The regularity of extensions of Diophantine triples or pairs

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Overview of Diophantine tuples (1)

Diophantus (3C) searched \( \{a_1, a_2, a_3, a_4\} \) s.t. \( a_i a_j + 1 = \square \) and found \( \{1/16, 33/16, 17/4, 105/16\} \)

- **Fermat** (17C) \( \{1, 3, 8, 120\} \)
- **Euler** (18C) \( \{a, b, a + b + 2r, 4r(r + a)(r + b)\} \) \( (r = \sqrt{ab} + 1) \)
  and \( \{1, 3, 8, 120, 777480/8288641\} \)

★ We restrict ourselves to tuples of “rational integers”

**Definition 1**

\[
a_1, \ldots, a_m \in \mathbb{Z}_{>0}
\]

\[
\{a_1, \ldots, a_m\} : \text{Diophantine } m\text{-tuple}
\]

\[
\text{def} \quad a_i a_j + 1 = \square \quad (1 \leq \forall i < \forall j \leq m)
\]

★ \( \{a_1, \ldots, a_m\} : D \) denotes “\( \{a_1, \ldots, a_m\} : \text{Diophantine } m\text{-tuple} \)”
Overview of Diophantine tuples (4)

- **Baker-Davenport** (’69)
  \[ \{1, 3, 8, d\} : D \implies d = 120 \quad (= d_+ \) \\
  Baker’s method on 3 logs

- **Arkin-Hoggatt-Strauss** (’79), **Gibbs** (’78)
  \[ \{a, b, c\} : D \implies \{a, b, c, d_+\} : D \quad “regular” \ Diophantine quadruple \]
  where \[ d_+ = a + b + c + 2abc + 2rst \]
  \[ ( r = \sqrt{ab + 1}, \ s = \sqrt{ac + 1}, \ t = \sqrt{bc + 1} ) \]
  \[ d = d_+ \] is a solution to the equation
  \[ (a + b - c - d)^2 = 4(ab + 1)(cd + 1) \]
  The other solution is \[ d_- = a + b + c + 2abc - 2rst \]
  (Note that \( 0 \leq d_- < c \) and \( d_- > 0 \iff c > a + b + 2r \))

**Conjecture 1**

\[ \{a, b, c, d\} : D \quad (a < b < c < d) \implies d = d_+ \]
(i.e. all Diophantine quadruples are regular)
Theorem (Dujella ’04)

(i) There exists no Diophantine sextuple
(ii) There exist at most finitely many Diophantine quintuples

Proof (i) Apply Bennett’s theorem based on Rickert’s theorem on simultaneous rational approximations of quadratic irrationals
(ii) Apply Baker’s method on 3 logs
In fact, it was shown that $d < 10^{2171}$ and $e < 10^{10^{26}}$

Theorem (He-Togbé-Ziegler ’19)
There does not exist a Diophantine quintuple

Proof Apply Baker’s method on 3 logs
\[\text{“A kit on linear forms in three logarithms” by Mignotte}\]
Main Theorem 1 (F-Miyazaki ’18)

\{a, b, c\} : D \ (a < b < c) : \text{fixed}

\[\Rightarrow \#\{d \mid \{a, b, c, d\} : D, \ c < d\} \leq 11\]

Main Theorem 2 (Cipu-F-Miyazaki ’18)

\{a, b, c\} : D \ (a < b < c) : \text{fixed}

\[\Rightarrow \#\{d \mid \{a, b, c, d\} : D, \ c < d\} \leq 8\]

\[\Rightarrow \#\{d \mid \{a, b, c, d\} : D, \ d_+ < d\} \leq 7\]

Main Theorem 3 (Cipu-Filipin-F preprint)

\[a \left( a + \frac{7}{2} - \frac{1}{2} \sqrt{4a + 13} \right) \leq b \leq 4a^2 + a + 2\sqrt{a}\]

\{a, b, c, d\} : D \ (b < c < d)

\[\Rightarrow d = d_+\]
**Determination of fundamental solutions (1)**

For a fixed \(\{a, b, c\} : D\; (a < b < c)\), assume \(\{a, b, c, d\} : D\; (c < d)\) and let

\[
ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2 \quad (r, s, t \in \mathbb{Z}_{>0})
\]
\[
ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2 \quad (x, y, z \in \mathbb{Z}_{>0})
\]

where \(x, y, z\) are considered to be unknowns. Eliminating \(d\) yields

\[
az^2 - cx^2 = a - c, \quad bz^2 - cy^2 = b - c
\]

By Nagell’s argument, any solutions \((z, x)\) and \((z, y)\) to the above are expressed as

\[
z \sqrt{a} + x \sqrt{c} = (z_0 \sqrt{a} + x_0 \sqrt{c})(s + \sqrt{ac})^m
\]
\[
z \sqrt{b} + y \sqrt{c} = (z_1 \sqrt{b} + y_1 \sqrt{c})(t + \sqrt{bc})^n
\]

for some \(|z_0|, |z_1|, x_0, y_1 \in \mathbb{Z}_{>0}\) and \(m, n \in \mathbb{Z}_{\geq0}\), where

\[
1 < |z_0| < \sqrt{\frac{c \sqrt{c}}{2 \sqrt{a}}}, \quad 1 < |z_1| < \sqrt{\frac{c \sqrt{c}}{2 \sqrt{b}}}
\]
Determination of fundamental solutions (2)

\[
\begin{align*}
z \sqrt{a} + x \sqrt{c} &= (z_0 \sqrt{a} + x_0 \sqrt{c})(s + \sqrt{ac})^m \\
z \sqrt{b} + y \sqrt{c} &= (z_1 \sqrt{b} + y_1 \sqrt{c})(t + \sqrt{bc})^n
\end{align*}
\]

