Diophantine quadruples in $\mathbb{Z}[i][X]$

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(joint work with Alan Filipin)

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Long history of the problem

- A set consisting of m positive integers such that the product of any two of its distinct elements increased by 1 is a perfect square is called **a Diophantine m-tuple**.
- **Diophantus of Alexandria** first studied this problem.
- How large such set can be?

- **Rational numbers:**
  - Diophantus found the first rational Diophantine quadruple $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$.
  - Euler found first quintuple, Gibbs [**?**] found some sextuples.
  - Dujella et al. [2017] proved that there exist infinitely many rational Diophantine sextuples.
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- **Integers:**
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  - There is also a stronger conjecture:

Conjecture 1

If $\{a, b, c, d\}$ is a Diophantine quadruple of integers and $d > \max\{a,b,c\}$, then $d = d_+ = a + b + c + 2(abc + \sqrt{(ab+1)(ac+1)(bc+1)})$.

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Definition 1

Let $m \geq 2$ and let R be a commutative ring with 1. Let $n \in R$ be a nonzero element and let $\{a_1, \ldots, a_m\}$ be a set of m distinct nonzero elements from R such that $a_i a_j + n$ is a square of an element of R for $1 \leq i < j \leq m$. The set $\{a_1, \ldots, a_m\}$ is called **a Diophantine** m**-tuple with the property** $D(n)$ or simply **a** $D(n)$**-**m**-tuple in** R.

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    - A lot of other variants of such a polynomial problem (Dujella, Fuchs, Tichy, Walsh, Jurasić, Kazalicki, Mikić, Sziksai, ...).

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- $R = \mathbb{R}[X]$
    - Filipin and Jurasic - there does not exist a Diophantine quintuple.
- $R = \mathbb{K}[X]$ ($\mathbb{K}$ - algebraically closed field of characteristic 0):
    - Dujella and Jurasić [**?**] - there does not exist a Diophantine 8-tuple.
    - Other similar results (Dujella and Luca [2007], Filipin and Jurasić [2016], ...).

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- We consider the case $R = \mathbb{Z}[i][X]$ and $n = 1$.
- Does there exist a Diophantine quintuple in $R = \mathbb{Z}[i][X]$?

Let $\{a, b, c\}$ be a $D(1)$-triple in $\mathbb{Z}[i][X]$ such that

$$ab + 1 = r^2, \ ac + 1 = s^2, \ bc + 1 = t^2,$$

where $r, s, t \in \mathbb{Z}[i][X]$.

Definition 2 (Gibbs)

A $D(1)$-**triple** $\{a, b, c\}$ in $\mathbb{Z}[i][X]$ is called **regular** if

$$(c - b - a)^2 = 4(ab + 1). \qquad (1)$$

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- From (1), we get

$$c = c_\pm = a + b \pm 2r,$$
$$ac_\pm + 1 = (a \pm r)^2,\ bc_\pm + 1 = (b \pm r)^2.$$

Definition 3 (Gibbs)

A $D(1)$-**quadruple** $\{a, b, c, d\}$ in R is called **regular** if

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- It is a quadratic equation in d with roots
$$d = d_{\pm} = a + b + c + 2(abc \pm rst).$$
- We denote by d_+ the polynomial with larger degree and by d_- the polynomial with smaller degree among the polynomials $d_{\pm}$.

- It is known that every $D(1)$-pair $\{a, b\}$ in in every ring R can be extended to a regular $D(1)$-quadruple in R:
$$\{a, b, a+b\pm 2r, 4r(a\pm r)(b\pm r)\}.$$

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- Our main result:

Theorem 1

Every $D(1)$-quadruple in $\mathbb{Z}[i][X]$ is regular.

One consequence of the main Theorem

Corolary 1

Every polynomial $D(-1)$-triple in $\mathbb{Z}[X]$ can be uniquely extended to $D(-1;1)$-quadruple in $\mathbb{Z}[X]$.

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Every polynomial $D(-1)$-triple in $\mathbb{Z}[X]$ can be uniquely extended to $D(-1;1)$-quadruple in $\mathbb{Z}[X]$.

- A set $\{a, b, c, d\}$ of four non-zero distinct polynomials from $\mathbb{Z}[X]$ is said to **have a property** $D(-1;1)$, or that it is **polynomial** $D(-1;1)$**-quadruple**, if $\{a, b, c\}$ is a polynomial $D(-1)$-triple and each of $ad + 1$, $bd + 1$ and $cd + 1$ is a square of polynomial from $\mathbb{Z}[X]$.

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- This improves result obtained by Blizanac Trebjesanin, Filipin and Jurasic [2018] under some additional conditions.

