Congruences for sporadic sequences and modular forms for non-congruence subgroups

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Elementary congruences

Denote by

$$F(n) = \sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 .$$

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Theorem (K.)

For all primes $p > 2$ we have

$$F\left(\frac{p-1}{2}\right) \equiv \begin{cases} 2(a^2 - 6b^2) \pmod{p} & \textit{if } p = a^2 + 6b^2 \\ 0 \pmod{p} & \textit{if } p \equiv 5, 11, 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Apéry's proof of the irrationality of $\zeta(3)$

In 1978 Roger Apéry proved that

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For that he defined sequences a_n and a'_n as a solutions of recursion

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0,$$

with initial conditions $(a_0, a_1) = (1, 5)$ and $(a'_0, a'_1) = (0, 6)$.

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He showed that for n sufficiently large relative to ϵ

$$|\zeta(3) - \frac{p_n}{q_n}| < \frac{1}{{q_n}^{\theta+\epsilon}}, \text{where } \frac{p_n}{q_n} = \frac{a'_n}{a_n} \text{ and } \theta = 1.080529\ldots,$$

which implies that $\zeta(3)$ is irrational.

Apéry's proof of the irrationality of $\zeta(3)$ cont.

One thing that is remarkable here is that a_n's are integers, i.e. $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$.

Apéry's proof of the irrationality of $\zeta(3)$ cont.

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Similarly for the proof of irrationality of $\zeta(2)$ he introduced numbers $b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ as a solutions of recursion

$$(n+1)^2 u_{n+1} - (11n^2 + 11n + 3)u_n - n^2 u_{n-1} = 0.$$

Zagier's sporadic sequences

Zagier performed a computer search on first 100 million triples $(A, B, C) \in \mathbb{Z}^3$ and found that the recursive relation generalizing b_n

$$(n+1)^2 u_{n+1} - (An^2 + An + B)u_n + Cn^2 u_{n-1} = 0,$$

with the initial conditions $u_{-1} = 0$ and $u_0 = 1$ has (non-degenerate i.e. $C(A^2 - 4C) \neq 0$) integral solution for only six more triples (whose solutions are so called sporadic sequences)

$$(0, 0, -16), (7, 2, -8), (9, 3, 27), (10, 3, 9), (12, 4, 32) \text{ and } (17, 6, 72).$$

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$(0, 0, -16), (7, 2, -8), (9, 3, 27), (10, 3, 9), (12, 4, 32)$ and $(17, 6, 72)$.

The sequence $F(n)$ corresponds to the triple $(17, 6, 72)$.

The previous work

Stienstra and Beukers proved congruences analogous to the one in the first slide for Apery numbers (and for two more sporadic sequences). Recently Osburn and Straub proved them for all sequences except for $F(n)$ - for which they made a conjecture.

Connection with geometry

Stienstra and Beukers showed that the generating functions of Apéry's numbers b_n (and Zagier for other sporadic sequences) are holomorphic solutions of Picard-Fuchs differential equation of some elliptic surface.

Picard-Fuchs differential equations for the Legendre family of elliptic curves

For $t \in \mathbb{C}$ let

$$E_t : \quad y^2 = x(x-1)(x-t),$$

be Legendre's family of elliptic curve with period integrals

$$\Omega_1(t) = \int_t^1 \frac{dx}{\sqrt{x(x-1)(x-t)}}, \quad \Omega_2(t) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}}.$$

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They satisfy Picard-Fuchs differential equation

$$t(t-1)\Omega''(t) + (2t-1)\Omega'(t) + \frac{1}{4}\Omega(t) = 0,$$

whose unique holomorphic solution at $t=0$ is hyperelliptic function

$$-\pi\Omega_2(t) = \sum_{n=0}^{\infty} \frac{((1/2)_n)^2}{n!} t^n.$$

Modular elliptic surface and sequence $F(n)$

Consider modular rational elliptic surface for $\Gamma_1(6)$

$$\mathcal{W} : (x+y)(x+z)(y+z) - 8xyz = \frac{1}{t}xyz,$$

with fibration $\phi : \mathcal{W} \to P^1$, $(x, y, z, t) \mapsto t$.