These equalities enable us to write \( z = v_m = w_n \), where

\[
\begin{align*}
v_0 &= z_0, & v_1 &= sz_0 + cx_0, & v_{m+2} &= 2sv_{m+1} - v_m \\
w_0 &= z_1, & w_1 &= tz_1 + cy_1, & w_{n+2} &= 2tw_{n+1} - w_n
\end{align*}
\]

The fundamental solutions \((z_0, x_0)\) and \((z_1, y_1)\) has been more or less determined:

**Theorem (Dujella ’04)**

1. \( m \equiv n \equiv 0 \pmod{2} \) with \( z_0 = z_1 \) and either \(|z_0| \in \{1, cr - st\}\) or \(|z_0| < \min \{0.869a^{-5/14}c^{9/14}, 0.972b^{-0.3}c^{0.7}\}\)
2. \( m \equiv 1, n \equiv 0 \pmod{2} \) with \(|z_0| = t, |z_1| = cr - st, z_0z_1 < 0\)
3. \( m \equiv 0, n \equiv 1 \pmod{2} \) with \(|z_0| = cr - st, |z_1| = s, z_0z_1 < 0\)
4. \( m \equiv n \equiv 0 \pmod{2} \) with \(|z_0| = t, |z_1| = s, z_0z_1 > 0\)
Theorem (Dujella ’04)

\[
\begin{align*}
(1) & \quad m \equiv n \equiv 0 \pmod{2} \implies z_0 = z_1 \text{ and either } |z_0| \in \{1, cr - st\} \quad \text{or} \quad |z_0| < \min\{0.869a^{-5/14}c^{9/14}, 0.972b^{-0.3}c^{0.7}\} \\
(2) & \quad m \equiv 1, \ n \equiv 0 \pmod{2} \implies |z_0| = t, |z_1| = cr - st, z_0z_1 < 0 \\
(3) & \quad m \equiv 0, \ n \equiv 1 \pmod{2} \implies |z_0| = cr - st, |z_1| = s, z_0z_1 < 0 \\
(4) & \quad m \equiv n \equiv 0 \pmod{2} \implies |z_0| = t, |z_1| = s, z_0z_1 > 0
\end{align*}
\]

The following completely determines the fundamental solutions:

Lemma 3.1 (F-Miyazaki ’18)

\[
\begin{align*}
(1) & \quad m \equiv n \equiv 0 \pmod{2} \implies z_0 = z_1 \text{ and } |z_0| \in \{1, cr - st\} \\
(2) & \quad m \equiv 1, \ n \equiv 0 \pmod{2} \implies |z_0| = t, |z_1| = cr - st, z_0z_1 < 0 \\
(3) & \quad m \equiv 0, \ n \equiv 1 \pmod{2} \implies |z_0| = cr - st, |z_1| = s, z_0z_1 < 0 \\
(4) & \quad m \equiv n \equiv 0 \pmod{2} \implies |z_0| = t, |z_1| = s, z_0z_1 > 0
\end{align*}
\]
Lemma 3.1 (F-Miyazaki ’18)

(1) $m \equiv n \equiv 0 \pmod{2} \implies z_0 = z_1$ and $|z_0| \in \{1, cr - st\}$
(2) $m \equiv 1, n \equiv 0 \pmod{2} \implies |z_0| = t, |z_1| = cr - st, z_0z_1 < 0$
(3) $m \equiv 0, n \equiv 1 \pmod{2} \implies |z_0| = cr - st, |z_1| = s, z_0z_1 < 0$
(4) $m \equiv n \equiv 0 \pmod{2} \implies |z_0| = t, |z_1| = s, z_0z_1 > 0$

Proof

Assume $m \equiv n \equiv 0 \pmod{2}$ with $|z_0| \notin \{1, cr - st\}$ and put

$$d_0 := \frac{(z_0^2 - 1)}{c} \quad \text{(} \implies |z_0| = 1 \implies d_0 = 0 \text{,} \quad |z_0| = cr - st \implies d_0 = d_\_ \text{)}$$

Then, $1 \leq d_0 < c$ and $\{a, b, d_0, c\}$ is an irregular Diophantine quadruple.

Examining “$v_m = w_n$” attached to $\{a, b, d_0, c\}$ closely, we see that $c$ is large enough compared to $a, b, d_0$.

