The regularity of extensions of Diophantine triples or pairs

Yasutsugu Fujita

College of Industrial Technology, Nihon University

Representation Theory XVI

Held at Inter-University Centre Dubrovnik
June 23–29, 2019

Overview of Diophantine tuples (1)

Diophantus (3C) searched "$\{a_1, a_2, a_3, a_4\}$ s.t. $a_i a_j + 1 = \square$" and found $\{1/16, 33/16, 17/4, 105/16\}$

- **Fermat** (17C) $\{1, 3, 8, 120\}$
- **Euler** (18C) $\{a, b, a+b+2r, 4r(r+a)(r+b)\}$ $(r = \sqrt{ab+1})$
  and $\{1, 3, 8, 120, 777480/8288641\}$

★ We restrict ourselves to tuples of "rational integers"

Definition 1

$a_1, \ldots, a_m \in \mathbb{Z}_{>0}$

$\{a_1, \cdots, a_m\}$: Diophantine m-tuple $\overset{\text{def}}{\iff}$ $a_i a_j + 1 = \square \quad (1 \leq \forall i < \forall j \leq m)$

★ $\{a_1, \ldots, a_m\} : D$ denotes "$\{a_1, \ldots, a_m\}$: Diophantine m-tuple"

Overview of Diophantine tuples (4)

- **Baker-Davenport** ('69)

  $\{1, 3, 8, d\} : D \implies d = 120 \ (= d_+)$ Baker's method on 3 logs

- **Arkin-Hoggatt-Strauss** ('79), **Gibbs** ('78)

  $\{a, b, c\} : D \implies \{a, b, c, d_+\} : D$ "regular" Diophantine quadruple

  where $d_+ = a + b + c + 2abc + 2rst$

  $(r = \sqrt{ab+1},\ s = \sqrt{ac+1},\ t = \sqrt{bc+1})$

$d = d_+$ is a solution to the equation

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1)$$

The other solution is $d_- = a + b + c + 2abc - 2rst$

(Note that $0 \leq d_- < c$ and $d_- > 0 \iff c > a + b + 2r$)

Conjecture 1

$\{a, b, c, d\} : D \ (a < b < c < d) \implies d = d_+$

(i.e. all Diophantine quadruples are regular)

Overview of Diophantine tuples (5)

Theorem (Dujella '04)

(i) There exists no Diophantine sextuple

(ii) There exist at most finitely many Diophantine quintuples

Proof (i) Apply **Bennett's theorem**

based on **Rickert's theorem** on simultaneous rational approximations of quadratic irrationals

(ii) Apply Baker's method on 3 logs

In fact, it was shown that $d < 10^{2171}$ and $e < 10^{10^{26}}$ □

Theorem (He-Togbé-Ziegler '19)

There does not exist a Diophantine quintuple

Proof Apply Baker's method on 3 logs

⇠ "A kit on linear forms in three logarithms" by Mignotte □

Main theorems

Main Theorem 1 (F-Miyazaki '18)

$\{a, b, c\} : D \ \ (a < b < c)$: fixed

$\Longrightarrow \ \#\{\, d \,|\, \{a, b, c, d\,\} : D,\ c < d\,\} \leq 11$

Main Theorem 2 (Cipu-F-Miyzaki '18)

$\{a, b, c\} : D \ \ (a < b < c)$: fixed

$\Longrightarrow \ \#\{\, d \,|\, \{a, b, c, d\} : D,\ c < d\,\} \leq 8$

$\Longrightarrow \ \#\{\, d \,|\, \{a, b, c, d\} : D,\ d_+ < d\,\} \leq 7$

Main Theorem 3 (Cipu-Filipin-F preprint)

$$a\left(a + \frac{7}{2} - \frac{1}{2}\sqrt{4a + 13}\right) \leq b \leq 4a^2 + a + 2\sqrt{a}$$

$\{a, b, c, d\} : D \ \ (b < c < d)$

$\Longrightarrow \ d = d_+$

Determination of fundamental solutions (1)

For a fixed $\{a, b, c\} : D \ (a < b < c)$, assume $\{a, b, c, d\} : D \ (c < d)$ and let

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2 \qquad (r,\ s,\ t \in \mathbb{Z}_{>0})$$
$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2 \qquad (x,\ y,\ z \in \mathbb{Z}_{>0})$$

where x, y, z are considered to be unknowns. Eliminating d yields

$$az^2 - cx^2 = a - c, \qquad bz^2 - cy^2 = b - c$$

By Nagell's argument, any solutions (z, x) and (z, y) to the above are expressed as

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m$$
$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n$$

for some $|z_0|$, $|z_1|$, x_0, $y_1 \in \mathbb{Z}_{>0}$ and m, $n \in \mathbb{Z}_{\geq 0}$, where

$$1 < |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}, \quad 1 < |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$$

Determination of fundamental solutions (2)

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m$$
$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n$$