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Picard-Fuchs differential equation associated to this elliptic surface

$$(8t+1)(9t+1)P(t)'' + t(144t+17)P(t)' + 6t(12t+1)P(t) = 0,$$

has a holomorphic solution around $t = 0$

$$P(t) = \sum_{n=0}^{\infty} (-1)^n F(n) t^n.$$

Modular forms

We can identify t with a modular function (for $\Gamma_0(6)$)

$$t(\tau) = \frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)}, \quad \tau \in \mathbb{H}$$

then $P(\tau) := \sum_{n=0}^{\infty}(-1)^n F(n) t(\tau)^n$ is a weight one modular form for $\Gamma_1(6)$.

The main idea

Proposition (Beukers)

Let p be a prime and $\omega(t) = \sum_{n=1}^{\infty} b_n t^{n-1} dt$ a differential form with $b_n \in \mathbb{Z}_p$. Let $t(q) = \sum_{n=1}^{\infty} A_n q^n, A_n \in \mathbb{Z}_p$, and suppose

$$\omega(t(q)) = \sum_{n=1}^{\infty} c_n q^{n-1} dq.$$

Suppose there exist $\alpha_p, \beta_p \in \mathbb{Z}_p$ with $p|\beta_p$ such that

$$b_{mp^r} - \alpha_p b_{mp^{r-1}} + \beta_p b_{mp^{r-2}} \equiv 0 \pmod{p^r}, \quad \forall m, r \in \mathbb{N}.$$

Then

$$c_{mp^r} - \alpha_p c_{mp^{r-1}} + \beta_p c_{mp^{r-2}} \equiv 0 \pmod{p^r}, \quad \forall m, r \in \mathbb{N}.$$

Moreover, if A_1 is p-adic unit then the second congruence implies the first, and we have that $b_p \equiv \alpha_p b_1 \pmod{p}$.

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Now consider a two cover $\mathcal{S}$ of $\mathcal{W}$, a K3-surface given by the equation

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where $t = s^2$. Then $s(\tau) = \sqrt{\frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)}}$ is a corresponding modular function for index two genus zero subgroup $\Gamma_2 \subset \Gamma_1(6)$

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Given prime $p > 2$, we apply the previous proposition to the differential form

$$\omega(s) = \sum_{n=1}^{\infty}(-1)^n F(n) s^{2n} ds,$$

and $s(q)$ - the q-expansion of modular function $s(\tau)$ (where $q = e^{\pi i \tau}$).

Congruences for $F(n)$ cont.

We obtain that

$$\omega(s(q)) = \sum_{n=0}^{\infty} c_n q^{n-1} dq,$$

where c_n are Fourier coefficients of weight 3 cusp form $g(\tau) \in S_3(\Gamma_2)$

$$g(q) = P(q) q \frac{d}{dq} s(q) = q + \frac{3}{2} q^3 - \frac{9}{8} q^5 - \frac{85}{16} q^7 - \frac{981}{128} q^9 + \cdots = \sum_{n=1}^{\infty} c_n q^n.$$

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Proposition (K.)

Let $p > 3$ be a prime. Then for all $m, r \in \mathbb{N}$, we have that

$$c_{mp^r} - \left(\frac{-1}{p}\right)\gamma(p)c_{mp^{r-1}} + \left(\frac{-6}{p}\right)p^2 c_{mp^{r-2}} \equiv 0 \pmod{p^{2r}},$$

where

$$\gamma(p) = \begin{cases} 2(a^2-6b^2) \pmod{p} & \textit{if } p = a^2+6b^2 \\ 0 \pmod{p} & \textit{if } p \equiv 5, 11, 13, 17, 19, 23 \pmod{24}. \end{cases}.$$

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For $m = 1$ and $r = 1$, it follows $c_p \equiv \left(\frac{-1}{p}\right)\gamma(p) \pmod{p}$, hence by the Theorem of Beukers

$$(-1)^{\frac{p-1}{2}} F\left(\frac{p-1}{2}\right) \equiv \left(\frac{-1}{p}\right)\gamma(p) \pmod{p}.$$

Atkin and Swinnerton-Dyer congruences

For a finite index noncongruece subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and a prime p, we say that weight k cusp form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in S_k(\Gamma, \overline{\mathbb{Z}_p})$ satisfy Atkin and Swinnerton-Dyer (ASD) congruence at p if there exist an algebraic integer A_p and a root of unity μ_p such that for all non-negative integers m and r we have

$$a_{mp^r} - A_p a_{mp^{r-1}} + \mu_p p^{k-1} a_{mp^{r-2}} \equiv 0 \pmod{p^{(k-1)r}}. \tag{1}$$

(In our example $a_n's$ and $A_p's$ are rational integers, and $\mu_p = \pm 1$.)

