Report on $D(-1)$-quadruples

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(joint work with N. C. Bonciocat and M. Mignotte)

Outline

Terminology

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$n = -1 \Longrightarrow$ pardi, tridi, quadi

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How large a $D(n) - m$-set can be?

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Dujella 2004 Any $D(n)-m$-set has

$$\begin{array}{ll} m \leq 31 & \text{if } 1 \leq |n| \leq 400 \\ m < 15.476 \log |n| & \text{if } \quad |n| > 400 \end{array}$$

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He-Togbé-Ziegler 2019 There is no $D(1)$-quintuple

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Bliznac Trebješanin-Filipin 2019 There is no $D(4)$-quintuple

A conjecture

Dujella 1993 if $n \notin \mathcal{S} := \{-4, -3, -1, 3, 5, 12, 20\}$ and $n \neq 4k + 2$ then there exists at least one $D(n)$-quadruple

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Conjecture for $n \in \mathcal{S}$ does not exist $D(n)$-quadruples

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How to find $D(-1)$-sets

$D(-1)$-quadruples

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Prologue
Terminology
Main problems
Existence
A conjecture
Classical approach to find quadi
Starting point
Alternative approach
Different viewpoint
A particular case
The general case
References

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Throughout $\{1, b, c, d\}$ will be a quadi with $1 < b < c < d$

r, s, t are the positive integers defined by $b - 1 = r^2$, $c - 1 = s^2$, $bc - 1 = t^2$

Existence of quadis

$\exists$ $D(-1)$-quadruples $\iff$ the system

$$z^2 - cx^2 = c - 1,\ bz^2 - cy^2 = c - b,\ y^2 - bx^2 = b - 1$$

is solvable in positive integers

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$\Longleftrightarrow$ $z = v_m = w_n$, where the integer sequences $(v_p)_{p\geq 0}$, $(w_p)_{p\geq 0}$ are given by explicit formulæ

$$v_p = \frac{s}{2}\left((s+\sqrt{c})^{2p} + (s-\sqrt{c})^{2p}\right),$$

$$w_p = \frac{s\sqrt{b}+\rho r\sqrt{c}}{2\sqrt{b}}(t+\sqrt{bc})^{2p} + \frac{s\sqrt{b}-\rho r\sqrt{c}}{2\sqrt{b}}(t-\sqrt{bc})^{2p}$$

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Similarly for x, y

Classical approach

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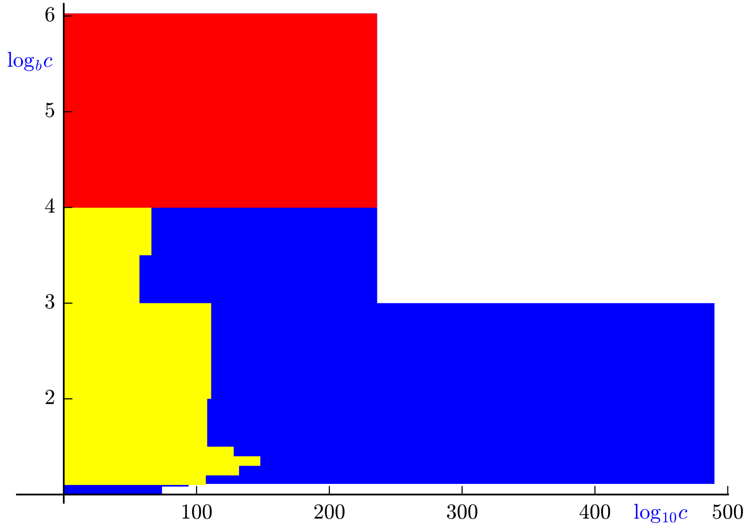
Long computations give necessary conditions for the existence of $D(-1)$-quadruples, including $b > 1.024 \cdot 10^{13}$ and $\max\{10^{14} b, b^{1.16}\} < c < \min\{9.6\, b^4, 10^{148}\}$

Where to search for quadis



Novel approach

Featuring the positive parameter

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$$f = t - rs$$

Squaring $f + rs = t$, one gets

$$r^2 + s^2 = 2frs + f^2 \qquad (*)$$

Our approach is essentially a study of solutions in positive integers to equation (*) in its various disguises, starting with $(s - rf)^2 - (f^2 - 1)r^2 = f^2$

New results

Theorem

There are no $D(-1)$–quadruples with $f = \gcd(r, s)$. In particular, there exists no $D(-1)$–quadruple for which the corresponding f has no prime divisor congruent to 1 modulo 4.

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Theorem

If $c \geq b^2$ then $c > 16b^3$.

Properties in the special case

When $\gcd(r, s) = f$, the cardinal equation has positive solutions

$$r = \frac{f(\gamma^k - \overline{\gamma}^k)}{\gamma - \overline{\gamma}}, \quad s = \frac{f(\gamma^{k+1} - \overline{\gamma}^{k+1})}{\gamma - \overline{\gamma}}, \quad k \in \mathbb{N},$$

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Experimental result: no quadi has $f \le 10^7$. In the special case no quadi has $f \le 10^9$

Second step in the proof of Theorem A

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$$v_m \equiv 2f^2 \pmod{8f^6}$$

$$w_n \equiv 2(\rho n + 1)f^2 + \left(\frac{4n^3 + 8n}{3}\rho + 4n^2\right) f^4 \pmod{8f^6}$$

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Observe that n is even and use this to deduce $n = 4f^2u$ for some positive integer u. Come back to the congruence to get $u \equiv 0 \pmod{f^2}$, so that $n \geq 4f^4$.

