p(a_5(t_k))$ is even unless

$$v_p\left(t_k\left(t_k^2-4t_k-3\right)\left(3t_k^2-4t_k-1\right)\left(t_k^3+8t_k^2+5t_k+4\right)\left(4t_k^3-5t_k^2+8t_k-1\right)\right) > 0.$$

As before, since the resultant is divisible only by 2 and 3 this cannot happen. The resultant of the polynomials $t\left(t^2-4t-3\right)\left(3t^2-4t-1\right)\left(t^3+8t^2+5t+4\right)\left(4t^3-5t^2+8t-1\right)$ and t^4+34t^2+1 is $2^{44}3^{16}$, hence we have the same conclusion if $v_p(t_k^4+34t_k^2+1) > 0$. As earlier, $v_2(t_k) < 0$ is not possible. If $v_2(s_k) > 0$, then $v_2(t_k) = 0$, and we find that

$$v_2\left(t_k\left(t_k^2-4t_k-3\right)\left(3t_k^2-4t_k-1\right)\left(t_k^3+8t_k^2+5t_k+4\right)\left(4t_k^3-5t_k^2+8t_k-1\right)\right) = 4.$$

Similarly, if $v_2(t_k^4+34t_k^2+1) > 0$, then $v_2(t_k) = 0$, and we have the same conclusion as above.

If $k \equiv 1, 2 \pmod 3$, then since $v_3(t_k - 5) > 1$ by a direct calculation we can show that
$v_3\left(3s_k^2\left(t_k^4 + 34t_k^2 + 1\right)^2\right) = 5$. Similarly we can check that $v_2(t_k) = 0$, $v_3(t_k^2 - 4t_k - 3) = 0$, $v_3(3t_k^2 - 4t_k - 1) \geq 2$, $v_3(t_k^3 + 8t_k^2 + 5t_k + 4) = 1$ and $v_3(4t_k^3 - 5t_k^2 + 8t_k - 1) = 2$ which implies that $v_3(a_5(t_k)) \geq 0$.

v) The claim for $a_6(t_k)$ is proved in a similar way.

Thank you for your attention