These equalities enable us to write $z = v_m = w_n$, where

$$v_0 = z_0, \quad v_1 = sz_0 + cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m$$
$$w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n$$

The fundamental solutions (z_0, x_0) and (z_1, y_1) has been more or less deteremined:

Theorem (Dujella '04)

(1) $m \equiv n \equiv 0 \pmod 2$ with $z_0 = z_1$ and either $|z_0| \in \{1, cr - st\}$
or $|z_0| < \min\{0.869a^{-5/14}c^{9/14}, 0.972b^{-0.3}c^{0.7}\}$
(2) $m \equiv 1,\ n \equiv 0 \pmod 2$ with $|z_0| = t$, $|z_1| = cr - st$, $z_0z_1 < 0$
(3) $m \equiv 0,\ n \equiv 1 \pmod 2$ with $|z_0| = cr - st$, $|z_1| = s$, $z_0z_1 < 0$
(4) $m \equiv n \equiv 0 \pmod 2$ with $|z_0| = t$, $|z_1| = s$, $z_0z_1 > 0$

Determination of fundamental solutions (3)

Theorem (Dujella '04)

(1) $m \equiv n \equiv 0 \pmod 2 \Longrightarrow z_0 = z_1$ and either $|z_0| \in \{1, cr - st\}$

or $|z_0| < \min\{0.869a^{-5/14}c^{9/14},\ 0.972b^{-0.3}c^{0.7}\}$

(2) $m \equiv 1,\ n \equiv 0 \pmod 2 \Longrightarrow |z_0| = t,\ |z_1| = cr - st,\ z_0 z_1 < 0$

(3) $m \equiv 0,\ n \equiv 1 \pmod 2 \Longrightarrow |z_0| = cr - st,\ |z_1| = s,\ z_0 z_1 < 0$

(4) $m \equiv n \equiv 0 \pmod 2 \Longrightarrow |z_0| = t,\ |z_1| = s,\ z_0 z_1 > 0$

The following completely determines the fundamental solutions:

Lemma 3.1 (F-Miyazaki '18)

(1) $m \equiv n \equiv 0 \pmod 2 \Longrightarrow z_0 = z_1$ and $|z_0| \in \{1, cr - st\}$

(2) $m \equiv 1,\ n \equiv 0 \pmod 2 \Longrightarrow |z_0| = t,\ |z_1| = cr - st,\ z_0 z_1 < 0$

(3) $m \equiv 0,\ n \equiv 1 \pmod 2 \Longrightarrow |z_0| = cr - st,\ |z_1| = s,\ z_0 z_1 < 0$

(4) $m \equiv n \equiv 0 \pmod 2 \Longrightarrow |z_0| = t,\ |z_1| = s,\ z_0 z_1 > 0$

Determination of fundamental solutions (4)

Lemma 3.1 (F-Miyazaki '18)

(1) $m \equiv n \equiv 0 \pmod 2 \Longrightarrow z_0 = z_1$ and $|z_0| \in \{1, cr - st\}$
(2) $m \equiv 1,\ n \equiv 0 \pmod 2 \Longrightarrow |z_0| = t,\ |z_1| = cr - st,\ z_0 z_1 < 0$
(3) $m \equiv 0,\ n \equiv 1 \pmod 2 \Longrightarrow |z_0| = cr - st,\ |z_1| = s,\ z_0 z_1 < 0$
(4) $m \equiv n \equiv 0 \pmod 2 \Longrightarrow |z_0| = t,\ |z_1| = s,\ z_0 z_1 > 0$

Proof Assume $m \equiv n \equiv 0 \pmod 2$ with $|z_0| \notin \{1, cr - st\}$ and put

$$d_0 := (z_0^2 - 1)/c \quad (\rightsquigarrow\ |z_0| = 1 \Rightarrow d_0 = 0,\ \ |z_0| = cr - st \Rightarrow d_0 = d_-\ )$$

Then, $1 \le d_0 < c$ and $\{a, b, d_0, c\}$ is an **irregular** Diophantine quadruple

Examining "$v_m = w_n$" attached to $\{a, b, d_0, c\}$ closely, we see that c is large enough compared to $a,\ b,\ d_0$

Thus, the following theorem based on Rickert's theorem is applicable:

Determination of fundamental solutions (5)

Lemma 3.1 (F-Miyazaki '18)

(1) $m \equiv n \equiv 0 \pmod 2 \Longrightarrow z_0 = z_1$ and $|z_0| \in \{1, cr - st\}$

Proof

Assume $m \equiv n \equiv 0 \pmod 2$ with $|z_0| \notin \{1, cr - st\}$

Then, $\{a, b, d_0, c\} : D \ \ (d_0 := (z_0^2 - 1)/c < c)$

Theorem (Cipu-F '15, cf. Rickert '93)