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This inequality and Matveev's theorem yield $f < 11300$

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Now it is preferable to use $y = U_n = u_l$, where

$$u_{l+2} = (4b-2)u_{l+1} - u_l, \qquad u_0 = r, \quad u_1 = (2b-1)r,$$
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The congruence mod r^2 gives $n \ge r - 2 = 2(f^2-1) > 1.999f^2$. From $f > 10^9$ it results $n > 10^{18}$, in contradiction with a previous result

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Write $f = f_1 f_2$, with f_1 the product of all the prime divisors of f which are congruent to 1 modulo 4, multiplicity included. From now on $f_1 \geq 5$

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$$r = \frac{f_2\left(\varepsilon\,\gamma^k - \overline{\varepsilon}\,\overline{\gamma}^k\right)}{\gamma - \overline{\gamma}}, \quad s = \frac{f_2\left(\varepsilon\,\gamma^{k+1} - \overline{\varepsilon}\,\overline{\gamma}^{k+1}\right)}{\gamma - \overline{\gamma}}, \quad k \geq 0,$$

with $\varepsilon = v_0 + u_0\sqrt{f^2 - 1}$ a fundamental solution to the generalized Pell equation $V^2 - (f^2 - 1)U^2 = f_1^2$ and $\overline{\varepsilon} = v_0 - u_0\sqrt{f^2 - 1}$.

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For $k \geq 1$ it holds $\gamma^2 - \dfrac{1}{2} < \dfrac{c}{b}$

$k = 0 \Longleftrightarrow c \geq b^2$

For $k \geq 1$ it holds

$$f^2 = \left\lceil \frac{c}{4b} \right\rceil .$$

Proof of Theorem B

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As $k = 0$, one has $r = f_2 u_0$, $s = f_2(v_0 + f u_0)$, with

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$$f_1^2 > 2u_0^2 f = 2u_0^2 f_1 f_2 \implies f_1 > 2u_0^2 f_2 = 2u_0 r$$
$$\implies f = f_1 f_2 > 2u_0 r f_2 = 2r^2$$

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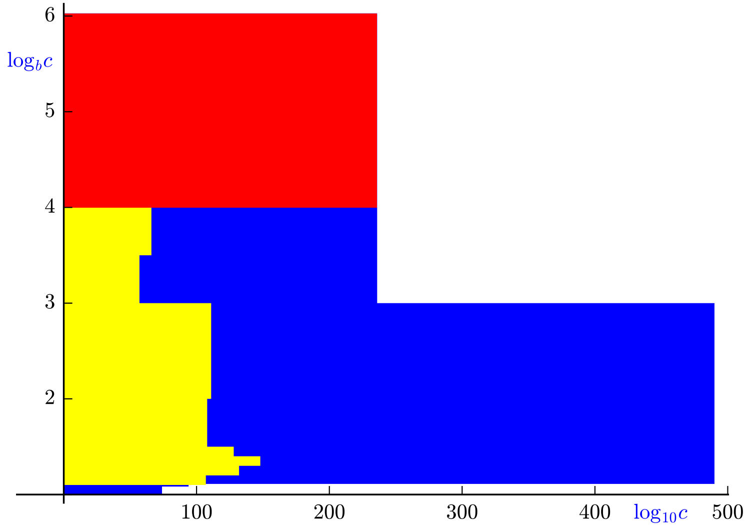
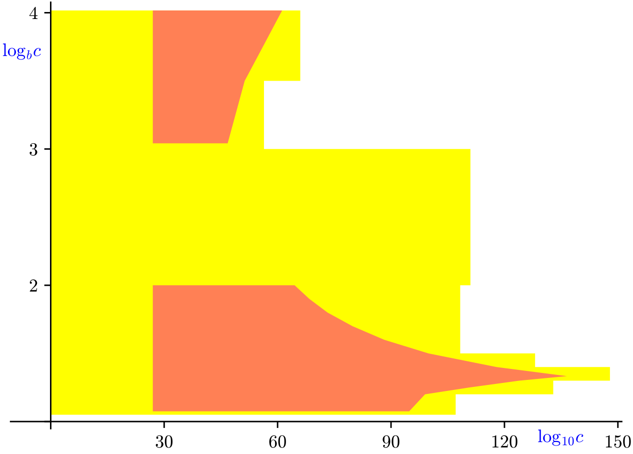
$$f_1^2 > 2u_0^2 f = 2u_0^2 f_1 f_2 \Longrightarrow f_1 > 2u_0^2 f_2 = 2u_0 r$$
$$\Longrightarrow f = f_1 f_2 > 2u_0 r f_2 = 2r^2$$

Put $f = 2r^2 + \delta$ in the cardinal equation to get a quadratic polynomial in s whose discriminant is square only for $\delta = 2r + 1$ or $\geq 4r + 2$

Where we started from



Where are we right now



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