Simplest quartic and simplest sextic Thue equations over imaginary quadratic fields

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Preliminaries

Let $F(x, y) \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree ≥ 3 and let $0 \neq \mu \in \mathbb{Z}$. Then equation of the form

$$F(x, y) = \mu \text{ in } x, y \in \mathbb{Z},$$

is called a Thue equation.

- In 1909 A. Thue proved that these equations admit only finitely many solutions.
- In 1967 A. Baker gave effective upper bounds for the solutions.
- Later on authors constructed numerical methods to reduce the bounds and to explicitly calculate the solutions.
- In 1990 Thomas investigated for the first time a parametrized family of Thue equations. Since then, several families have been studied.

Let M be an algebraic number field with ring of integers $\mathbb{Z}_M$. As a generalization of Thue equations we consider relative Thue equations of type

$$F(x,y) = \mu \text{ in } x, y \in \mathbb{Z}_M,$$

where $F(x,y) \in \mathbb{Z}_M[x,y]$ is an irreducible binary form of degree ≥ 3 and $0 \neq \mu \in \mathbb{Z}_M$.

- Using Baker's method S. V. Kotov and V. G. Sprindžuk gave the first effective upper bounds for the solutions of relative Thue equations. Their theorem was extended by several authors.
- Applying Baker's method, reduction and enumeration algorithms I. Gaál and M. Pohst gave an efficient algorithm for solving relative Thue equations.
- Authors considered infinite parametric families of Thue equations in the relative case, as well. Up to now all these families were considered mostly over imaginary quadratic fields.

Totally real Thue inequalities over imaginary quadratic fields

- Let $m \geq 1$ be a square-free positive integer and let $M = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field with ring of integers $\mathbb{Z}_M$.
- Let $F(x, y) \in \mathbb{Z}_M[x, y]$ be an irreducible binary form of degree $n \geq 3$ where the roots of $F(x, 1)$ are all real.
- In 2018 Gaál, Remete and J. gave an efficient algorithm to reduce the resolution of relative Thue inequalities of type

$$|F(x, y)| \leq K \;\; \text{in} \;\; x, y \in \mathbb{Z}_M$$

to the resolution of (absolute) Thue inequalities of type

$$|F(x, y)| \leq k \;\; \text{in} \;\; x, y \in \mathbb{Z}.$$

- Let $K \geq 1$. We consider the relative Thue inequality

$$|F(x,y)| \leq K \text{ in } x, y \in \mathbb{Z}_M.$$

- Let $f(x) = F(x,1)$. Assume $f(x)$ that has leading coefficient 1 and distinct real roots $\alpha_1, \ldots, \alpha_n$. Set

$$A = \min_{i \neq j} |\alpha_i - \alpha_j|, \quad B = \min_i \prod_{j \neq i} |\alpha_j - \alpha_i|.$$

- Let $0 < \varepsilon < 1,\ 0 < \eta < 1.$ Set

$$C = \max\left\{\frac{K}{(1-\varepsilon)^{n-1}B}, 1\right\},$$

$$C_1 = \max\left\{\frac{K^{1/n}}{\varepsilon A},\ (2C)^{1/(n-2)}\right\}, \quad C_2 = \max\left\{\frac{K^{1/n}}{\varepsilon A},\ C^{1/(n-2)}\right\},$$

$$D = \left(\frac{K}{\eta(1-\varepsilon)^{n-1}AB}\right)^{1/n}, \ E = \frac{(1+\eta)^{n-1}K}{(1-\varepsilon)^{n-1}}.$$

Remark 1

For given Thue inequality $|F(x,y)| \leq K$, parameters A, B, K and n are constants and

$$\begin{aligned} C_1 &= C_1(\varepsilon), \quad C_2 = C_2(\varepsilon), \\ D &= D(\varepsilon,\eta), \quad E = E(\varepsilon,\eta). \end{aligned}$$

If $m \equiv 1, 2 \pmod 4$, then $x, y \in \mathbb{Z}_M$ can be written as

$$x = x_1 + x_2 i\sqrt{m}, \quad y = y_1 + y_2 i\sqrt{m}, \text{ with } x_1, x_2, y_1, y_2 \in \mathbb{Z}.$$

If $m \equiv 3 \pmod 4$, then $x, y \in \mathbb{Z}_M$ can be written as

$$x = \frac{(2x_1 + x_2) + x_2 i\sqrt{m}}{2}, \qquad y = \frac{(2y_1 + y_2) + y_2 i\sqrt{m}}{2}$$

with $x_1, x_2, y_1, y_2 \in \mathbb{Z}$.

