Existence of cuspidal automorphic forms for reductive groups over number fields

Goran Muić

June 27, 2019

joint work with Allen Moy: **A. Moy, G. Muić, On existence of generic cusp forms on semisimple algebraic groups. Trans. Amer. Math. Soc. 370 (2018), no. 7, 47314757**

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we show existence of variety of generic (and more general) automorphic cuspidal representations with local components in particular Bernstein classes **not necessarily supercuspidal**

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possibly degenerate Fourier coefficients of automorphic cuspidal forms are important for the theory of automorphic L–functions in adelic and in a more classical settings such as Siegel modular forms (Ginzburg, Soudry, Jiang, Gan, Savin, Ikeda, ...)

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for almost all places of k, G is a group scheme over $\mathcal{O}_v$, and $G(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup of $G(k_v)$

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let K_∞ and $\mathfrak{g}_\infty$ be a maximal compact subgroup and the (real) Lie algebra of G_∞, respectively

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we remark that the space $C_c^\infty(G(k) \setminus G(\mathbb{A}))$ is a subspace of $C^\infty(G(\mathbb{A}))$ consisting of all functions which are $G(k)$–invariant on the left and which are compactly supported modulo $G(k)$

introduce the Hilbert space $L^2(G(k) \setminus G(\mathbb{A}))$, where the inner product is the usual Petersson inner product

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if P is a k–parabolic subgroups of G, then we denote by U_P the unipotent radical of P. For $\varphi \in L^2(G(k) \setminus G(\mathbb{A}))$, the constant term is a function

$$\varphi_P(g) = \int_{U_P(k) \backslash U_P(\mathbb{A})} \varphi(ug) du$$

defined almost everywhere on $G(\mathbb{A})$

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$\implies L^2_{cusp}(G(k) \setminus G(\mathbb{A})) = \hat{\oplus}_j \overline{V}_j$ a Hilbert direct sum of irreducible representations of $G(\mathcal{A})$.

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a Bernstein class $\mathfrak{M}_v$ determined by (M_v, ρ_v), where M_v is a Levi subgroup of $G(k_v)$ and ρ_v is an (irreducible) supercuspidal representation of M_v, is a equivalence class of such pairs

(M_v, ρ_v) and (M'_v, ρ'_v) are equivalent if we can find $g_v \in G(k_v)$ and an unramified character χ_v of M'_v such that $M'_v = g_v M_v g_v^{-1}$ and $\rho'_v \simeq \chi_v \rho_v^{g_v}$ i.e.,

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by definition, $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v)$ is the largest $G(k_v)$–submodule of $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$ such that its every irreducible subquotient is a subquotient of $Ind_{P_v}^{G(k_v)}(\chi_v \rho_v)$, for some unramified character χ_v of M_v. Here P_v is an arbitrary parabolic subgroup of $G(k_v)$ containing M_v as a Levi subgroup

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we can iterate this for v ranging over a finite set of places, and as a result, we arrive at the question of non–triviality of a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v \in T)$, where $T \subset V_f$ is **a finite and non–empty** set of places.

Theorem. Let T be a finite set of places of k such that G is unramified over k_v for $v \in V_f - T$. For $v \in T$, let $\mathfrak{M}_v$ be a Bernstein's class of $G(k_v)$ determined by (M_v, ρ_v). We define $\mathfrak{P}$ to be the set of all k–parabolic subgroups P such that a Levi factor of $P(k_v)$ contains a $G(k_v)$–conjugate of M_v for all $v \in T$. Then we have the following:

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Theorem. Let T be a finite set of places of k such that G is unramified over k_v for $v \in V_f - T$. For $v \in T$, let $\mathfrak{M}_v$ be a Bernstein's class of $G(k_v)$ determined by (M_v, ρ_v). We define $\mathfrak{P}$ to be the set of all k–parabolic subgroups P such that a Levi factor of $P(k_v)$ contains a $G(k_v)$–conjugate of M_v for all $v \in T$. Then we have the following:
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some of the results can also be obtained by using more delicate techniques of Arthur–Selberg trace formula as I was informed by Shin

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an irreducible closed subrepresentation $\mathfrak{U}$ of $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ is (ψ, U)–generic if

$$\mathfrak{U} \not\subset L^2_{cusp,\ (\psi,U)\text{–degenerate}}(G(k) \setminus G(\mathbb{A})).$$

Main Theorem: Assume that G is a semisimple algebraic group defined over a number field k. Let U be a unipotent k-subgroup. Let $\psi : U(k) \setminus U(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ be a (unitary) character. Let S be a finite set of places, containing V_∞, large enough such that G and ψ are unramified for $v \notin S$ (in particular, ψ_v is trivial on $U(\mathcal{O}_v)$). For each finite place $v \in S$, let $\mathfrak{M}_v$ be a $(\psi_v, U(k_v))$–generic Bernstein's class (i.e., there is a $(\psi_v, U(k_v))$–generic irreducible representation which belongs to that class such that the following holds:

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if P is a k–parabolic subgroup of G such that a Levi subgroup of $P(k_v)$ contains a conjugate of a Levi subgroup defining $\mathfrak{M}_v$ for all finite v in S, then $P = G$.

Then, there exists an irreducible subspace in $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ which is (ψ, U)–generic such that its K–finite vectors $\pi_\infty \otimes_{v \in V_f} \pi_v$ satisfy the following:

Then, there exists an irreducible subspace in $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ which is (ψ, U)–generic such that its K–finite vectors $\pi_\infty \otimes_{v \in V_f} \pi_v$ satisfy the following:

(i) π_v is unramified for $v \notin S$.

(ii) π_v belongs to the class $\mathfrak{M}_v$ for all finite $v \in S$.

(iii) π_v is $(\psi_v, U(k_v))$–generic for all finite v.

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THANK YOU!