Result of Scholl

In the case when the space of cusp forms is one dimensional and generated by $f(\tau)$ (which is the case for $S_3(\Gamma_2)$ and $g(\tau)$), Scholl proved that the ASD congruence holds for all but finitely many p.

Action of Frobenius - de Rham cohomology

The congruences were obtained by embedding the module of cusp forms (in our case of weight 3) into certain de Rham cohomology group $DR(\Gamma)$ which is the de Rham realization of the motive associated to the relevant space of modular forms.

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At a good prime p, crystalline theory endows $DR(\Gamma) \otimes \mathbb{Z}_p$ with a Frobenius endomorphism whose action on q-expansion gives rise to Atkin and Swinnerton-Dyer congruences, i.e. congruence (1) holds, if

$$T^2 - A_p T + \mu_p p^2$$

is a characteristic polynomial of Frobenius acting on $DR(\Gamma) \otimes \mathbb{Z}_p$.

Action of Frobenius - ℓ-adic cohomology

To calculate the trace of Frobenius A_p, following Scholl, we associate to the subgroup Γ_2 a strictly compatible family of ℓ-adic Galois representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $\tilde{\rho}_\ell$, that is isomorphic to ℓ-adic realization of the motive associated to the space of cusp forms $S_3(\Gamma_2)$. Then

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$$A_p = trace(\tilde{\rho}_2(Frob_p)) \text{ and } \mu_p = \det(\tilde{\rho}_2(Frob_p)).$$

Explicit description

Let $X(\Gamma_2)^0$ be the complement in $X(\Gamma_2)$ of the cusps. Denote by i the inclusion of $X(\Gamma_2)^0$ into $X(\Gamma_2)$, and by $h' : \mathcal{S} \to X(\Gamma_2)^0$ the restriction of elliptic surface $h : \mathcal{S} \to X(\Gamma_2)$ to $X(\Gamma_2)^0$. For a prime ℓ we obtain a sheaf

$$\mathcal{F}_\ell = R^1 h'_* \mathbb{Q}_\ell$$

on $X(\Gamma_2)^0$, and also sheaf $i_*\mathcal{F}_\ell$ on $X(\Gamma_2)$ (here $\mathbb{Q}_\ell$ is the constant sheaf on the elliptic surface $\mathcal{S}$, and R^1 is derived functor). The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the $\mathbb{Q}_\ell$-vector space

$$W = H^1_{et}(X(\Gamma_2) \otimes \overline{\mathbb{Q}}, i_*\mathcal{F}_\ell)$$

defines ℓ-adic representation ρ_ℓ.

The third family of ℓ-adic representation

For $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$ let

$$f(\tau) = \sum_{n=0}^{\infty} = q - 2q^2 + 3q^3 + \cdots = \sum_{n=0}^{\infty} \gamma(n) q^n \in S_3\left(\Gamma_0(24), \left(\frac{-6}{\cdot}\right)\right)$$

be a newform. Then for prime p

$$\gamma(p) = \begin{cases} 2(a^2 - 6b^2) \text{ if } p = a^2 + 6b^2 \\ 0 \pmod{p} \text{ if } p \equiv 3, 11, 13, 17, 19, 23 \pmod{24}. \end{cases}.$$

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Denote by ρ'_ℓ a two dimensional ℓ-adic Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ attached to the newform $f(\tau) \otimes \left(\frac{-1}{\cdot}\right)$ by the work of Deligne. Hence,

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$$\mathrm{trace}(\rho'_\ell(\mathrm{Frob}_p)) = \left(\frac{-1}{p}\right)\gamma(p) \text{ and } \det(\rho'_\ell(\mathrm{Frob}_p)) = \left(\frac{-24}{p}\right)p^2,$$

for prime $p \neq 2, 3$ and ℓ.