Theorem 1 (Gaál, Remete and J.)

Let $(x, y) \in \mathbb{Z}_M^2$ *be a solution of* $|F(x, y)| \le K$. *Assume that*

$$
\begin{aligned}
|y| &> C_1 \quad \text{if} \quad m \equiv 3 \pmod 4, \\
|y| &> C_2 \quad \text{if} \quad m \equiv 1, 2 \pmod 4.
\end{aligned}
$$

Then

$$x_2 y_1 = x_1 y_2.$$

I. Let $m \equiv 3 \pmod 4$.

IA1. If $2y_1 + y_2 = 0$, then $2x_1 + x_2 = 0$ and $|F(x_2, y_2)| \le \dfrac{2^n K}{(\sqrt{m})^n}$.

IA2. If $|2y_1 + y_2| \ge 2D$, then $|F(2x_1 + x_2, 2y_1 + y_2)| \le 2^n E$.

IB1. If $y_2 = 0$, then $x_2 = 0$ and $|F(x_1, y_1)| \le K$.

IB2. If $|y_2| \ge \dfrac{2}{\sqrt{m}} D$, then $|F(x_2, y_2)| \le \dfrac{2^n}{(\sqrt{m})^n} E$.

II. Let $m \equiv 1, 2 \pmod 4$.

IIA1.	If $y_1 = 0$, then $x_1 = 0$ and $\|F(x_2, y_2)\| \leq \dfrac{K}{(\sqrt{m})^n}$.
IIA2.	If $\|y_1\| \geq D$, then $\|F(x_1, y_1)\| \leq E$.
IIB1.	If $y_2 = 0$, then $x_2 = 0$ and $\|F(x_1, y_1)\| \leq K$.
IIB2.	If $\|y_2\| \geq \dfrac{D}{\sqrt{m}}$, then $\|F(x_2, y_2)\| \leq \dfrac{E}{(\sqrt{m})^n}$.

How to apply Theorem 1

Useful hints for a practical application of Theorem 1 on a totally real relative Thue inequality $|F(x,y)| \leq K$:

Case I. $(m \equiv 3 \pmod 4)$. We distinguish two cases:

$$\begin{array}{lccl} \textbf{1.} & |y| & \leq & C_1 \\ \textbf{2.} & |y| & > & C_1 \quad \text{(Theorem 1)} \end{array}$$

1. Let $|y| \leq C_1$.

- If $|y| \leq C_1$, then we have only finitely many possible values for y and hence for y_1, y_2, as well;
- For each possible y and for all integers $\mu \in \mathbb{Z}_M$ with $|\mu| \leq K$ we calculate the roots of the equation $F(x,y) - \mu = 0$ in x;
- For such a root x we calculate the corresponding x_1, x_2 from $x = \frac{(2x_1+x_2)+x_2 i\sqrt{m}}{2}$. If x_1, x_2 are integers, then $x \in \mathbb{Z}_M$ and (x,y) is a solution.

2. Let $|y| > C_1$.

In this case we can apply **Theorem 1**. We distinguish two cases:

$$\begin{array}{lrcl} \textbf{2.a} & |2y_1 + y_2| & < & 2D \\ \textbf{2.b} & |2y_1 + y_2| & \geq & 2D \end{array}$$

2.a Let $|2y_1 + y_2| < 2D$. Then we have:

i) If $|y_2| < 2D/\sqrt{m}$, then we have only finitely many values for y_1, y_2, and we proceed as in 1.

ii) If $|y_2| \geq 2D/\sqrt{m}$, then we use **IB2.** We solve

$$F(x_2, y_2) = k$$

for all $k \in \mathbb{Z}$ with $|k| \leq 2^n E/(\sqrt{m})^n$. For each possible (x_2, y_2) we determine the possible values of y_1 that satisfy $|2y_1 + y_2| < 2D$. We substitute x_2, y_1, y_2 into

$$x_2 y_1 = x_1 y_2$$

to see if there exist corresponding integer x_1.

Note: If $2y_1 + y_2 = 0$ or $y_2 = 0$, then it is useful to apply **IA1** or **IB1**, respectively.