Thus, the following theorem based on Rickert’s theorem is applicable:
**Determination of fundamental solutions (5)**

**Lemma 3.1 (F-Miyazaki ’18)**

(1) \( m \equiv n \equiv 0 \pmod{2} \implies z_0 = z_1 \text{ and } |z_0| \in \{1, cr - st\} \)

**Proof**

Assume \( m \equiv n \equiv 0 \pmod{2} \) with \(|z_0| \notin \{1, cr - st\}\).

Then, \(\{a, b, d_0, c\} : D \ (d_0 := (z_0^2 - 1)/c < c)\)

**Theorem (Cipu-F ’15, cf. Rickert ’93)**

\[
N > 3.706a'b^2(b - a)^2 \quad (a' = \max\{b - a, a\}) \quad \text{and} \quad ab \mid N
\]

\[
\theta_1 = \sqrt{1 + a/N}, \quad \theta_2 = \sqrt{1 + b/N}
\]

\[
\implies \max \{ |\theta_1 - p_1/q|, |\theta_2 - p_2/q| \} > (1.413 \cdot 10^{28} a'bN/a)^{-1} q^{-\lambda}
\]

for any \( p_1, p_2, q (> 0) \), where \( \lambda = 1 + \frac{\log(10a^{-1}a'bN)}{\log(2.699a^{-1}b^{-1}(b-a)^{-2}N^2)} < 2 \)

Combining this theorem with the lower bounds for solutions obtained by “the congruent method” due to Dujella leads us to a contradiction \(\square\)
Bounding the number of $d$’s corresponding to $(z_0, z_1)$

Put $N(z_0, z_1) := \#\{d | \{a, b, c, d\} : D, d > d_+, d \text{ corresponds to } (z_0, z_1)\}$

**Theorem 3.2**

(i) $N(z_0, z_1) \leq 2$

(ii) $(z_0, z_1) \in \{(st - cr, st - cr), (st - cr, s), (t, s)\} \implies N(z_0, z_1) \leq 1$

**Proof** (i) Assume $z = v_m = w_n$ has 3 solutions $(m_i, n_i)$ ($i \in \{0, 1, 2\}$, $2 < m_0 < m_1 < m_2$) belonging to the same class of solutions, and put

$$\Lambda_i := m_i \log \xi - n_i \log \eta + \log \mu$$

where $\xi = s + \sqrt{ac}$, $\eta = t + \sqrt{bc}$, $\mu = \frac{\sqrt{b} (x_0 \sqrt{c} + z_0 \sqrt{a})}{\sqrt{a} (y_1 \sqrt{c} + z_1 \sqrt{b})}$

$\star$ $m_0 > 2$ implies $d > d_+$ (and then we have $m_0 \geq 4$)

If $mn > 0$, then $0 < \Lambda_i < \kappa \xi^{-2m_i}$, where $\kappa = \begin{cases} 6 \sqrt{ac} & \text{if } m_i \geq 4 \\ 6 & \text{if } |z_0| = 1 \\ 2.001c/b & \text{if } z_0 = st - cr \\ 1/(2ab) & \text{if } z_0 = t \end{cases}$
Okazaki’s gap principle

Lemma 3.3 (cf. Bennett-Cipu-Mignotte-Okazaki ’06)

\[ m_2 - m_1 > \Lambda_0^{-1} \Delta \log \eta, \quad \text{where} \quad \Delta = \left| \begin{array}{cc} n_1 - n_0 & n_2 - n_1 \\ m_1 - m_0 & m_2 - m_1 \end{array} \right| > 0 \]

In particular, if \( m_0n_0 > 0 \), then

\[ m_2 - m_1 > \kappa^{-1}(4ac)^{m_0} \Delta \log \eta \]

Proof of Lemma 3.3

Put \( p_i := m_i \log \xi, \quad q_i := n_i \log \eta \)

Then \( \Lambda_i = m_i \log \xi - n_i \log \eta + \log \mu = p_i - q_i + \log \mu \)

The equality \( v_{m_i} = w_{n_i} \) implies that \( (p_i, q_i)’\)’s are on the curve

\[ C : \frac{(x_0 \sqrt{c} + z_0 \sqrt{a}) e^p - (x_0 \sqrt{c} - z_0 \sqrt{a}) e^{-p}}{\sqrt{a}} = \frac{(y_1 \sqrt{c} + z_1 \sqrt{b}) e^q - (y_1 \sqrt{c} - z_1 \sqrt{b}) e^{-q}}{\sqrt{b}} \]

Since \( \frac{dq}{dp} > 1 \) and \( \frac{d^2 q}{dp^2} < 0 \), we obtain

\[ 0 < \frac{q_1 - q_0}{p_1 - p_0} - \frac{q_2 - q_1}{p_2 - p_1} < \frac{q_1 - q_0 - p_1 + p_0}{p_1 - p_0} = \frac{\Lambda_0 - \Lambda_1}{p_1 - p_0} < \frac{\Lambda_0}{p_1 - p_0} \]

\[ \square \]
Rickert’s theorem applied by Bennett (1)

Lemma 3.3 (cf. Bennett-Cipu-Mignotte-Okazaki ’06)

\[ m_2 - m_1 > \kappa^{-1}(4ac)^{m_0} \triangle \log \eta \]

Now the assertion \( N(z_0, z_1) \leq 2 \) can be shown by using Lemma 3.3 and Baker’s method on linear forms in 3 logarithms.