$N > 3.706 a' b^2 (b-a)^2 \ \ (a' = \max\{b-a, a\})$ and $ab \mid N$

$\theta_1 = \sqrt{1 + a/N}, \ \theta_2 = \sqrt{1 + b/N}$

$\Longrightarrow \max\{|\theta_1 - p_1/q|, |\theta_2 - p_2/q|\} > (1.413 \cdot 10^{28} a' bN/a)^{-1} q^{-\lambda}$

for any $p_1, p_2, q \, (> 0)$, where $\lambda = 1 + \frac{\log(10a^{-1}a'bN)}{\log(2.699a^{-1}b^{-1}(b-a)^{-2}N^2)} < 2$

Combining this theorem with the lower bounds for solutions obtained by "the congruent method" due to Dujella leads us to a contradiction □

Bounding the number of d's corresponding to (z_0, z_1)

Put $N(z_0, z_1) := \#\{d \mid \{a, b, c, d\} : D,\ d > d_+,\ d \text{ corresponds to } (z_0, z_1)\}$

Theorem 3.2

(i) $N(z_0, z_1) \le 2$

(ii) $(z_0, z_1) \in \{(st - cr, st - cr),\ (st - cr, s),\ (t, s)\} \Longrightarrow N(z_0, z_1) \le 1$

Proof (i) **Assume $z = v_m = w_n$ has 3 solutions (m_i, n_i)** $(i \in \{0, 1, 2\}$, $2 < m_0 < m_1 < m_2)$ belonging to the same class of solutions, and put

$$\Lambda_i := m_i \log \xi - n_i \log \eta + \log \mu$$

where $\xi = s + \sqrt{ac}$, $\eta = t + \sqrt{bc}$, $\mu = \dfrac{\sqrt{b}\,(x_0\sqrt{c} + z_0\sqrt{a})}{\sqrt{a}\,(y_1\sqrt{c} + z_1\sqrt{b})}$

★ $m_0 > 2$ implies $d > d_+$ (and then we have $m_0 \ge 4$)

If $mn > 0$, then $0 < \Lambda_i < \kappa\,\xi^{-2m_i}$, where $\kappa = \begin{cases} 6\sqrt{ac} & \text{if } m_i \ge 4 \\ 6 & \text{if } |z_0| = 1 \\ 2.001c/b & \text{if } z_0 = st - cr \\ 1/(2ab) & \text{if } z_0 = t \end{cases}$

Okazaki's gap principle

Lemma 3.3 (cf. Bennett-Cipu-Mignotte-Okazaki '06)

$\boldsymbol{m_2 - m_1 > \Lambda_0^{-1} \Delta \log \eta}$, where $\Delta = \begin{vmatrix} n_1 - n_0 & n_2 - n_1 \\ m_1 - m_0 & m_2 - m_1 \end{vmatrix} > 0$

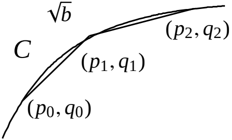
In particular, if $m_0 n_0 > 0$, then $m_2 - m_1 > \kappa^{-1}(4ac)^{m_0} \Delta \log \eta$

Proof of Lemma 3.3 Put $p_i := m_i \log \xi$, $q_i := n_i \log \eta$

Then $\Lambda_i = m_i \log \xi - n_i \log \eta + \log \mu = p_i - q_i + \log \mu$

The equality $v_{m_i} = w_{n_i}$ implies that (p_i, q_i)'s are on the curve

$$C : \frac{(x_0\sqrt{c} + z_0\sqrt{a})\,e^{p} - (x_0\sqrt{c} - z_0\sqrt{a})\,e^{-p}}{\sqrt{a}} = \frac{(y_1\sqrt{c} + z_1\sqrt{b})\,e^{q} - (y_1\sqrt{c} - z_1\sqrt{b})\,e^{-q}}{\sqrt{b}}$$



Since $\dfrac{dq}{dp} > 1$ and $\dfrac{d^2q}{dp^2} < 0$, we obtain

$$0 < \frac{q_1 - q_0}{p_1 - p_0} - \frac{q_2 - q_1}{p_2 - p_1} < \frac{q_1 - q_0 - p_1 + p_0}{p_1 - p_0} = \frac{\Lambda_0 - \Lambda_1}{p_1 - p_0} < \frac{\Lambda_0}{p_1 - p_0} \qquad \square$$

Rickert's theorem applied by Bennett (1)

Lemma 3.3 (cf. Bennett-Cipu-Mignotte-Okazaki '06)

$$m_2 - m_1 > \kappa^{-1}(4ac)^{m_0}\Delta \log \eta$$

Now the assertion $N(z_0, z_1) \leq 2$ can be shown by using Lemma 3.3 and

Baker's method on linear forms in 3 logarithms

This completes the proof of (i)

(ii) $(z_0, z_1) \in \{(st - cr, st - cr),\ (st - cr, s),\ (t, s)\} \Longrightarrow N(z_0, z_1) \leq 1$

Assume $z = v_m = w_n$ has 2 solutions (m_i, n_i) $(i \in \{1, 2\},\ 2 < m_1 < m_2)$

Note that $m_0 = 1,\ 2$ is a solution to $z = v_m = w_n$ for $z_0 = t,\ st - cr$, resp.

