Extensions of a $D(4)$-triple

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Diophantine $D(n)$-m-tuples

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Let $n \in \mathbb{Z}$. Diophantine $D(n)$-m-tuple is a set of m positive integers $\{a_1, \ldots, a_m\}$, such that $a_i a_j + n$ is a perfect square for each $1 \leq i < j \leq m$.

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  - for $n = 1$, quadruple in rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$

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- Fermat: for $n = 1$, $\{1, 3, 8, 120\}$ quadruple in positive integers
- Euler:
  - for $n = 1$, infinitely many quadruple in positive integers $\{a, b, a + b + 2r, 4r(r + a)(r + b)\}$, where $r^2 = ab + 1$
  - for $n = 1$, rational quintuple $\{1, 3, 8, 120, \frac{777480}{8288641}\}$

Comparison of cases $n = 1$ and $n = 4$

Case $\mathbf{n = 1}$

- pair $\{a, b\}$ extended with $c = a + b + 2r$
- triple $\{a, b, c\}$ extended with

$$d_+ = a + b + c + 2(abc + rst)$$

Case $\mathbf{n = 4}$

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Known results in the $D(4)$-case

Let $\{a, b, c, d\}$ be a $D(4)$-quadruple such that $a < b < c < d_+ < d$, i.e. an irregular quadruple.

Baćić, Filipin (2013) Then $b > 10^4$.

Filipin (2017) Then $b \geq a + 57\sqrt{a}$.

Filipin (2008) Then $c < \max\{7b^{11}, 10^{26}\}$.

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Results for some specific parametric families of pairs and triples.

Dujella, Ramasamy (2005) If $\{a, b, c\} = \{F_{2k}, 5F_{2k}, 4F_{2k+2}\}$ then $d = d_+ = 4L_{2k}F_{4k+2}$.

Filipin, He, Togbe (2012) If $\{a, b, c\} = \{k, A^2k + 4A, (A+1)^2k + 4(A+1)\}$, $k \in \mathbb{Z}$ and $1 \leq A \leq 22$ or $A \geq 51767$ then $d = d_+$.

Filipin, Baćić (2013) If $\{a, b\} = \{k-2, k+2\}$, $k \geq 3$ then $d = d_+$.

New results in the $D(1)$ case

Theorem (Fujita, Miyazaki (2018))

Any fixed Diophantine triple can only be extended to a Diophantine quadruple in **at most 11** *ways by joining a fourth element exceeding the maximal element in the triple.*

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Theorem (Cipu, Fujita, Miyazaki (2018))

Any fixed Diophantine triple can only be extended to a Diophantine quadruple in **at most 8 ways** *by joining a fourth element exceeding the maximal element in the triple.*

Main result in the $D(4)$ case

Theorem

Any $D(4)$-triple can be extended to a $D(4)$-quadruple with $d > \max\{a, b, c\}$ in at most 8 ways.
A regular $D(4)$-triple $\{a, b, c\}$ can be extended to a $D(4)$-quadruple with $d > \max\{a, b, c\}$ in at most 4 ways.

System of pellian equations

Let us observe a fixed $D(4)$-pair $\{a, b\}$, $a < b$. Then

$$ab + 4 = r^2, \quad r \in \mathbb{N}.$$

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$$ab + 4 = r^2, \quad r \in \mathbb{N}.$$

If we extend it with c to a triple $\{a, b, c\}$, $a < b < c$, then there exist $s, t \in \mathbb{N}$ s.t.

$$ac + 4 = s^2, \quad bc + 4 = t^2.$$

These equations yield a Pellian equation with unknowns (s, t)

$$bs^2 - at^2 = 4(b - a).$$

System of pellian equations

A solution (s, t) can be described as an element of a sequence (s_ν, t_ν), $\nu \in \mathbb{N}_0$, where (s_0, t_0) is a fundamental solution of the Pellian equation, and

$$t_\nu\sqrt{a} + s_\nu\sqrt{b} = (t_0\sqrt{a} + s_0\sqrt{b})\left(\frac{r + \sqrt{ab}}{2}\right)^\nu .$$

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Any fundamental solution (s_0, t_0) satisfies inequalities

$$2 \leq s_0 < \sqrt{r+2}, \qquad 2 \leq |t_0| \leq \sqrt{\frac{b\sqrt{b}}{\sqrt{a}}}.$$