This completes the proof of (i)

(ii) \((z_0, z_1) \in \{(st - cr, st - cr), (st - cr, s), (t, s)\} \implies N(z_0, z_1) \leq 1\)

Assume \( z = v_m = w_n \) has 2 solutions \((m_i, n_i) \) \((i \in \{1, 2\}, 2 < m_1 < m_2)\)

Note that \( m_0 = 1, 2 \) is a solution to \( z = v_m = w_n \) for \( z_0 = t, st - cr \), resp.

★ Since \( m_0 \) is small, we need other ingredients
(z_0, z_1) \in \{(st - cr, st - cr), (st - cr, s), (t, s)\} \implies N(z_0, z_1) \leq 1

Consider the linear form \( \Gamma := \Lambda_2 - \Lambda_1 = (m_2 - m_1) \log \xi - (n_2 - n_1) \log \eta \)

Then \( 0 < |\Gamma| < \kappa \xi^{-2m_1} \)

In order that Baker’s method can work, we need \( m_2 - m_1 \sim m_1 \)

i.e. \( m_2 (n_2) \) is not so larger than \( m_1 (n_1) \)

Theorem 3.4 (cf. Bennett ’98 (based on Rickert ’93))

\[ a_1 := a(c - b), a_2 := b(c - a), N := abz^2 \]
\[ (u := c - b, v := c - a, w := b - a, a'_1 := \max\{a_1, a_2 - a_1\}) \]
\[ \theta_1 := \sqrt{1 + a_1/N}, \theta_2 := \sqrt{1 + a_2/N} \]
\[ N \geq 10^5 a_2 \implies \max\{|\theta_1 - p_1/q|, |\theta_2 - p_2/q|\} > (32.01 a'_1 a_2 u N/a_1)^{-1} q^{-\lambda} \]

for \( \forall p_1, p_2, q (> 0) \), where \( a'_1 = \max\{a_1, a_2 - a_1\}, \lambda = 1 + \frac{\log\left(\frac{16a'_1 a_2 u N}{a_1}\right)}{\log\left(\frac{1.6874 N^2}{a_1 a_2 (a_2 - a_1) u w}\right)} < 2 \)
Rickert’s theorem applied by Bennett (3)

\[(z_0, z_1) \in \{(st - cr, st - cr), (st - cr, s), (t, s)\} \implies N(z_0, z_1) \leq 1\]

**Theorem 3.4 (cf. Bennett ’98 (based on Rickert ’93))**

\[a_1 := a(c - b), \ a_2 := b(c - a), \ N := abz^2 \ \ (u := c - b, \ a' := \max\{a_1, a_2 - a_1\})\]

\[\theta_1 := \sqrt{1 + a_1/N}, \ \theta_2 := \sqrt{1 + a_2/N}\]

\[N \geq 10^5 a_2 \implies \max\{|\theta_1 - p_1/q|, |\theta_2 - p_2/q|\} > (32.01a'_1a_2uN/a_1)^{-1} q^{-\lambda}\]

Combining Theorem 3.4 \((q = abz(1)z(2))\) with the trivial estimate

\[\max\left\{|\theta_1 - \frac{acy(1)y(2)}{abz(1)z(2)}|, \left|\theta_2 - \frac{bcx(1)x(2)}{abz(1)z(2)}\right|\right\} < \frac{c^{3/2}}{2a^{3/2}} z(2)^{-2}\]  \hspace{1cm} (1)

(where \((x(i), y(i), z(i)) \ (i \in \{1, 2\})\) are positive solutions to

\[az^2 - cx^2 = a - c, \ bz^2 - cy^2 = b - c\]

we obtain:
Proof of Main Theorem 1 (1)

\[(z_0, z_1) \in \{ (st - cr, st - cr), (st - cr, s), (t, s) \} \implies N(z_0, z_1) \leq 1\]

Lemma 3.5

- \(n_1 \geq 8 \implies n_2 < 148n_1\)
- \(n_1 = 7\) with \(z_1 = s \implies n_2 \leq 462 (\leq 66n_1)\)

Proof of Theorem 3.2 (ii) Note \((m_0, z_0) \in \{ (2, st - cr), (1, t) \}\) for \((z_0, z_1) \in \{ (st - cr, st - cr), (st - cr, s), (t, s) \}\)

Lemma 3.3 \[m_2 - m_1 > \kappa^{-1}(4ac)^{m_0}A \log \eta, \quad \kappa = \begin{cases} 2.001c/b & \text{if } z_0 = st - cr \\ 1/(2ab) & \text{if } z_0 = t \end{cases}\]

shows that \[
\frac{m_2 - m_1}{\log \eta} > 30a^2bc > 30b^2 \tag{2}
\]

Since we can show \(m, n \geq 7\) for the above \(z_0\), we may apply Lemma 3.5
Proof of Main Theorem 1 (2)

\[(z_0, z_1) \in \{(st - cr, st - cr), (st - cr, s), (t, s)\} \implies N(z_0, z_1) \leq 1\]