2.b If $|2y_1 + y_2| \geq 2D$, then we use **IA2.** We calculate the solutions

$$X = 2x_1 + x_2, \; Y = 2y_1 + y_2$$

of

$$F(X, Y) = k$$

for all $k \in \mathbb{Z}$ with $|k| \leq 2^n E$. For each possible (X, Y) we have:

i) If $|y_2| < 2D/\sqrt{m}$, then there are only finitely many possible values for y_2. We determine corresponding y_1 from Y. Then we substitute $y_1, y_2, x_2 = X - 2x_1$ into

$$x_2 y_1 = x_1 y_2$$

and test if a corresponding x_1 is in $\mathbb{Z}$.

ii) If $|y_2| \geq 2D/\sqrt{m}$, then we use **IB2.** We solve

$$F(x_2, y_2) = k$$

for $|k| \leq 2^n E/(\sqrt{m})^n$. We determine x_1, y_1 from x_2, y_2 and X, Y.

Note: If $y_2 = 0$, then we can also use **IB1**.

Remark 2

-
- *For solving absolute Thue equations $F(x, y) = k$ for certain values $k \in \mathbb{Z}$ one can efficiently apply Kash and Magma.*
- *An appropriate choice of the parameters ε and η of Theorem 1 makes the resolution much easier:*
  - *It is worthy to keep $C_1 = C_1(\varepsilon)$ and also $D = D(\varepsilon, \eta)$ small, to avoid extensive tests of small possible solutions. On the other hand, if $E = E(\varepsilon, \eta)$ is small, then there are fewer Thue equations (over $\mathbb{Z}$) to be solved.*
  - *We can not make all these parameters simultaneously small, therefore we need to make a compromise, taking into consideration also the value of K, the degree n of the binary form $F(x, y)$ and the value of m (which also determines the number of Thue equations to be solved).*
  - *Usually it is worthy to try several values of ε and η before applying Theorem 1.*

Simplest quartic and simplest sextic Thue inequalities over imaginary quadratic fields

Let t be an rational integer parameter and let

$$\begin{aligned} f_t^{(3)}(x) &= x^3-(t-1)x^2-(t+2)x-1, \\ f_t^{(4)}(x) &= x^4-tx^3-6x^2+tx+1, \\ f_t^{(6)}(x) &= x^6-2tx^5-(5t+15)x^4-20x^3+5tx^2+(2t+6)x+1. \end{aligned}$$

The infinite parametric families of number fields generated by the roots of these polynomials are called simplest cubic, simplest quartic and simplest sextic fields, respectively.

- It was shown by G. Lettl, A. Pethő and P. Voutier that these are all parametric families of number fields which are totally real cyclic with Galois group generated by a mapping of type $x \mapsto \frac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{Z}$.

Let t be an rational integer parameter and let

$$F_t^{(3)}(x,y) = x^3 - (t-1)x^2y - (t+2)xy^2 - y^3,$$
$$F_t^{(4)}(x,y) = x^4 - tx^3y - 6x^2y^2 + txy^3 + y^4,$$
$$F_t^{(6)}(x,y) = x^6 - 2tx^5y - (5t+15)x^4y^2 - 20x^3y^3 + 5tx^2y^4 + (2t+6)xy^5 + y^6.$$

be corresponding binary forms.
In 2019, by using Theorem 1 and the corresponding results in the absolute case, Gaál, Remete and J. gave all solutions of the infinite parametric family of simplest quartic and simplest sextic relative Thue inequalities

$$|F_t^{(4)}(x,y)| \leq 1 \text{ in } x,y \in \mathbb{Z}_M$$

and

$$|F_t^{(6)}(x,y)| \leq 1 \text{ in } x,y \in \mathbb{Z}_M,$$

where $\mathbb{Z}_M$ is a ring of integers of an imaginary quadratic number field $M = \mathbb{Q}(\sqrt{-m})$.

The infinite parametric family of simplest cubic relative Thue inequalities

$$|F_t^{(3)}(x,y)| \leq 1$$

was already solved. More generally, the family of relative Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu,$$

where the parameters t, the root of unity μ and the solutions x and y are integers in the same imaginary quadratic number field $M = \mathbb{Q}(\sqrt{-m})$ was completely solved:

- In 2002, C.Heuberger, A. Pethő and R.F.Tichy gave the solutions for large values of $|t|$;
- In 2006, C.Heuberger gave the solutions for all parameters t;
- Heuberger's result was extended by P. Kirschenhofer, C.M. Lampl and J. Thuswaldner in 2007 involving also a wider class of rights hand sides.