System of pellian equations

If we consider an extension of a $D(4)$-triple $\{a, b, c\}$ to a $D(4)$-quadruple with an element d then there must exist $x, y, z \in \mathbb{N}$

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2.$$

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Solution z is an element of sequences v_m and w_n - solutions of these equations,

$$v_0 = z_0, \; v_1 = \frac{1}{2}\left(sz_0 + cx_0\right), \; v_{m+2} = sv_{m+1} - v_m,$$
$$w_0 = z_1, \; w_1 = \frac{1}{2}\left(tz_1 + cy_1\right), \; w_{n+2} = tw_{n+1} - w_n.$$

A regular quadruple

An extension of a triple $\{a, b, c\}$ to a quadruple with a larger element does exist:

$$d_{+} = a + b + c + \frac{1}{2}(abc + rst).$$

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Lemma

Let $\{a, b, c\}$ be a $D(4)$-triple such that $z = v_m = w_n$ has a solution (m, n) for which $d = d_+ = \frac{z^2-4}{c}$. Then only one of the following cases can occur:

i) $(m, n) = (2, 2)$ *and* $z_0 = z_1 = \frac{1}{2}(st - cr)$,
ii) $(m, n) = (1, 2)$ *and* $z_0 = t$, $z_1 = \frac{1}{2}(st - cr)$,
iii) $(m, n) = (2, 1)$ *and* $z_0 = \frac{1}{2}(st - cr)$, $z_1 = s$,
iv) $(m, n) = (1, 1)$ *and* $z_0 = t$, $z_1 = s$.

Fundamental solutions

Filipin (2009)

Suppose that $\{a, b, c, d\}$ is a $D(4)$-quadruple with $a < b < c < d$ and that w_m and v_n are defined as before. If $z = v_m = w_n$ has a solution with

i) m and n both even, then $z_0 = z_1$ and
$|z_0| = 2$ or $|z_0| = \frac{1}{2}(cr - st)$ or $z_0 < 1.608a^{-5/14}c^{9/14}$.

ii) m odd and n even, then $z_0 z_1 < 0$ and
$|z_0| = t$ and $|z_1| = \frac{1}{2}(cr - st)$.

iii) m even and n odd, then $z_0 z_1 < 0$ and
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Fundamental solutions - new result

Theorem

Suppose that $\{a, b, c, d\}$ *is a* $D(4)$*-quadruple with* $a < b < c < d$ *and that* w_m *and* v_n *are defined as before. If* $z = v_m = w_n$ *has a solution with*

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Moreover, if $d > d_+$*, case ii) cannot occur.*

Outline of classical methods

- congruence method
- Baker's method on linear forms in logarithms- Matveev's theorem and Laurent's theorem,
- Rickert's theorem on simultaneous rational approximation of irrationals
- Baker and Davenport's reduction method

Numerical lower bound on an element b

Lemma

If $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$ and $b \leq 10^5$ then $d = d_+$.

Lemma (Dujella, Pethö)

Assume that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of a real number κ such that $q > 6M$ and let

$$\eta = \|\mu q\| - M \cdot \|\kappa q\|,$$

where $\| \cdot \|$ denotes the distance from the nearest integer. If $\eta > 0$, then the inequality

$$0 < J\kappa - K + \mu < AB^{-J}$$

has no solution in integers J and K with $\frac{\log(Aq/\eta)}{\log B} \leq J \leq M$.

Bound on c in the terms of b

Theorem

Let $\{a, b, c, d\}$ be a $D(4)$-quadruple and $a < b < c < d$. Then

i) *if $b < 2a$ and $c \geq 890b^4$ or*

ii) *if $2a \leq b \leq 12a$ and $c \geq 1613b^4$ or*

iii) *if $b > 12a$ and $c \geq 39247b^4$*

we must have $d = d_+$.

Counting a number of extensions

Following idea by Okazaki:
Let us assume that there are 3 solutions to the equation $z = v_m = w_n$ which belong to the same fundamental solution, denote (m_i, n_i), $i = 0, 1, 2$ and $\Lambda_0 = m_0 \log \frac{s+\sqrt{ac}}{2} - n_0 \log \frac{t+\sqrt{bc}}{2} + \log \mu$, then

$$m_2 - m_1 > \Lambda_0^{-1} \Delta \log \frac{t + \sqrt{bc}}{2},$$

where $\Delta = \begin{vmatrix} n_1 - n_0 & n_2 - n_1 \\ m_1 - m_0 & m_2 - m_1 \end{vmatrix} > 0.$