**Lemma 3.5**

- \(n_1 \geq 8 \implies n_2 < 148n_1\)
- \(n_1 = 7 \text{ with } z_1 = s \implies n_2 \leq 462 \leq 66n_1\)

\[
\frac{m_2 - m_1}{\log \eta} > 30a^2bc > 30b^2 \tag{2}
\]

Noting \(n_i - 1 \leq m_i \leq 2n_i + 1\), Lemma 3.5 implies \(m_2 - m_1 < 338m_1\), which together with Baker’s method on 2 logs (Laurent’s theorem ’08) implies

\[
\frac{m_2 - m_1}{\log \eta} < 1.8 \cdot 10^7
\]

which contradicts (2) with \(b > 4000\) (we know “\(d > d_+ \implies b > 4000\)"

This completes the proof of (ii)
Proof of Main Theorem 1

Recall $1 \leq |z_0| < \sqrt{\frac{c \sqrt{c}}{2 \sqrt{a}}}$, $1 \leq |z_1| < \sqrt{\frac{c \sqrt{c}}{2 \sqrt{b}}}$

which imply

(i) $|z_0| = cr - st \implies c < 4 \tau^{-4} ab^2$

(ii) $|z_1| = cr - st \implies c < 4 \tau^{-4} a^2 b$

(iii) $|z_0| = t \implies c > 4 ab^2$

(iv) $|z_1| = s \implies c > 4 a^2 b$

where $\tau = \frac{\sqrt{ab}}{r} (1 - \frac{a+b+1/c}{c})$ ($< 1$)

We consider several cases separately, and show $N \leq 10$ in each case

For example, if $4ab^2 < c < 4 \tau^{-4} ab^2$, then $|z_1| \neq cr - st$ and

$$N \leq N(1, 1) + N(-1, -1) + N(st - cr, s) + N(cr - st, -s) + N(t, s) + N(-t, -s)$$

$$\leq 2 + 2 + 1 + 2 + 1 + 2 = 10$$

$\Box$
Recall the system of Pellian equations $az^2 - cx^2 = a - c$, $bz^2 - cy^2 = b - c$, any solutions of which can be expressed as $z = v_m = w_n$, where

$$
v_0 = z_0, \quad v_1 = sz_0 + cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m
$$

$$
w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n
$$

Denote by $\{v_{z_0,m}\}$, $\{w_{z_1,n}\}$ the recurrence sequences $\{v_m\}$, $\{w_n\}$ with the initial terms $z_0$, $z_1$, resp.

The following is the key lemma

**Lemma 4.1**

$\forall m \geq 0$

$$
v_{cr-st,m} = v_{-t,m+1}, \quad v_{st-cr,m+1} = v_{t,m}
$$

$\forall n \geq 0$

$$
w_{cr-st,n} = w_{-s,n+1}, \quad w_{st-cr,n+1} = w_{s,n}
$$

**Proof**

$$z_0 = cr - st \implies v_0 = cr - st, \quad v_1 = 2c(rs - at) - t$$

$$z_0 = -t \implies v_1 = cr - st, \quad v_2 = 2c(rs - at) - t$$

$$\implies v_{cr-st,m} = v_{-t,m+1}$$
Proof of Main Theorem 2 (1)

**Lemma 4.1**

\[ v_{cr-st,m} = v_{-t,m+1}, \quad v_{st-cr,m+1} = v_{t,m} \quad \text{for all} \quad m \geq 0 \]

\[ w_{cr-st,n} = w_{-s,n+1}, \quad w_{st-cr,n+1} = w_{s,n} \quad \text{for all} \quad n \geq 0 \]

**Proof of Main Theorem 2**

In the preceding section, we bounded the number \( N \) dividing several cases by considering whether \( c \) is greater than \( 4ab^2 \) and so on, where we were able to eliminate some cases because of the inequalities \( 1 \leq |z_0| < \sqrt{\frac{c \sqrt{c}}{2 \sqrt{a}}} \), \( 1 \leq |z_1| < \sqrt{\frac{c \sqrt{c}}{2 \sqrt{b}}} \).

Now we **count** \( N \) using not these inequalities but **Lemma 4.1**

Denote by \( N'(z_0, z_1) \) the number of solutions to \( v_m = w_n \) with \( m > 2 \) \((\Leftrightarrow \quad d > d_+)\), where we are not assuming the inequalities above. Then

\[ N \leq N'(-1, -1) + N'(1, 1) + N'(z_0^-, z_1^-) + N'(z_0^+, z_1^+) \]

where \((z_0^-, z_1^-) \in \{(cr - st, cr - st), (-t, cr - st), (cr - st, -s), (-t, -s)\}\) and \((z_0^+, z_1^+) \in \{(st - cr, st - cr), (t, st - cr), (st - cr, s), (t, s)\}\)

**★** \((z_0^-, z_1^-), (z_0^+, z_1^+)\) attain \( d_- \), \( d_+ \) for some \( m, n \leq 2 \), resp.
Proof of Main Theorem 2 (2)

\[ N \leq N'(-1, -1) + N'(1, 1) + N'(z_0^-, z_1^-) + N'(z_0^+, z_1^+) \]

where \((z_0^-, z_1^-) \in \{(cr - st, cr - st), (-t, cr - st), (cr - st, -s), (-t, -s)\}\) and \((z_0^+, z_1^+) \in \{(st - cr, st - cr), (t, st - cr), (st - cr, s), (t, s)\}\)