Simplest quartic Thue inequalities over imaginary quadratic fields

Theorem 2 (Gaál, Remete and J.)

Let $t \in \mathbb{Z}$ with $t \neq -3, 0, 3$. All solutions of

$$|F_t^{(4)}(x,y)| \leq 1 \ \text{ in } \ x, y \in \mathbb{Z}_M,$$

up to sign, are given by:

for any m and any t: $(x,y) = (0,0), (0,1), (1,0)$;

for any m and $t = 1$: $(x,y) = (1,2), (2,-1)$;

for any m and $t = -1$: $(x,y) = (2,1), (-1,2)$;

for any m and $t = 4$: $(x,y) = (2,3), (3,-2)$;

for any m and $t = -4$: $(x,y) = (3,2), (-2,3)$;

for $m = 1$ *and any t:* $(x,y) = (0,i), (i,0)$;

for $m = 3$ *and any t:* $(x,y) = (\omega,0), (0,\omega), (1-\omega,0), (0,1-\omega)$;

for $m = 1$ *and* $t = 1$*:* $(x, y) = (i, 2i), (2i, -i)$;

for $m = 1$ *and* $t = -1$*:* $(x, y) = (2i, i), (-i, 2i)$;

for $m = 1$ *and* $t = 4$*:* $(x, y) = (2i, 3i), (3i, -2i)$;

for $m = 1$ *and* $t = -4$*:* $(x, y) = (3i, 2i), (-2i, 3i)$*;*

for $m = 3$ *and* $t = 1$*:*
$(x, y) = (2\omega - 2, -\omega + 1), (\omega - 1, 2\omega - 2), (-2\omega, \omega), (\omega, 2\omega)$*;*

for $m = 3$ *and* $t = -1$*:*
$(x, y) = (-\omega + 1, 2\omega - 2), (2\omega - 2, \omega - 1), (\omega, -2\omega), (2\omega, \omega)$;

for $m = 3$ *and* $t = 4$*:*
$(x, y) = (3\omega - 3, -2\omega + 2), (2\omega - 2, 3\omega - 3), (2\omega, 3\omega), (3\omega, -2\omega)$*;*

for $m = 3$ *and* $t = -4$*:*
$(x, y) = (-2\omega + 2, 3\omega - 3), (3\omega - 3, 2\omega - 2), (3\omega, 2\omega), (-2\omega, 3\omega)$,

where $\omega = (1 + i\sqrt{3})/2$*.*

Sketch of proof:

We use Theorem 1 and the corresponding results in the absolute case:

Lemma 1 (Chen and Voutier)

Let $t \in \mathbb{Z}$ with $t \geq 1, t \neq 3$. All solutions of

$$F_t^{(4)}(x, y) = \pm 1 \quad \text{in } x, y \in \mathbb{Z}$$

are given by

$$(x, y) = (\pm 1, 0), (0, \pm 1).$$

Further, for $t = 1$ we have

$$(x, y) = (1, 2), (-1, -2), (2, -1), (-2, 1),$$

and for $t = 4$ we have

$$(x, y) = (2, 3), (-2, -3), (3, -2), (-3, 2).$$

Lemma 2 (Lettl, Pethő and Voutier)

Let $t \in \mathbb{Z}$, $t \geq 58$ and consider the primitive solutions of

$$\left| F_t^{(4)}(x,y) \right| \leq 6t+7 \quad \text{in } x,y \in \mathbb{Z}. \tag{1}$$

If (x,y) is a solution of (1), then every pair in the orbit

$$\{(x,y),(y,-x),(-x,-y),(-y,x)\}$$

is also a solution. Every such orbit has a solution with $y > 0$, $-y \leq x \leq y$. If such an orbit contains a primitive solution, then all solutions in this orbit are primitive. All solutions of the above inequality with $y > 0, -y \leq x \leq y$ are

$$(x,y) = (0,1), (\pm 1, 1), (\pm 1, 2).$$

- We exclude the parameters $t = -3, 0, 3$ for which the binary form $F_t^{(4)}(x, y)$ is reducible over $\mathbb{Z}$.
- Since
$$F_t^{(4)}(x, y) = F_{-t}^{(4)}(y, x),$$
it is enough to solve Thue inequality only for $t > 0$.
Also, for given t, we have
$$F_t^{(4)}(x, y) = F_t^{(4)}(-x, -y) = F_t^{(4)}(y, -x) = F_t^{(4)}(-y, x).$$
Therefore, if $(x, y) \in \mathbb{Z}_M^2$ is a solution, then $(y, -x)$, $(-y, x)$, $(-x, -y)$ are solutions, too.