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where $\Delta = \begin{vmatrix} n_1 - n_0 & n_2 - n_1 \\ m_1 - m_0 & m_2 - m_1 \end{vmatrix} > 0.$

If $c > b > 10^5$, then

$$m_0 \leq 2.$$

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Then:

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- so

$$N = N_{ee} = N(2, 2) + N(-2, -2) \leq \mathbf{1} + 2 = 3.$$

"Shifting" a sequence

Observe

v_0	v_1	v_2	v_3
t	$\frac{1}{2}(cr+st)$	$\frac{1}{2}((s^2-1)t+crs)$	$\frac{1}{2}(cr(s^2-1)+st(ac+1))$
$-\frac{1}{2}(cr-st)$	t	$\frac{1}{2}(cr+st)$	$\frac{1}{2}((s^2-1)t+crs)$

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t	$\frac{1}{2}(cr+st)$	$\frac{1}{2}((s^2-1)t+crs)$	$\frac{1}{2}(cr(s^2-1)+st(ac+1))$
$-\frac{1}{2}(cr-st)$	t	$\frac{1}{2}(cr+st)$	$\frac{1}{2}((s^2-1)t+crs)$

Let $\{v_{z_0,m}\}$ denote a sequence $\{v_m\}$ with an initial value z_0.

$$v_{\frac{1}{2}(cr-st),m} = v_{-t,m+1}, \quad v_{-\frac{1}{2}(cr-st),m+1} = v_{t,m}$$

for each $m \geq 0$ and

$$w_{\frac{1}{2}(cr-st),n} = w_{-s,n+1}, \quad w_{-\frac{1}{2}(cr-st),n+1} = w_{s,n}$$

for each $n \geq 0$.

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$$N = N(-2,-2) + N(2,2) + N'(z_0^-, z_1^-) + N'(z_0^+, z_1^+)$$

where
$(z_0^+, z_1^+) \in \{ \left(\frac{1}{2}(st - cr), \frac{1}{2}(st - cr)\right), \left(t, \frac{1}{2}(st - cr)\right), \ \left(\frac{1}{2}(st - cr), s\right),$
$(t, s)\}$ and $(z_0^-, z_1^-) = (-z_0^+, -z_1^+)$.

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$(t, s)\}$ and $(z_0^-, z_1^-) = (-z_0^+, -z_1^+)$.

- $N'(z_0^+, z_1^+) \leq 2$ and $N'(z_0^-, z_1^-) \leq 2$
- $N'(z_0^+, z_1^+) = N'(t, s) \leq 1$ for $c > b^2$

Counting a number of extensions

Theorem

Let $\{a, b, c\}$ be a $D(4)$-triple with $a < b < c$.

i) *If $c = a + b + 2r$, then $N \leq 3$.*

ii) *If $a + b + 2r \neq c < b^2$, then $N \leq 7$.*

iii) *If $b^2 < c < 39247b^4$, then $N \leq 6$.*

iv) *If $c \geq 39247b^4$, then $N = 0$.*

Counting a number of extensions

Proposition

Let $\{a, b\}$ *be a* $D(4)$*-pair with* $a < b$. *Let* $c = c_\nu^\tau$ *given by*

$$c = c_\nu^\tau = \frac{4}{ab}\left\{\left(\frac{\sqrt{b}+\tau\sqrt{a}}{2}\right)^2\left(\frac{r+\sqrt{ab}}{2}\right)^{2\nu} + \left(\frac{\sqrt{b}-\tau\sqrt{a}}{2}\right)^2\left(\frac{r-\sqrt{ab}}{2}\right)^{2\nu} - \frac{a+b}{2}\right\}$$

where $\tau \in \{1, -1\}$ *and* $\nu \in \mathbb{N}$.

i) *If* $c = c_1^\tau$ *for some* τ, *then* $N \leq 3$.

ii) *If* $c_2^+ \leq c \leq c_4^+$ *then* $N \leq 6$.

iii) *If* $c = c_2^-$ *and* $a \geq 2$ *then* $N \leq 6$ *and if* $a = 1$ *then* $N \leq 7$.

iv) *If* $c \geq c_5^-$ *or* $c \geq c_4^-$ *and* $a \geq 35$ *then* $N = 0$.

Corollary

Let $\{a, b, c\}$ *be a* $D(4)$*-triple. If* $a < b \leq 6.85a$ *then* $N \leq 6$.

Thank you for your attention.