★ Since m_0 is small, we need other ingredients

Rickert's theorem applied by Bennett (2)

$$(z_0, z_1) \in \{(st - cr, st - cr),\ (st - cr, s),\ (t, s)\} \Longrightarrow N(z_0, z_1) \le 1$$

Consider the linear form $\Gamma := \Lambda_2 - \Lambda_1 = (m_2 - m_1) \log \xi - (n_2 - n_1) \log \eta$

Then $0 < |\Gamma| < \kappa \xi^{-2m_1}$

In order that Baker's method can work, we need $m_2 - m_1 \sim m_1$
i.e. m_2 (n_2) is not so larger than m_1 (n_1)

Theorem 3.4 (cf. Bennett '98 (based on Rickert '93))

$a_1 := a(c-b)$, $a_2 := b(c-a)$, $N := abz^2$

$(u := c-b,\ v := c-a,\ w := b-a,\ a_1' := \max\{a_1, a_2 - a_1\})$

$\theta_1 := \sqrt{1 + a_1/N}$, $\theta_2 := \sqrt{1 + a_2/N}$

$$N \ge 10^5 a_2 \implies \max\{|\theta_1 - p_1/q|, |\theta_2 - p_2/q|\} > (32.01 a_1' a_2 u N / a_1)^{-1} q^{-\lambda}$$

for $\forall p_1, p_2, q (> 0)$, where $a_1' = \max\{a_1, a_2 - a_1\}$, $\lambda = 1 + \dfrac{\log\left(\frac{16 a_1' a_2 u N}{a_1}\right)}{\log\left(\frac{1.6874 N^2}{a_1 a_2 (a_2 - a_1) u v w}\right)} < 2$

Rickert's theorem applied by Bennett (3)

$(z_0, z_1) \in \{(st-cr, st-cr),\ (st-cr, s),\ (t, s)\} \Longrightarrow N(z_0, z_1) \le 1$

Theorem 3.4 (cf. Bennett '98 (based on Rickert '93))

$a_1 := a(c-b)$, $a_2 := b(c-a)$, $N := abz^2$ $(u := c-b,\ a' := \max\{a_1, a_2 - a_1\})$

$\theta_1 := \sqrt{1 + a_1/N}$, $\theta_2 := \sqrt{1 + a_2/N}$

$N \ge 10^5 a_2 \Longrightarrow \max\{|\theta_1 - p_1/q|, |\theta_2 - p_2/q|\} > (32.01 a_1' a_2 u N/a_1)^{-1} q^{-\lambda}$

Combining Theorem 3.4 ($q = abz_{(1)}z_{(2)}$) with the trivial estimate

$$\max\left\{\left|\theta_1 - \frac{acy_{(1)}y_{(2)}}{abz_{(1)}z_{(2)}}\right|, \left|\theta_2 - \frac{bcx_{(1)}x_{(2)}}{abz_{(1)}z_{(2)}}\right|\right\} < \frac{c^{3/2}}{2a^{3/2}} z_{(2)}^{-2} \qquad (1)$$

(where $(x_{(i)}, y_{(i)}, z_{(i)})$ $(i \in \{1,2\})$ are positive solutions to

$az^2 - cx^2 = a - c,\ bz^2 - cy^2 = b - c$)

we obtain:

Proof of Main Theorem 1 (1)

$$(z_0, z_1) \in \{(st - cr, st - cr),\ (st - cr, s),\ (t, s)\} \Longrightarrow N(z_0, z_1) \le 1$$

Lemma 3.5

- $n_1 \ge 8 \Longrightarrow n_2 < 148 n_1$
- $n_1 = 7$ with $z_1 = s \Longrightarrow n_2 \le 462\,(\le 66 n_1)$

Proof of Theorem 3.2 (ii) Note $(m_0, z_0) \in \{(2, st - cr), (1, t)\}$ for $(z_0, z_1) \in \{(st - cr, st - cr),\ (st - cr, s),\ (t, s)\}$

Lemma 3.3 $m_2 - m_1 > \kappa^{-1}(4ac)^{m_0} \Delta \log \eta$, $\kappa = \begin{cases} 2.001c/b & \text{if } z_0 = st - cr \\ 1/(2ab) & \text{if } z_0 = t \end{cases}$

shows that

$$\frac{m_2 - m_1}{\log \eta} > 30a^2bc > 30b^2 \tag{2}$$

Since we can show $m,\ n \ge 7$ for the above z_0, we may apply Lemma 3.5

Proof of Main Theorem 1 (2)

$(z_0, z_1) \in \{(st - cr, st - cr),\ (st - cr, s),\ (t, s)\} \implies N(z_0, z_1) \leq 1$