- In order to apply Theorem 1 first we need to determine

$$A = \min_{i \neq j} |\alpha_i - \alpha_j| \quad \text{and} \quad B = \min_i \prod_{j \neq i} |\alpha_j - \alpha_i|.$$

Using the estimates for the roots $\alpha_1, \ldots, \alpha_4$ of the polynomial $F_t^{(4)}(x, 1)$ given by G. Lettl, A. Pethő and P. Voutier, we obtain

$$A > 0.9833 \quad \text{and} \quad B > 58.1 \quad \text{for} \quad t \geq 58.$$

Calculating the roots for $0 < t < 58$ we obtain

$$A > 0.8320 \quad \text{and} \quad B > 4.6114 \quad \text{for any } t > 0, \; t \neq 3.$$

- By Theorem 1, the resolution of our relative Thue inequality reduces to the resolution of (absolute) Thue inequalities of type

$$\left|F_t^{(4)}(x,y)\right| < k,$$

where

$$k = \frac{2^4}{(\sqrt{m})^4},\ \frac{2^4 E}{(\sqrt{m})^4},\ \ldots\ \left(\text{or}\ \frac{1}{(\sqrt{m})^4},\ \frac{E}{(\sqrt{m})^4}, \ldots\right) \tag{2}$$

- We want to use Lemma 1 and Lemma 2 for as many pairs (t, m) as possible in order to have fewer (absolute) Thue equations to solve in Magma.
- Since the values of k in (2) are decreasing functions in m, the cases:
  - $m \neq 1, 3$
  - $m = 1$
  - $m = 3$

  are considered separately with an appropriate choice of the parameters ε and η.

Case $m = 3$

We consider the cases $t \geq 58$ and $0 < t < 58$ separately since for $t \geq 58$ we can use Lemma 2.

- First we assume $t \geq 58$. Then

$$A > 0.9833 \quad \text{and} \quad B > 58.1.$$

We set

$$\varepsilon = 0.6273, \qquad \eta = 0.0361,$$

and obtain

$$C_1 < 1.622.$$

Now, Theorem 1 implies:

Corollary 1

Let $(x, y) \in \mathbb{Z}_M^2$ *be a solution of* $\left|F_t^{(4)}(x, y)\right| \leq 1$ *and* $m = 3$*. Assume that* $t \geq 58$ *and*

$$|y| > 1.622.$$

Then

$$x_2 y_1 = x_1 y_2.$$

Further

IA1. If $2y_1 + y_2 = 0,$ then $2x_1 + x_2 = 0$ and $|F_t^{(4)}(x_2, y_2)| \leq 1.778.$

IA2. If $|2y_1 + y_2| \geq 3.499,$ then $|F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2)| \leq 343.754.$

IB1. If $y_2 = 0,$ then $x_2 = 0$ and $|F_t^{(4)}(x_1, y_1)| \leq 1.$

IB2. If $|y_2| \geq 0.989,$ then $|F_t^{(4)}(x_2, y_2)| \leq 38.95.$

1. Assume $|y| > 1.622$. Then, by the **Corollary 1**, we have:

a) If $2y_1 + y_2 = 0$, then by **IA1**, $2x_1 + x_2 = 0$ and

$$|F_t^{(4)}(x_2, y_2)| \leq 1.778.$$

By **Lemma 1** this inequality implies

$$(x_2, y_2) = (0,0), (0, \pm 1), (\pm 1, 0).$$

Since $2y_1 + y_2 = 0$ and $2x_1 + x_2 = 0$ we obtain that only possibility is $(x, y) = (0, 0)$ which contradicts $|y| > 1.622$.

If $|2y_1 + y_2| \geq 3.499$, then **IA2** implies

$$\left|F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2)\right| \leq 343.754.$$

Using **Lemma 2** we can easily list all primitive and non-primitive solutions of this inequality and we always have $|2y_1 + y_2| \leq 4$.

Therefore only $|2y_1 + y_2| = 1, 2, 3, 4$ is possible.

b) Using **IB1** and **IB2** we similarly obtain that only $|y_2| = 1, 2$ is possible.

c) The equations $|2y_1 + y_2| = 1, 2, 3, 4$ and $|y_2| = 1, 2$ leave only a few possible values for (y_1, y_2).