Sketch of the proof of Theorem 1

We partially follow the strategies used by Dujella and Fuchs [2004] for $\mathbb{Z}[X]$, by Filipin and Jurasic for $\mathbb{R}[X]$ and by Dujella and Jurasic [2010] for $\mathbb{C}[X]$.

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1. Since we do not have the relation "$<$" between the elements of $\mathbb{Z}[i][X]$, we use the relation "$\leq$" between their degrees.
2. We transform the problem of extending a $D(1)$-triple $\{a, b, c\}$ to a $D(1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[i][X]$ into **solving a system of simultaneous Pellian equations**, which furthermore transforms to **finding intersections of binary recurrent sequences of polynomials**.

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5 We **describe all possible initial terms of the observed recurring sequences**.

- For some initial terms we obtain the same elements of sequences, but with shifted indices and with different degrees of polynomials (overlaps "by form").
- Hence, some cases are reduced to the study of the other ones, which saves some time and which is some improvement from the proofs of previous analogous results.

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6 Finally, we prove the Theorem 1.

Reduction to intersections of recursive sequences

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- Notation:
  - Let α, β, γ denote the degrees of polynomials a, b, c, respectively.
  - Assume that $0 \leq \alpha \leq \beta \leq \gamma$ and $\beta, \gamma > 0$.

Let

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2, \tag{3}$$

where $x, y, z \in \mathbb{Z}[i][X]$.

Eliminating d from (3), we get the **system of Pellian equations**

$$az^2 - cx^2 = a - c, \tag{4}$$

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- We look for solutions (z, x) and (z, y) of (4) and (5), respectively.

Estimates of the initial values of recurrent sequences

Lemma1 (Adapted result of Dujella and Luca [2007])

Let (z, x) and (z, y) be solutions, with $x, y, z \in \mathbb{Z}[i][X]$, of (4) and (5), respectively. Then there exist solutions (z_0, x_0) and (z_1, y_1), with $z_0, x_0, z_1, y_1 \in \mathbb{Z}[i][X]$, of (4) and (5), respectively, such that:

$$\deg(z_0) \le \frac{3\gamma - \alpha}{4}, \qquad \deg(x_0) \le \frac{\alpha + \gamma}{4},$$
$$\deg(z_1) \le \frac{3\gamma - \beta}{4}, \qquad \deg(y_1) \le \frac{\beta + \gamma}{4}.$$

There also exist non-negative integers m *and* n such that

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m,$$
$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n.$$

- Most of the statements of Lemma 1 follow directly from Dujella and Luca [2007] (for $\mathbb{K}[X]$, where $\mathbb{K}$ is algebraically closed field of characteristic 0).

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- By Lemma 1, $z = v_m = w_n$, where the sequences (v_m) and (w_n) are, for $m, n \geq 0$, defined by

$$v_0 = z_0, \quad v_1 = sz_0 + cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m, \tag{6}$$

$$w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n. \tag{7}$$

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- Hence, we reduced the problem of finding extensions of $D(1)$-triple $\{a, b, c\}$ to solving the equation $v_m = w_n$ in $m, n \geq 0$.

- We use the expression

$$c = a + b + d_- + 2(abd_- \mp ruv),$$

where $u = at \pm rs$, $v = bs \pm rt$, $w = cr \pm st$ and $ad_- + 1 = u^2$, $bd_- + 1 = v^2$, $cd_- + 1 = w^2$, obtained by Jones [1977] for $\mathbb{Z}[X]$.

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- For $d_- \neq 0$, by Dujella, Fuchs and Luca [2008], we have $\deg(d_-) \leq \gamma - \alpha - \beta$, i.e. $\gamma \geq \alpha + \beta$. We prove:

Lemma 2

Let $\{a, b, c\}$ be a $D(1)$-triple in $\mathbb{Z}[i][X]$. Then $d_- = 0$ or $\deg(d_-) = \gamma - \alpha - \beta$.

- By the proof of Lemma 2, if $\beta = \gamma$ then $d_{-} = 0$ or $d_{-} = a = \pm i$.

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  - In this case $\{a, b, c\} = \{\pm i, \pm ti \pm i, \mp ti \pm i\}$.
  - Such an example is a $D(1)$-triple

$$\{\pm i, \pm 4iX^2 \mp 4X, \mp 4iX^2 \pm 4X \pm 2i\}.$$

- We assume that $\{a, b, c, d'\}$, with $\deg(d') = \delta$ and $\gamma \leq \delta$, is an irregular $D(1)$-quadruple with minimal δ among all irregular $D(1)$-quadruples in $\mathbb{Z}[i][X]$ and we try to prove that such quadruple does not exist.

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    - By Dujella and Jurasić [2010], $\deg(d') \geq \frac{3\beta+5\gamma}{2}$.