In the same way as in the preceding section, one can show:

\[ N'(-1, -1), \; N'(1, 1), \; N'(z_0^-, z_1^-) \leq 2 \]

using Lemma 3.3 with Baker’s method on 3 logs, and

\[ N'(z_0^+, z_1^+) \leq 1 \]

using Lemma 3.3 with Rickert’s theorem and Baker’s method on 2 logs

Therefore

\[ N \leq 2 + 2 + 2 + 1 = 7 \]

\(\square\)
Overview of the relevant results (1)

- **Baker-Davenport (’69)**
  \[
  \{1, 3, 8, d\} : D \implies d = 120 (= d_+) 
  \]
  Baker’s method on 3 logs

- **Dujella (’97)**
  \[
  \{k - 1, k + 1, 4k, d\} : D \implies d = 16k^3 - 4k (= d_+) 
  \]
  Rickert’s theorem

- **Dujella-Pethő (’98)**
  \[
  \{1, 3, c, d\} : D \quad (c < d) \implies d = d_+ 
  \]
  Baker’s method on 3 logs

- **Dujella (’99)**
  \[
  \{F_{2k}, F_{2k+2}, F_{2k+4}, d\} : D \implies d = 4F_{2k+1}F_{2k+2}F_{2k+3} (= d_+) 
  \]
  Baker’s method on 3 logs
Overview of the relevant results (2)

- **F (’08)**
  \[
  \{k - 1, k + 1, c, d\} : D \ (16k^3 - 4k < c < d) \implies d = d_+
  \]
  Rickert's theorem twice
  \[
  (\implies \nexists\{k - 1, k + 1, c, d, e\} : D)
  \]

- **Bugeaud-Dujella-Mignotte (’07)**
  \[
  \{k - 1, k + 1, c, d\} : D \ (c = 16k^3 - 4k < d)
  \implies d = 64k^5 - 48k^3 + 8k (= d_+)
  \]
  Baker's method on 3 logs with refined congruence method
  using “\(s = \sqrt{ac + 1} \sim t = \sqrt{bc + 1}\)”

\[
\implies \{k - 1, k + 1, c, d\} : D \ (c < d) \implies d = d_+
\]
Overview of the relevant results (3)

- **Bugeaud-Dujella-Mignotte** ('07) \{ \( k - 1, k + 1 \), \( 16k^3 - 4k \)\}
  Baker’s method on 3 logs with refined congruence method
  using \( s = \sqrt{ac + 1} \sim t = \sqrt{bc + 1} \)

- **He-Togbé** ('09, '12)
  \( \{ k, A^2k + 2A, (A + 1)^2k + 2(A + 1), d \} : D \) with
  \( 2 \leq A \leq 10 \) Rickert
  \( A \geq 52330 \) Baker on 2 logs
  \( r = \sqrt{ab + 1} \sim s = \sqrt{ac + 1} \)
  \( \implies d = d_+ \)

- **Cipu-F-Mignotte** ('18)
  \( \{ k, A^2k \pm 2A, (A + 1)^2k \pm 2(A + 1), d \} : D \implies d = d_+ \)
  Proof \( k \geq 240.24(A + 1) + 740 \) optimization of Rickert’s theorem
  \( A \geq 2811 \) Baker on 2 logs
  \( \implies d = d_+ \)

The remaining cases can be checked by the reduction method
The possibilities for the third element $c$ induced by $\{a, b\}$

For a fixed $\{a, b\} : D (a < b)$, assume $\{a, b, c\} : D (b < c)$ and let

\[ ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2 \quad (r, s, t \in \mathbb{Z}_{>0}) \]

Eliminating $c$ yields \[ at^2 - bs^2 = a - b \]

By Nagell’s arguments, any solution $(t, s)$ to the above is expressed as

\[ t \sqrt{a} + s \sqrt{b} = (t_0 \sqrt{a} + s_0 \sqrt{b})(r + \sqrt{ab})^\nu \]

for some $|t_0|, s_0, \nu \in \mathbb{Z}_{>0}$, where $|t_0| < \sqrt{\frac{b \sqrt{b}}{2 \sqrt{a}}}, s_0 < \sqrt{(r + 1)/2}$

Each $(t_0, s_0)$ (often) gives a sequence of possible third elements $c$’s

E.g., $(t_0, s_0) = (\pm 1, 1)$ correspond to $c = c_\nu^\tau$ ($\tau \in \{\pm\}$), where

\[ c_\nu^\tau = \frac{1}{4ab} \left\{ (\sqrt{b} + \tau \sqrt{a})^2(r + \sqrt{ab})^{2\nu} + (\sqrt{b} - \tau \sqrt{a})^2(r - \sqrt{ab})^{2\nu} - 2(a + b) \right\} \]

$\Rightarrow$ We always have $\{a, b, c_\nu^\tau\} : D$
Outline

Overview of Diophantine tuples and main theorems
The number of extensions of Diophantine triples I
The number of extensions of Diophantine triples II
The extendibility problem