2. If $|x| > 1.622$, then we similarly obtain $|2x_1 + x_2| = 1, 2, 3, 4$ and $|x_2| = 1, 2$ since if (x, y) is a solution, then also is $(y, -x)$.
3. Therefore we have:
   - **i)** If $|x| > 1.622$ and $|y| > 1.622$, then we test the finite set
   $$|2x_1 + x_2| = 1, 2, 3, 4, \quad |x_2| = 1, 2, \quad |2y_1 + y_2| = 1, 2, 3, 4, \quad |y_2| = 1, 2.$$
   - **ii)** If $|x| > 1.622$ and $|y| \leq 1.622$, then we test the finite set
   $$|2x_1 + x_2| = 1, 2, 3, 4, \quad |x_2| = 1, 2, \quad |y| \leq 1.622.$$
   - **iii)** If $|x| \leq 1.622$ and $|y| > 1.622$, then we test the finite set
   $$|2y_1 + y_2| = 1, 2, 3, 4, \quad |y_2| = 1, 2, \quad |x| \leq 1.622.$$
   - **iv)** If $|x| \leq 1.622$ and $|y| \leq 1.622$, then we have to test only finitely many possibilities for (x, y) .

All solutions, up to sign, for $m = 3$ and $t \geq 58$are:

$$(x, y) = (0, 0), (1, 0), (0, 1), (\omega, 0), (0, \omega), (1 - \omega, 0), (0, 1 - \omega),$$

where $\omega = \frac{1+i\sqrt{3}}{2}$.

- Now we assume $m = 3$ and $0 < t < 58$. Then $A > 0.8320$ and $B > 4.6114$. If we set

$$\varepsilon = 0.0348, \qquad \eta = 0.0005,$$

then Theorem 1 implies:

Corollary 2

Let $(x, y) \in \mathbb{Z}_M^2$ be a solution of $\left|F_t^{(4)}(x, y)\right| \leq 1$ and $m = 3$. Assume that $0 < t < 58$ and

$$|y| > 34.539.$$

Then

$$x_2 y_1 = x_1 y_2.$$

Further

IA1. If $2y_1 + y_2 = 0$, then $2x_1 + x_2 = 0$ and $|F_t^{(4)}(x_2, y_2)| \leq 1.778$.

IA2. If $|2y_1 + y_2| \geq 9.814$, , then $|F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2)| \leq 17.821$.

IB1. If $y_2 = 0$, then $x_2 = 0$ and $|F_t^{(4)}(x_1, y_1)| \leq 1$.

IB2. If $|y_2| \geq 5.666$, then $|F_t^{(4)}(x_2, y_2)| \leq 1.981$.

1. Assume $|y| > 34.539$. Then by the **Corollary 2** we have:

   a) If $y_2 = 0$ or $|y_2| \geq 5.666$, then by **Lemma 1** and **IB1** or **IB2,** respectively, we obtain a contradiction.

   Therefore only $|y_2| = 1, 2, 3, 4, 5$ is possible.

   b) If $2y_1 + y_2 = 0$ then by **Lemma 1** and **IA1,** we obtain a contradiction with $|y| > 34.539$.If $|2y_1 + y_2| \geq 9.814$, then **IA2** implies

$$|F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2)| \leq 17.821.$$

   Using **Magma** we solve this inequality for all $0 < t < 58$. All these solutions contradict $|2y_1 + y_2| \geq 9.814$.

   Therefore only $|2y_1 + y_2| = 1, 2, ..., 9$ is possible.

   c) In the set $|y_2| = 1, 2, 3, 4, 5$, $|2y_1 + y_2| = 1, 2, ..., 9$, all corresponding y have absolute values less than 34.539 which is in contradiction with $|y| > 34.539$.

   Therefore only $|y| \leq 34.539$ is possible.