$\rho_\ell \cong \tilde{\rho}_\ell$ and $\rho_\ell \cong \rho'_\ell$

To prove ASD congruence for $g(\tau)$ it is enough to show that representations ρ'_ℓ and $\tilde{\rho}_\ell$ are isomorphic. We prove that by showing that both of them are isomorphic to ρ_ℓ.

Serre-Faltings method $\rightarrow \rho_\ell \cong \rho'_\ell$

Theorem (Serre, Scholl)

For a finite set of primes S of $\mathbb{Q}$, let $\chi_1, \ldots, \chi_r$ be a maximal independent set of quadratic characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside S, and G a subset of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the map $(\chi_1, \ldots, \chi_r) : G \rightarrow (\mathbb{Z}/2\mathbb{Z})^r$ is surjective.
Let $\sigma, \sigma' : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathbb{Q}_2)$ be continuous semisimple representation unramified away from S, whose images are pro-2-groups. If for every $g \in G$

$$\mathrm{tr}(\sigma(g)) = \mathrm{tr}(\sigma'(g)) \text{ and } \det(\sigma(g)) = \det(\sigma'(g)),$$

then σ and σ' are isomorphic.

Point counting

Theorem

Let $q = p^s$ be a power of prime $p \neq 2, 3, \ell$. The following are true:

(1) *We have that*

$$\mathrm{tr}(Frob_q | W) = -\sum_{t \in X(\Gamma_j)(\mathbb{F}_q)} \mathrm{tr}(Frob_q | (i_* \mathcal{F}_\ell)_t).$$

(2) *If the fiber $E_t := h^{-1}(t)$ is smooth, then*

$$\mathrm{tr}(\mathrm{Frob}_q | (i_* \mathcal{F}_\ell)_t) = \mathrm{tr}(\mathrm{Frob}_q | H^1(E_t, \mathbb{Q}_\ell)) = q + 1 - \# E_t(\mathbb{F}_q).$$

(3) *If the fiber E_t^j is singular, then*

$$\mathrm{tr}(\mathrm{Frob}_q | (i_* \mathcal{F}_\ell)_t) = \begin{cases} 1 & \text{fiber is split multiplicative,} \\ -1 & \text{fiber is nonsplit multiplicative,} \\ 0 & \text{fiber is additive.} \end{cases}$$

Explicit calculation

K3 surface

$$\mathcal{S} : (x+y)(x+z)(y+z) - 8xyz = \frac{1}{s^2}xyz,$$

has Weierstrass model

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We apply the previous theorem to $S = \{2, 3\}$, characters $\chi_1(Frob_p) = \left(\frac{-1}{p}\right)$, $\chi_2(Frob_p) = \left(\frac{2}{p}\right)$, $\chi_3(Frob_p) = \left(\frac{3}{p}\right)$,and $G = \{Frob_p : 31 \leq p \leq 73, \text{ for } p \text{ prime}\}$.

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$$c_{mp^r} - \chi(p)\left(\frac{-1}{p}\right)\gamma(p)c_{mp^{r-1}} + \left(\frac{-6}{p}\right)p^2 c_{mp^{r-2}} \equiv 0 \pmod{p^{2r}},$$

does not hold for some choice of m and r.

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does not hold for some choice of m and r.

Hence the main theorem follows.

Work in progress

Considering three covers of the starting elliptic surface $\mathcal{W}$, we study congruences for $F(\frac{p-1}{3})$ when $p \equiv 1 \pmod 3$.

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Conjecture

For prime $p \equiv 1 \pmod 3$ we have

$$F\left(\frac{p-1}{3}\right) \equiv A(p) \cdot \frac{1+\sqrt{-3}}{2} \pmod p,$$

where $A(p) = -Trace(c_h(p))/2$ or $Trace(c_h(p))$, depending on the splitting behavior of p in $\mathbb{Q}(\sqrt{-3}, \sqrt[3]{3})$.

Thank you for your attention!