Overview of the relevant results

- **Filipin-Togbé-F ('14)**
  \[ \{k, 4k \pm 4, c, d\} : \mathcal{D} \quad (c_2^+ \neq c < d) \implies d = c_{\nu+1}^\tau (= d_+) \]
  - \( c = c_1^\tau \) known by **Dujella ('97) and He-Togbé ('09)**
  - \( c = c_2^- \) Baker’s method on **2 logs**
  - \( c \geq c_3^- \) Rickert’s theorem

- **He-Pu-Shen-Togbé ('18)**
  \[ \{k, 4k \pm 4, c, d\} : \mathcal{D} \quad (c = c_2^+ < d) \implies d = c_3^+ (= d_+) \]
  new kind of application of Baker’s method on **2 logs**
  “\( m \ mod \ r, n \ mod \ r : \text{small constant} \)” (e.g. \( m \equiv n \equiv \pm 1 \ (\text{mod} \ r) \))

- **Cipu-Filipin-F ('19 (?))**
  \[ \{A^2k, 4A^4k \pm 4A, c, d\} : \mathcal{D} \quad (c = c_\nu^\tau < d) \implies d = c_{\nu+1}^\tau (= d_+) \]
  - \( c = c_1^\tau \) known by **Cipu-F-Mignotte ('18)**
  - \( c = c_2^\tau \) Baker’s method on **2 logs**
  - **[He-Togbé]** for \( m, n : \text{even} \), **[He-Pu-Shen-Togbé]** for \( m, n : \text{odd} \)
  - \( c \geq c_3^- \) Rickert’s theorem
Corollary and key lemma

Main Theorem 3 (Cipu-Filipin-F preprint)

\[ a \left( a + \frac{7}{2} - \frac{1}{2} \sqrt{4a + 13} \right) \leq b \leq 4a^2 + a + 2 \sqrt{a} \]  

\{a, b, c, d\} : D \ (b < c < d)  
\[
\implies d = d_+ 
\]

Corollary 5.1

\{A^2k, 4A^4k \pm 4A, c, d\} : D \ (c < d) \ \text{with} \ k \in \{1, 2, 3, 4\}  
\[
\implies d = c_\nu^{\tau} (= d_+) \quad \leftarrow \text{unnecessary to assume} \quad c = c_\nu^{\tau} 
\]

Lemma 5.2 (Key lemma)

\{a, b, c\} : D \ with \ (3)  
\[
\implies \{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \ \text{for some} \ T \in \mathbb{Z}_{\geq 1}  
\text{or} \quad c = c_\nu^{\tau} \ \text{for some} \ \nu \in \mathbb{Z}_{\geq 1} \ \text{and} \ \tau \in \{\pm\} 
\]
Key lemma (1)

**Lemma 5.2 (Key lemma)**

\[
a \left( a + \frac{7}{2} - \frac{1}{2} \sqrt{4a + 13} \right) \leq b \leq 4a^2 + a + 2 \sqrt{a} \tag{3}
\]

\[\implies \{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \text{ for some } T \in \mathbb{Z}_{\geq 1}
\]

or \(c = c^\tau_\nu\) for some \(\nu \in \mathbb{Z}_{\geq 1}\) and \(\tau \in \{\pm\}\)

**Proof** Suppose \(c \neq c^\tau_\nu\) and let \(c = \gamma_\nu\)

\[t \sqrt{a} + s \sqrt{b} = (t_0 \sqrt{a} + s_0 \sqrt{b})(r + \sqrt{ab})^\nu\]

We may assume \(\exists \nu_0\) s.t. \(\gamma_{\nu_0} < b < \gamma_{\nu_0+1}\), and put \(c' := \gamma_{\nu_0}\)

Then we see that \(\{a, b', c', b\} : D\) is regular for some \(b' \in \mathbb{Z}_{\geq 1}\)

Thus \(b = a + b' + c' + 2ab'c' + 2r's'T\)

with \(r' = \sqrt{ab'} + 1, \ s' = \sqrt{ac'} + 1, \ T = \sqrt{b'c'} + 1\)

Noting \(4ab'c' + a + b' + c' < b < 4ab'c' + 4 \max\{a, b', c'\}\)

we can negate inequalities (3), except one exceptional case
Lemma 5.2 (Key lemma)

\[ a \left( a + \frac{7}{2} - \frac{1}{2} \sqrt{4a + 13} \right) \leq b \leq 4a^2 + a + 2 \sqrt{a} \]  

\[ \implies \{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \text{ for some } T \in \mathbb{Z}_{\geq 1} \]

\text{or } c = c_{\tau}^\nu \text{ for some } \nu \in \mathbb{Z}_{\geq 1} \text{ and } \tau \in \{\pm\}

Proof

\[ 4ab'c' + a + b' + c' < b < 4ab'c' + 4 \max\{a, b', c'\} \]