Lemma 3.5

- $n_1 \geq 8 \implies n_2 < 148n_1$
- $n_1 = 7$ with $z_1 = s \implies n_2 \leq 462\,(\leq 66n_1)$

$$\frac{m_2 - m_1}{\log \eta} > 30a^2bc > 30b^2 \qquad (2)$$

Noting $n_i - 1 \leq m_i \leq 2n_i + 1$, Lemma 3.5 implies $m_2 - m_1 < 338m_1$, which together with Baker's method on **2 logs** (Laurent's theorem '08) implies

$$\frac{m_2 - m_1}{\log \eta} < 1.8 \cdot 10^7$$

which contradicts (2) with $b > 4000$ (we know "$d > d_+ \implies b > 4000$")
This completes the proof of (ii) □

Proof of Main Theorem 1 (3)

Proof of Main Theorem 1 $N := \#\{d > d_+ \mid \{a,b,c,d\} : D\} \leq 10$

Recall $1 \leq |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}$, $1 \leq |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$
which imply

(i) $|z_0| = cr - st \implies c < 4\tau^{-4}ab^2$

(ii) $|z_1| = cr - st \implies c < 4\tau^{-4}a^2b$

(iii) $|z_0| = t \implies c > 4ab^2$

(iv) $|z_1| = s \implies c > 4a^2b$

where $\tau = \frac{\sqrt{ab}}{r}(1 - \frac{a+b+1/c}{c})$ (< 1)

We consider several cases separately, and show $N \leq 10$ in each case

For example, if $4ab^2 < c < 4\tau^{-4}ab^2$, then $|z_1| \neq cr - st$ and

$$N \leq N(1,1) + N(-1,-1) + N(st-cr, s) + N(cr-st, -s) + N(t,s) + N(-t,-s)$$

$$\leq 2 + 2 + 1 + 2 + 1 + 2 = 10$$

□

Auxiliary lemma

Recall the system of Pellian equations $az^2 - cx^2 = a - c$, $bz^2 - cy^2 = b - c$ any solutions of which can be expressed as $z = v_m = w_n$, where

$$v_0 = z_0,\ v_1 = sz_0 + cx_0,\ v_{m+2} = 2sv_{m+1} - v_m$$
$$w_0 = z_1,\ w_1 = tz_1 + cy_1,\ w_{n+2} = 2tw_{n+1} - w_n$$

Denote by $\{v_{z_0,m}\}$, $\{w_{z_1,n}\}$ the recurrence sequences $\{v_m\}$, $\{w_n\}$ with the initial terms z_0, z_1, resp.
The following is the key lemma

Lemma 4.1

$v_{cr-st,m} = v_{-t,m+1}$, $v_{st-cr,m+1} = v_{t,m}$ for all $m \geq 0$
$w_{cr-st,n} = w_{-s,n+1}$, $w_{st-cr,n+1} = w_{s,n}$ for all $n \geq 0$

Proof $z_0 = cr - st \implies v_0 = cr - st,\ v_1 = 2c(rs - at) - t$
$z_0 = -t \implies v_1 = cr - st,\ v_2 = 2c(rs - at) - t$
$\rightsquigarrow v_{cr-st,m} = v_{-t,m+1}$ □

Proof of Main Theorem 2 (1)

Lemma 4.1

$v_{cr-st,m} = v_{-t,m+1}$, $v_{st-cr,m+1} = v_{t,m}$ for all $m \geq 0$
$w_{cr-st,n} = w_{-s,n+1}$, $w_{st-cr,n+1} = w_{s,n}$ for all $n \geq 0$

Proof of Main Theorem 2 In the preceding section, we bounded the number N dividing several cases by considering whether c is greater than $4ab^2$ and so on, where we were able to eliminate some cases because of the inequalities $1 \leq |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}$, $1 \leq |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$

Now we **count N using** not these inequalities but **Lemma 4.1**
Denote by $N'(z_0, z_1)$ the number of solutions to $v_m = w_n$ with $m > 2$ ($\Leftrightarrow d > d_+$), where we are not assuming the inequalities above. Then

$$N \leq N'(-1,-1) + N'(1,1) + N'(z_0^-, z_1^-) + N'(z_0^+, z_1^+)$$

where $(z_0^-, z_1^-) \in \{(cr-st, cr-st), (-t, cr-st), (cr-st, -s), (-t, -s)\}$ and $(z_0^+, z_1^+) \in \{(st-cr, st-cr), (t, st-cr), (st-cr, s), (t, s)\}$

★ (z_0^-, z_1^-), (z_0^+, z_1^+) attain d_-, d_+ for some $m, n \leq 2$, resp.