- We prove that that if $m, n \in \{0, 1\}$ then from $v_m = w_n$ we can obtain polynomial $D(1)$-quadruples:
  a) $\{a, b, c, d_-\}$,
  b) $\{0, a, b, c\}$,
  c) $\{\pm i, \pm i, b, c\}$,
  d) $\{a, b, c, d_+\}$ and $\gamma \geq \alpha + 2\beta$.

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  - A $D(1)$-quadruple with a relaxed condition that its elements need not be distinct and need not be non-zero is called (regular or irregular) **improper** $D(1)$**-quadruple**.
- We use this result in our proofs.
  - By the mininmality assumption, whenever we get d such that $\deg(d) < \delta$ then we may conlude that $d = d_-$ or $d = 0 \neq d_-$ or $d = \pm i \neq d_-$.

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  - Similar gap principle is known in classical case and in a polynomial variants of the problem of Diophantus, but we obtained more information about possible polynomial $D(1)$-triples.
  - We consider $d_0, d_1 \in \mathbb{Z}[i][X]$, where

$$ad_0 + 1 = x_0^2, \quad cd_0 + 1 = z_0^2$$

and

$$bd_1 + 1 = y_1^2, \quad cd_1 + 1 = z_1^2.$$

Gap principle for degrees of polynomials a, b and c

We describe all possible relations between α, β and γ.

Lemma 3

Let $\{a, b, c\}$ be a $D(1)$-triple in $\mathbb{Z}[i][X]$. We have:

1. If $d_- = 0$, then $z_0 = z_1 = \pm 1$. In this case $c = a + b \pm 2r$ and $\beta = \gamma$.
2. a) If $d_- = a = \pm i$, then $(z_0, z_1) = (\pm s, \pm s)$, $\alpha = 0$, $\beta = \gamma$ and $c = -b \pm 2i$.
   b) If $d_- \in \mathbb{Z}[i] \setminus \{0, a\}$, then $z_0 = z_1 = \pm cr \pm st$, $\alpha > 0$ and $\gamma = \alpha + \beta$.
3. If $\deg(d_-) > 0$, then we have the following possibilities:
   a) $z_0 = z_1 = \pm cr \pm st$, with $\deg(d_-) \leq \alpha$, $\alpha > 0$ and $\alpha + \beta < \gamma \leq 2\alpha + \beta$,
   b) $(z_0, z_1) = (\pm cr \pm st, \pm s)$, where $\alpha \leq \deg(d_-) \leq \beta$, $\alpha \geq 0$ and $2\alpha + \beta \leq \gamma \leq \alpha + 2\beta$,
   c) $(z_0, z_1) = (\pm t, \pm cr \pm st)$, with $\deg(d_-) = \alpha$, $\alpha = \beta$ and $\gamma = 3\alpha$,
   d) $(z_0, z_1) = (\pm t, \pm s)$, where $\beta \leq \deg(d_-) < \gamma$, $\alpha \geq 0$ and $\gamma \geq \alpha + 2\beta$.

- To prove that, we consider the equation $v_m = w_n$ from which the solution $d = d_-$ arises from.

- To prove that, we consider the equation $v_m = w_n$ from which the solution $d = d_-$ arises from.
- We distinguish the cases by possible form of d_- and we use the fact that if $d = d_-$, then $v_m = w_n = \pm w$ for $m, n \in \{0, 1\}$, which we also proved.

- We described more precisely one special case of a $D(1)$-triple and we adjust the result of Dujella and Fuchs [2004] to the situation in $\mathbb{Z}[i][X]$:

- We described more precisely one special case of a $D(1)$-triple and we adjust the result of Dujella and Fuchs [2004] to the situation in $\mathbb{Z}[i][X]$:

Lemma 4

Let $\{a, b, c\}$ be a $D(1)$-triple in $\mathbb{Z}[i][X]$ with $\beta < \gamma = \alpha + 2\beta$. Then, $\{a, b, d_-, c\}$ has elements

$$\{a, b, a + b \pm 2r, 4r(a \pm r)(b \pm r)\}$$

or

$$\{\pm i, b, -b \pm 2i, \mp 4b^2 i \pm 8b \pm 5i\}.$$

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- We distinguish the cases depending on the parity of indices m and n in the recurring sequences (v_m) and (w_n).
- By Dujella and Fuchs [2004]:

Lemma 5

Let the sequences (v_m) and (w_n) be given by (6) and (7). Then,

$$\begin{aligned} v_{2m} &\equiv z_0 \pmod{2c}, \quad v_{2m+1} \equiv sz_0 + cx_0 \pmod{2c}, \\ w_{2n} &\equiv z_1 \pmod{2c}, \quad w_{2n+1} \equiv tz_1 + cy_1 \pmod{2c}. \end{aligned}$$

Dujella and Jurasić [2010] described all possible relations between the initial terms z_0 and z_1 of the recurring sequences (v_m) and (w_n) in $\mathbb{C}[X]$.