2. Similarly as in previous case we obtain $|x| \leq 34.539$.

3. Enumerating all x, y with properties $|x| \leq 34.539$ and $|y| \leq 34.539$, we obtain, up to sign, the following solutions:

for $m = 3$ and any $0 < t < 58$:
$(x, y) = (0, 0),\ (1, 0),\ (0, 1),\ (\omega, 0),\ (0, \omega),\ (1 - \omega, 0),\ (0, 1 - \omega),$

for $m = 3$ and $t = 1$:
$(x, y) = (1, 2),\ (2, -1),\ (2\omega - 2, -\omega + 1),\ (\omega - 1, 2\omega - 2),$
$(-2\omega, \omega),\ (\omega, 2\omega)$

for $m = 3$ and $t = 4$:
$(x, y) = (2, 3),\ (3, -2),\ (3\omega - 3, -2\omega + 2),\ (2\omega - 2, 3\omega - 3),$
$(2\omega, 3\omega),\ (3\omega, -2\omega).$

Therefore we have proved Theorem 2 for all $t \in \mathbb{Z}$ with $t \neq -3, 0, 3$ and $m = 3$.

Case $m = 1$

In this case we set

$$\varepsilon = 0.1792, \qquad \eta = 0.0308,$$

for all $t > 0$, $t \neq 3$. By **Theorem 1** and **Lemma 1**, we obtain, up to sign, the following solutions:

for $m = 1$ and any t: $(x, y) = (0, 0), (0, 1), (1, 0), (0, i), (i, 0)$,

for $m = 1$ and $t = 1$: $(x, y) = (1, 2), (i, 2i), (2, -1), (2i, -i)$,

for $m = 1$ and $t = 4$: $(x, y) = (2, 3), (2i, 3i), (3, -2), (3i, -2i)$.

Therefore we have proved Theorem 2 for all $t \in \mathbb{Z}$ with $t \neq -3, 0, 3$ and for $m = 1$.

Case $m \neq 1, 3$

In this case we set

$$\varepsilon = 0.1924, \qquad \eta = 0.169,$$

for all $t > 0$, $t \neq 3$ and all $m \neq 1, 3$. By **Theorem 1** and **Lemma 1**, we obtain up to sign the following solutions:

for any m and any t: $(x, y) = (0, 0), (0, 1), (1, 0)$,

for any m and $t = 1$: $(x, y) = (1, 2), (2, -1)$,

for any m and $t = 4$: $(x, y) = (2, 3), (3, -2)$.

Therefore we have proved Theorem 2 for all $t \in \mathbb{Z}$ with $t \neq -3, 0, 3$ and for all square-free positive integers m.

Simplest sextic Thue inequalities over imaginary quadratic fields

Theorem 3 (Gaál, Remete and J.)

Let $t \in \mathbb{Z}$ *with* $t \neq -8, -3, 0, 5$. *All solutions of*

$$|F_t^{(6)}(x, y)| \leq 1 \ \text{ in } \ x, y \in \mathbb{Z}_M,$$

up to sign, are given by:

for any m *and any* t: $(x, y) = (0, 0), (0, 1), (1, 0), (1, -1)$;

for $m = 1$ *and any* t: $(x, y) = (0, i), (i, 0), (i, -i)$;

for $m = 3$ *and any* t:

$(x, y) = (\omega, 0), (0, \omega), (\omega, -\omega), (1 - \omega, 0), (0, 1 - \omega), (\omega - 1, -\omega + 1)$.

where $\omega = (1 + i\sqrt{3})/2$.

We have proved Theorem 3 analogously as Theorem 2. We have used Theorem 1 and the corresponding results in the absolute case:

Lemma 3 (Hoshi)

Let $t \in \mathbb{Z}$ with $t \neq -8, -3, 0, 5$. All solutions of

$$F_t^{(6)}(x, y) = \pm 1 \quad \text{in } x, y \in \mathbb{Z}$$

are given by

$$(x, y) = (\pm 1, 0),\ (0, \pm 1),\ (1, -1),\ (-1, 1).$$

Lemma 4 (Lettl, Pethő and Voutier)

Let $t \in \mathbb{Z}$, $t \geq 89$ and consider the primitive solutions of

$$\left| F_t^{(6)}(x, y) \right| \leq 120t + 323 \quad \text{in } x, y \in \mathbb{Z}. \tag{2}$$

If (x, y) is a solution of (2), then every pair in the orbit

$$\{(x, y), (-y, x + y), (-x - y, x), (-x, -y), (y, -x - y), (x + y, -x)\}$$

is also a solution. Every such orbit has a solution with $y > 0$, $-y/2 \leq x \leq y$. If such an orbit contains a primitive solution, then all solutions in this orbit are primitive. All solutions of the above inequality with $y > 0$, $-y/2 \leq x \leq y$ are

$$(x, y) = (0, 1),\ (1, 1),\ (1, 2),\ (-1, 3).$$