Proof of Main Theorem 2 (2)

$$N \le N'(-1,-1) + N'(1,1) + N'(z_0^-, z_1^-) + N'(z_0^+, z_1^+)$$

where $(z_0^-, z_1^-) \in \{(cr - st, cr - st), (-t, cr - st), (cr - st, -s), (-t, -s)\}$ and $(z_0^+, z_1^+) \in \{(st - cr, st - cr), (t, st - cr), (st - cr, s), (t, s)\}$

In the same way as in the preceding section, one can show

$$N'(-1,-1),\ N'(1,1),\ N'(z_0^-, z_1^-) \le 2$$

using Lemma 3.3 with Baker's method on 3 logs, and

$$N'(z_0^+, z_1^+) \le 1$$

using Lemma 3.3 with Rickert's theorem and Baker's method on **2 logs**
Therefore

$$N \le 2 + 2 + 2 + 1 = 7$$

□

Overview of the relevant results (1)

- **Baker-Davenport** ('69)

  $\{1, 3, 8, d\} : D \implies d = 120\,(= d_+)$

  Baker's method on 3 logs

- **Dujella** ('97)

  $\{k-1, k+1, 4k, d\} : D \implies d = 16k^3 - 4k\,(= d_+)$

  Rickert's theorem

- **Dujella-Pethő** ('98)

  $\{1, 3, c, d\} : D \quad (c < d) \implies d = d_+$

  Baker's method on 3 logs

- **Dujella** ('99)

  $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\} : D \implies d = 4F_{2k+1}F_{2k+2}F_{2k+3}\,(= d_+)$

  Baker's method on 3 logs

Overview of the relevant results (2)

- **F** ('08)

  $\{k-1, k+1, c, d\} : D \quad (16k^3 - 4k < c < d) \implies d = d_+$

  Rickert's theorem twice

  $(\rightsquigarrow \nexists \{k-1, k+1, c, d, e\} : D)$

- **Bugeaud-Dujella-Mignotte** ('07)

  $\{k-1, k+1, c, d\} : D \quad (c = 16k^3 - 4k < d)$

  $\implies d = 64k^5 - 48k^3 + 8k \; (= d_+)$

  Baker's method on 3 logs with refined congruence method using "$s = \sqrt{ac+1} \sim t = \sqrt{bc+1}$"

$\rightsquigarrow \quad \{k-1, k+1, c, d\} : D \quad (c < d) \implies d = d_+$

Overview of the relevant results (3)

- **Bugeaud-Dujella-Mignotte** ('07) $\{k-1, k+1, 16k^3-4k\}$
  Baker's method on 3 logs with refined congruence method
  using "$s=\sqrt{ac+1} \sim t=\sqrt{bc+1}$"
- **He-Togbé** ('09, '12)

  $\{k, A^2k+2A, (A+1)^2k+2(A+1), d\} : D$ with

  or $2 \le A \le 10$ Rickert
  $A \ge 52330$ Baker on **2 logs** "$r=\sqrt{ab+1} \sim s=\sqrt{ac+1}$"

  $\Longrightarrow d = d_+$
- **Cipu-F-Mignotte** ('18)

  $\{k, A^2k \pm 2A, (A+1)^2k \pm 2(A+1), d\} : D \Longrightarrow d = d_+$

  Proof $k \ge 240.24(A+1)+740$ optimization of Rickert's theorem
  $A \ge 2811$ Baker on **2 logs**
  $\Longrightarrow d = d_+$

  The remaining cases can be checked by the reduction method □

The possibilities for the third element c induced by $\{a, b\}$

For a fixed $\{a, b\} : D$ $(a < b)$, assume $\{a, b, c\} : D$ $(b < c)$ and let

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2 \qquad (r,\ s,\ t \in \mathbb{Z}_{>0})$$

Eliminating c yields $at^2 - bs^2 = a - b$

By Nagell's argumens, any solution (t, s) to the above is expressed as

$$t\sqrt{a} + s\sqrt{b} = (t_0\sqrt{a} + s_0\sqrt{b})(r + \sqrt{ab})^{\nu}$$

for some $|t_0|,\ s_0,\ \nu \in \mathbb{Z}_{>0}$, where $|t_0| < \sqrt{\frac{b\sqrt{b}}{2\sqrt{a}}},\ s_0 < \sqrt{(r+1)/2}$

Each (t_0, s_0) (often) gives a sequence of possible third elements c's

E.g., $(t_0, s_0) = (\pm 1, 1)$ correspond to $c = c_{\nu}^{\tau}$ $(\tau \in \{\pm\})$, where

$$c_{\nu}^{\tau} = \frac{1}{4ab}\left\{(\sqrt{b} + \tau\sqrt{a})^2(r + \sqrt{ab})^{2\nu} + (\sqrt{b} - \tau\sqrt{a})^2(r - \sqrt{ab})^{2\nu} - 2(a+b)\right\}$$

$\rightsquigarrow$ We always have $\{a, b, c_{\nu}^{\tau}\} : D$

Overview of the relevant results (4)

- **Filipin-Togbé-F** ('14)

  $\{k, 4k \pm 4, c, d\} : D \ (c_2^+ \neq c < d) \implies d = c_{\nu+1}^\tau \,(= d_+)$

  $c = c_1^\tau$ known by **Dujella** ('97) and **He-Togbé** ('09)
  $c = c_2^-$ Baker's method on **2 logs**
  $c \geq c_3^-$ Rickert's theorem

- **He-Pu-Shen-Togbé** ('18)

  $\{k, 4k \pm 4, c, d\} : D \ (c = c_2^+ < d) \implies d = c_3^+ \,(= d_+)$

  new kind of application of Baker's method on **2 logs**
  "$m \mod r,\ n \mod r$: small constant" (e.g. $m \equiv n \equiv \pm 1 \pmod r$)

- **Cipu-Filipin-F** ('19 (?))