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- This is more precise version:

Lemma 6

1) If $v_{2m} = w_{2n}$, then $z_0 = z_1$.
2) If $v_{2m+1} = w_{2n}$, then either $(z_0, z_1) = (\pm 1, \pm s)$ or $(z_0, z_1) = (\pm s, \pm 1)$ or $z_1 = sz_0 \pm cx_0$, where x_0 is not constant.
3) If $v_{2m} = w_{2n+1}$, then either $(z_0, z_1) = (\pm t, \pm 1)$ or $(z_0, z_1) = (\pm s, \pm 1)$ or $(z_0, z_1) = (\pm 1, \pm 1)$ or $z_0 = tz_1 \pm cy_1$, where y_1 is not constant.
4) If $v_{2m+1} = w_{2n+1}$, then either $(z_0, z_1) = (\pm 1, \pm cr \pm st)$ or $(z_0, z_1) = (\pm cr \pm st, \pm 1)$ or $sz_0 \pm cx_0 = tz_1 \pm cy_1$, where x_0 and y_1 are not constant and polynomials on both sides of the equation have degree less than γ.

- We examine which possibilities from Lemma 6 exist in $\mathbb{Z}[i][X]$ and which initial terms appear from those possibilities.

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  - In the following lemma for each possibility of initial terms we have relations between degrees α, β and γ which admit that possibility.
  - For one particular triple $\{a, b, c\}$ there can be more initial terms, depending on degrees of polynomials a, b and c.

Lemma 7

1) If $v_{2m} = w_{2n}$, then either

- **a)** $z_0 = z_1 = \pm 1$ or
- **b)** $z_0 = z_1 = \pm s$ and $\alpha = 0$ or
- **c)** $z_0 = z_1 = \pm cr \pm st$ and $\alpha > 0$, $\alpha + \beta \leq \gamma \leq 2\alpha + \beta$.

2) If $v_{2m+1} = w_{2n}$, then $(z_0, z_1) = (\pm t, \pm cr \pm st)$ and $\alpha = \beta$, $\gamma = 3\alpha$.

3) If $v_{2m} = w_{2n+1}$, then either

- **a)** $(z_0, z_1) = (\pm s, \pm 1)$ and $\alpha = 0$, $\beta = \gamma$ or
- **b)** $(z_0, z_1) = (\pm cr \pm st, \pm s)$ and $\alpha \geq 0$, $2\alpha + \beta \leq \gamma \leq \alpha + 2\beta$ (special case:
  - **1)** $(z_0, z_1) = (\pm s, \pm s)$ and $\alpha = 0$, $\beta = \gamma$).

4) If $v_{2m+1} = w_{2n+1}$, then $(z_0, z_1) = (\pm t, \pm s)$ and $\gamma \geq \alpha + 2\beta$.

Since in $\mathbb{Z}[i][X]$ we do not have the relation "$<$" between the elements, there are more possibilities to examine for z_0 and z_1 than in $\mathbb{R}[X]$, but less possibilities really hold.

Lemma 8

Let $v_{z_0,m}$ be the m-th term of the sequence $(v_m)_{m\geq 0}$ with initial term z_0 and $w_{z_1,n}$ the n-th term of the sequence $(w_n)_{n\geq 0}$ with initial term z_1. Then

$$
\begin{aligned}
v_{t,m} &= -v_{cr-st,m+1}, & v_{-t,m+1} &= -v_{-cr+st,m},\\
v_{t,m+1} &= v_{cr+st,m}, & v_{-t,m} &= v_{-cr-st,m+1},\\
w_{s,n} &= -w_{cr-st,n+1}, & w_{-s,n+1} &= -w_{-cr+st,n},\\
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- This lemma is a suitable version for $\mathbb{Z}[i][X]$ of Lemma 2.3 by Cipu, Fujita, Miyazaki [2018].

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    - In the proof of the Theorem 1 we considered the case 1.c) for all possible combinations of degrees.

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  - We use important relations, obtained by Dujella and Fuchs [2004] (we consider congruences in $\mathbb{Z}[i][X]$):

Lemma 9

Let the sequences $(v_m)_{m\geq 0}$ and $(w_n)_{n\geq 0}$ be given by (6) and (7). Then,

$$\begin{aligned}
v_{2m} &\equiv z_0 + 2c(az_0m^2 + sx_0m) \pmod{8c^2},\\
v_{2m+1} &\equiv sz_0 + c[2asz_0m(m+1) + x_0(2m+1)] \pmod{4c^2},\\
w_{2n} &\equiv z_1 + 2c(bz_1n^2 + ty_1n) \pmod{8c^2},\\
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\end{aligned}$$

- **Conclusion** - every Diophantine quadruple in $\mathbb{Z}[i][X]$ is regular.

Thank you for your attention!