- \[ a \leq b'c' \implies b > 4ab'c' + a + b' + c' > 4a^2 + a + 2 \sqrt{a}, \text{ contradicting (3)} \]
- \[ a > b'c' \implies \]
  - \[ a \neq b' + c' + 2T \implies a > 4b'c' + b' + c' \]
    \[ \implies b < a (a + 7/2 + \sqrt{4a + 13}/2), \text{ contradicting (3)} \]
  - \[ a = b' + c' + 2T \implies b'c' < b' + c' + 2T \]

Setting \( B := \min\{b', c'\} \) and \( C := \max\{b', c'\} \), we obtain \( B = 1 \)

Then \( T = \sqrt{b'c' + 1} = \sqrt{C + 1} \), \( a = C + 1 + 2 \sqrt{C + 1} \) and

\[ a = T^2 + 2T, \ b = 4T^4 + 8T^3 - 4T \]

\[ \square \]
The case \( \{a, b\} \neq \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \)

\[
c_1^\tau = a + b + 2\tau r \\
( b < c_1^\tau \implies \tau = + )
\]

\[
c_2^\tau = 4(a + b)(ab + 1) + 4\tau r(2ab + 1) \sim 4ab^2
\]

\[
c_3^\tau = 16a^2b^2(a + b + 2\tau r) + 8ab(3a + 3b + 4\tau r) + 3(3a + 3b + 2\tau r)
\sim 16a^2b^3
\]

Proof of Main Theorem 3

- \( c = c_1^+ \) Baker on 2 logs “\( r = \sqrt{ab + 1} \sim s = \sqrt{ac + 1} \)” [He-Togbé]
- \( c = c_2^\tau \) Baker on 2 logs
  - \( m, n : \text{even} \implies \alpha = s + \sqrt{ac} \sim \beta^2 = (r + \sqrt{ab})^2 \)” [He-Togbé]
  - \( m, n : \text{odd} \implies m \equiv n \equiv \pm 1 \pmod{r} \)” [He-Pu-Shen-Togbé]
- \( c \geq c_3^- \) Rickert
The case \( \{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \) (1)

The case \( \{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \)

We may assume \( c \neq c^T \).

From \( ac + 1 = s^2, bc + 1 = t^2 \), we have \( at^2 - bs^2 = a - b \)

\[ t\sqrt{a} + s\sqrt{b} = (t_0\sqrt{a} + s_0\sqrt{b})(s + \sqrt{ab})' \]

\[ 1 < s_0 < T\sqrt{T} + 2 \quad (\text{by Nagell's argument}) \]

\[ (t_0, s_0) = (\pm(2T^2 + 2T - 1), T + 1) \]

\[ c = \gamma_1^T, \quad \text{where} \quad (\gamma_0 = \gamma_0^T = 1, \quad \gamma_1^- = T^2 - 1 < b) \]

\[ \gamma_1^+ = 16T^6 + 64T^5 + 72T^4 - 31T^2 - 4T + 3 \]
\[ \gamma_2^- = 16T^8 + 64T^7 + 48T^6 - 80T^5 - 88T^4 + 32T^3 + 36T^2 - 4T - 3 \]
\[ \gamma_2^+ > 256T^{12} + 2048T^{11} \]
The case \( \{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \) (2)

\[ \begin{align*}
&\text{The case } \{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\} \\
&\quad \bullet \quad c = \gamma_1^+ = 16T^6 + 64T^5 + 72T^4 - 31T^2 - 4T + 3 \\
&\quad \text{We get } m \equiv n \equiv 0 \pmod{2} \text{ and } z_0 = z_1 \in \{\pm 1\} \\
&\quad \bullet \quad m > 3.9999b^{-1/2}c^{1/2}T^{1/2} \quad (\text{assuming } T > 10^{10}) \\
&\quad \quad \quad \Leftarrow \quad \text{coming from } a \equiv b \equiv 0 \pmod{T} \\
&\quad \quad \quad \quad \text{(cf. known in general: } m > b^{-1/2}c^{1/2}) \\
&\quad \bullet \quad \text{Baker on 3 logs } [\text{Matveev '98}] \\
&\quad \quad \quad \quad \Lambda = m \log \alpha - n \log \beta' - \log \chi' \\
&\quad \quad \quad \quad \quad \Leftarrow \quad \text{“Kummer condition” should be checked:} \\
&\quad \quad \quad \quad \quad \quad [K(\sqrt{\alpha}, \sqrt{\beta'}, \sqrt{\chi'}) : K] = 2^3 = 8 \\
&\quad \quad \quad \quad \quad \quad \quad \left( K := \mathbb{Q}(\sqrt{ac}, \sqrt{bc}), \alpha = s + \sqrt{ac}, \beta' = t + \sqrt{bc}, \chi' = \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})} \right) \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \sim T < 5.146 \cdot 10^{10} \quad (\text{cf. } T < 1.8 \cdot 10^{11} \text{ by using } [\text{Aleksentsev '08}]) \\
&\quad \bullet \quad c = \gamma_2^- \quad \text{Baker on 2 logs } \beta^2 = (r + \sqrt{ab})^2 \sim \beta' = t + \sqrt{bc} \quad [\text{He-Togbé}] \\
&\quad \bullet \quad c \geq \gamma_2^+ \quad \text{Rickert}
\end{align*} \]
Thank you so much for your kind attention!!