  $\{A^2k,\ 4A^4k \pm 4A,\ c,\ d\} : D \ (c = c_\nu^\tau < d) \implies d = c_{\nu+1}^\tau \,(= d_+)$

  $c = c_1^\tau$ known by **Cipu-F-Mignotte** ('18)
  $c = c_2^\tau$ Baker's method on **2 logs**
  [He-Togbé] for m, n : even, **[He-Pu-Shen-Togbé]** for m, n : odd
  $c \geq c_3^-$ Rickert's theorem

Corollary and key lemma

Main Theorem 3 (Cipu-Filipin-F preprint)

$$a\left(a+\frac{7}{2}-\frac{1}{2}\sqrt{4a+13}\right)\le b\le 4a^2+a+2\sqrt{a} \qquad (3)$$

$\{a,b,c,d\} : D \ \ (b<c<d)$

$\Longrightarrow\ d = d_+$

Corollary 5.1

$\{A^2k,\ 4A^4k\pm 4A,\ c,\ d\} : D \ \ (c<d)$ with $k\in\{1,2,3,4\}$

$\Longrightarrow\ d = c^{\tau}_{\nu+1}\ (= d_+)$ ←-- unnecessry to assume $c = c^{\tau}_{\nu}$

Lemma 5.2 (Key lemma)

$\{a,b,c\} : D$ with (3)

$\Longrightarrow\ \{a,b\} = \{T^2+2T, 4T^4+8T^3-4T\}$ for some $T\in\mathbb{Z}_{\ge 1}$

or $c = c^{\tau}_{\nu}$ for some $\nu\in\mathbb{Z}_{\ge 1}$ and $\tau\in\{\pm\}$

Key lemma (1)

Lemma 5.2 (Key lemma)

$$a\left(a+\tfrac{7}{2}-\tfrac{1}{2}\sqrt{4a+13}\right) \le b \le 4a^2+a+2\sqrt{a} \qquad (3)$$

$\Longrightarrow$ $\{a,b\}=\{T^2+2T, 4T^4+8T^3-4T\}$ for some $T\in\mathbb{Z}_{\ge 1}$
or $c=c_\nu^\tau$ for some $\nu\in\mathbb{Z}_{\ge 1}$ and $\tau\in\{\pm\}$

Proof Suppose $c\neq c_\nu^\tau$ and let $\boldsymbol{c=\gamma_\nu}$ $\leftarrow$-- $t\sqrt{a}+s\sqrt{b}=(t_0\sqrt{a}+s_0\sqrt{b})(r+\sqrt{ab})^\nu$

We may assume $\exists\,\nu_0$ s.t. $\gamma_{\nu_0}<b<\gamma_{\nu_0+1}$, and put $c':=\gamma_{\nu_0}$

Then we see that $\{a,b',c',b\}:D$ is regular for some $b'\in\mathbb{Z}_{\ge 1}$

Thus $\boldsymbol{b=a+b'+c'+2ab'c'+2r's'T}$

with $r'=\sqrt{ab'+1}$, $s'=\sqrt{ac'+1}$, $T=\sqrt{b'c'+1}$

Noting $4ab'c'+a+b'+c'<b<4ab'c'+4\max\{a,b',c'\}$

we can negate inequalities (3), except one exceptional case

Key lemma (2)

Lemma 5.2 (Key lemma)

$$a\left(a+\frac{7}{2}-\frac{1}{2}\sqrt{4a+13}\right)\le b\le 4a^2+a+2\sqrt{a} \qquad (3)$$

$\Longrightarrow$ $\{a,b\}=\{T^2+2T, 4T^4+8T^3-4T\}$ for some $T\in\mathbb{Z}_{\ge 1}$

or $c=c_\nu^\tau$ for some $\nu\in\mathbb{Z}_{\ge 1}$ and $\tau\in\{\pm\}$

Proof $4ab'c'+a+b'+c' < b < 4ab'c'+4\max\{a,b',c'\}$

- $a\le b'c' \Rightarrow b > 4ab'c'+a+b'+c' > 4a^2+a+2\sqrt{a}$, contradicting (3)
- $a > b'c' \Rightarrow$
  - ◇ $a\ne b'+c'+2T \Rightarrow a > 4b'c'+b'+c'$
    $\Rightarrow b < a(a+7/2+\sqrt{4a+13}/2)$, contradicting (3)
  - ◇ $a = b'+c'+2T \Rightarrow b'c' < b'+c'+2T$

    Setting $B:=\min\{b',c'\}$ and $C:=\max\{b',c'\}$, we obtain $B=1$

    Then $T=\sqrt{b'c'+1}=\sqrt{C+1}$, $a=C+1+2\sqrt{C+1}$ and

    $a=T^2+2T,\ \ b=4T^4+8T^3-4T$ □

The case $\{a,b\} \neq \{T^2+2T, 4T^4+8T^3-4T\}$

$$c_1^\tau = a+b+2\tau r \qquad (\, b < c_1^\tau \implies \tau = + \,)$$

$$c_2^\tau = 4(a+b)(ab+1) + 4\tau r(2ab+1) \sim 4ab^2$$

$$c_3^\tau = 16a^2b^2(a+b+2\tau r) + 8ab(3a+3b+4\tau r) + 3(3a+3b+2\tau r)$$

$$\sim 16a^2b^3$$

Proof of Main Theorem 3

The case $\{a,b\} \neq \{T^2+2T, 4T^4+8T^3-4T\}$

- $c = c_1^+$ Baker on **2 logs** "$r = \sqrt{ab+1} \sim s = \sqrt{ac+1}$" **[He-Togbé]**
- $c = c_2^\tau$ Baker on **2 logs**
  - m, n : even $\implies$ "$\alpha = s + \sqrt{ac} \sim \beta^2 = (r + \sqrt{ab})^2$" **[He-Togbé]**
  - m, n : odd $\implies$ "$m \equiv n \equiv \pm 1 \pmod{r}$" **[He-Pu-Shen-Togbé]**
- $c \geq c_3^-$ Rickert

The case $\{a,b\} = \{T^2+2T, 4T^4+8T^3-4T\}$ (1)

The case $\{a,b\} = \{T^2+2T, 4T^4+8T^3-4T\}$

We may assume $c \neq c_\nu^\tau$

From $ac+1=s^2$, $bc+1=t^2$, we have $at^2 - bs^2 = a-b$

$\rightsquigarrow t\sqrt{a} + s\sqrt{b} = (t_0\sqrt{a} + s_0\sqrt{b})(s+\sqrt{ab})^\nu$

$1 \le s_0 < T\sqrt{T+2}$ (by Nagell's argument)

$\rightsquigarrow (t_0, s_0) = (\pm(2T^2+2T-1), T+1)$

$\rightsquigarrow c = \gamma_\nu^\tau$, where $(\gamma_0 = \gamma_0^\tau = 1,\ \gamma_1^- = T^2 - 1 < b)$

$\gamma_1^+ = 16T^6 + 64T^5 + 72T^4 - 31T^2 - 4T + 3$

$\gamma_2^- = 16T^8 + 64T^7 + 48T^6 - 80T^5 - 88T^4 + 32T^3 + 36T^2 - 4T - 3$

$\gamma_2^+ > 256T^{12} + 2048T^{11}$

The case $\{a,b\}=\{T^2+2T, 4T^4+8T^3-4T\}$ (2)

The case $\{a,b\}=\{T^2+2T, 4T^4+8T^3-4T\}$

- $c=\gamma_1^+=16T^6+64T^5+72T^4-31T^2-4T+3$
  We get $m\equiv n\equiv 0 \pmod 2$ and $z_0=z_1\in\{\pm 1\}$
  - $m>3.9999b^{-1/2}c^{1/2}T^{1/2}$ (assuming $T>10^{10}$)
    $\longleftarrow$ coming from $a\equiv b\equiv 0 \pmod T$
    (cf. known in general: $m>b^{-1/2}c^{1/2}$)
  - Baker on 3 logs **[Matveev '98]** $\Lambda = m\log\alpha - n\log\beta' - \log\chi'$
    $\longleftarrow$ "Kummer condition" should be checked:
    $[K(\sqrt{\alpha},\sqrt{\beta'},\sqrt{\chi'}):K]=2^3=8$
    $\left(K:=\mathbb{Q}(\sqrt{ac},\sqrt{bc}),\ \alpha=s+\sqrt{ac},\ \beta'=t+\sqrt{bc},\ \chi'=\dfrac{\sqrt{b}(\sqrt{c}\pm\sqrt{a})}{\sqrt{a}(\sqrt{c}\pm\sqrt{b})}\right)$
    $\rightsquigarrow T<5.146\cdot 10^{10}$ (cf. $T<1.8\cdot 10^{11}$ by using **[Aleksentsev '08]**)
- $c=\gamma_2^-$ Baker on **2 logs** "$\beta^2=(r+\sqrt{ab})^2\sim\beta'=t+\sqrt{bc}$" **[He-Togbé]**
- $c\geq\gamma_2^+$ Rickert $\square$

Thank you so much for your kind attention !!