On classification of unitary highest weight modules

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Hermitian symmetric spaces

$G/Q \simeq G_0/K$ where G is complex and Q is a parabolic subgroup with *abelian* nilradical and Levi part of Q is complexification of the maximal compact subgroup K

Cartan decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_+ \qquad \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}_+$$

compact roots Φ_c & noncompact roots Φ_n

$$\rho = \rho_\mathfrak{k} + \rho_n$$
$$W = W_\mathfrak{k} W^\mathfrak{k}$$

Classification of HSS

G	G_0	K
$SL(p+q,\mathbb{C})$	$SU(p,q)$	$S(U(p)\times U(q))$
$SO(p+2,\mathbb{C})$	$SO(2,p)$	$S(O(p)\times O(2))$
$SO(2n,\mathbb{C})$	$SO^*(2n)$	$U(n)$
$Sp(2n,\mathbb{C})$	$Sp(n,\mathbb{R})$	$U(n)$
$E_6^{\mathbb{C}}$	E_6^{-14}	$\mathrm{Spin}(10)\times SO(2)$
$E_7^{\mathbb{C}}$	E_7^{-25}	$E_6\times SO(2)$

(Plus covers of these.)

Unitarizable highest weight modules

$$\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+, \quad \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}_+$$

finite-dimensional $\mathfrak{k}$-module F_λ of highest weight λ

generalizde Verma module $M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q})} F_\lambda$

its unique maximal submodule $J(\lambda) \leq M(\lambda)$

and the simple quotient $L(\lambda) \simeq M(\lambda)/J(\lambda)$

Shapovalov form $\langle X \cdot u \,|\, v \rangle = \langle u \,|\, \sigma(X) \cdot v \rangle$ where σ is minus the conjugate with respect to the real form $\mathfrak{g}_0$

[EHW83]Thomas Enright, Roger Howe, and Nolan Wallach. “A classification of unitary highest weight modules”. In: *Representation theory of reductive groups (Park City, Utah, 1982)*. Vol. 40. Progr. Math. Boston, MA: Birkhäuser Boston, 1983, pp. 97–143

[EJ90]Thomas J. Enright and Anthony Joseph. “An intrinsic analysis of unitarizable highest weight modules”. In: *Mathematische Annalen* 288.1 (Dec. 1990), pp. 571–594

[Jak83] Hans Plesner Jakobsen. “Hermitian symmetric spaces and their unitary highest weight modules”. In: *Journal of Functional Analysis* 52.3 (July 1, 1983), pp. 385–412

Classification of UHW Modules

$$M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q})} F_\lambda \simeq \mathfrak{U}(\mathfrak{p}_-) \otimes_{\mathbb{C}} F_\lambda = S(\mathfrak{p}-) \otimes F_{\lambda_0} \otimes \mathbb{C}_z$$

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β ... maximal non-compact root

any weight $\lambda \in \mathfrak{h}^*$ can be written uniquely as $\lambda = \lambda_0 + z\zeta$ where

$$\zeta \perp \Phi_c, \qquad \langle \zeta, \beta^\vee \rangle = 1 \qquad \& \qquad \langle \lambda_0 + \rho, \beta \rangle = 0$$

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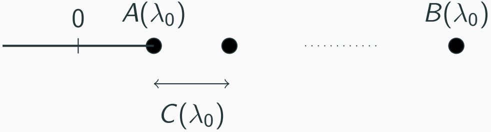
set of $z \in \mathbb{C}$ for which the simple factor of Verma module $L(\lambda)$ is unitarizable:



$A(\lambda_0)$, $B(\lambda_0)$ and $C(\lambda_0)$ are real numbers expressible in terms of certain root systems $Q(\lambda_0)$ and $R(\lambda_0)$ associated to λ_0

Classification of UHW Modules — continued

The level of reduction of a simple module $L(\lambda) \simeq M(\lambda)/J(\lambda)$ is the first natural number k for which $J(\lambda) \cap M(\lambda)^k \neq 0$, where
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For $\lambda = \lambda_0 + (B(\lambda_0) - kC(\lambda_0))\zeta$ we have $l(\lambda) = k + 1$

Example: $G = SU(p, q)$

Using standard coordinates and setting $n = p + q$, we write

$$\lambda = (\lambda_1, \dots, \lambda_p \,|\, \lambda_{p+1}, \dots, \lambda_n).$$

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then $Q(\lambda_0) = R(\lambda_0)$ is the root system built on the first i and the last j coordinates.

Furthermore, $A(\lambda_0) = \max\{i, j\}$, while $B(\lambda_0) = i + j - 1$.

Our results

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The possibility of organizing the modules into cones was first noticed by Davidson-Enright-Stanke.

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The possibility of organizing the modules into cones was first noticed by Davidson-Enright-Stanke.

The proof in EHW is based on one instance of Dirac inequality. We use a different version of the Dirac inequality and obtain a simpler and more natural proof.

Dirac operator

Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to the Killing form B.

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D is independent of b_i and K-invariant.

Dirac operator

D^2 is the spin Laplacian (Parthasarathy):

$$D^2 = -(\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho\|^2) + (\mathrm{Cas}_{\mathfrak{k}_\Delta} + \|\rho_{\mathfrak{k}}\|^2).$$

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$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g}) \quad \text{and} \quad \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p}).$$

Dirac operator

$$D = \sum_i b_i \otimes d_i \quad \in \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{p})$$

D acts on $M \otimes S$, where

- M is $(\mathfrak{g}, K)$-module
- S is the spin module for $C(\mathfrak{p})$ ($S = \bigwedge \mathfrak{p}^+$, $\mathfrak{p}^+$ acts by wedging and $\mathfrak{p}^-$ acts by contracting)

Parthasarathy's Dirac inequality

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By the formula for D^2, the inequality becomes explicit on any K-type $F_\tau \subset M \otimes S$:

$$\|\tau + \rho_{\mathfrak{k}}\|^2 \geq \|\Lambda\|^2,$$

where $\Lambda \in \mathfrak{h}^*$ corresponds to the infinitesimal character of M via the Harish-Chandra isomorphism.

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(For $L(\lambda)$ we have $\Lambda = \lambda + \rho$.)

Each F_τ is contained in some $F_\mu \otimes F_\nu \subset M \otimes S$. All ν are of the form $\sigma\rho - \rho_{\mathfrak{k}}$, where $\sigma \in W$ is such that $\sigma\rho$ is $\mathfrak{k}$-dominant, i.e. $\sigma \in W^{\mathfrak{k}}$.

Parthasarathy-Varadarajan-Rao component

For fixed $F_\mu \otimes F_\nu$, the critical F_τ is the PRV component $\tau = (\mu + \nu^-)^+$, where ν^- is the lowest weight of F_ν and $(\cdot)^+$ denotes the $\mathfrak{k}$-dominant $W_\mathfrak{k}$-conjugate.

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The PRV component is characterized by having the smallest $\|\tau + \rho_\mathfrak{k}\|^2$ over all $F_\tau \subset F_\mu \otimes F_\nu$.

In particular, for $\nu = \rho_n$ (the weight of the 1-dimensional $\mathfrak{k}$-module $\bigwedge^{\dim \mathfrak{p}} \mathfrak{p}^+ \subset S$), we see that if $M = L(\lambda)$ is unitary, then

$$\|\mu + \rho\|^2 \geq \|\Lambda\|^2 = \|\lambda + \rho\|^2$$

for any K-type $F_\mu \subset L(\lambda)$.

EHW criterion

EHW proved that $L(\lambda)$ is unitary if and only if

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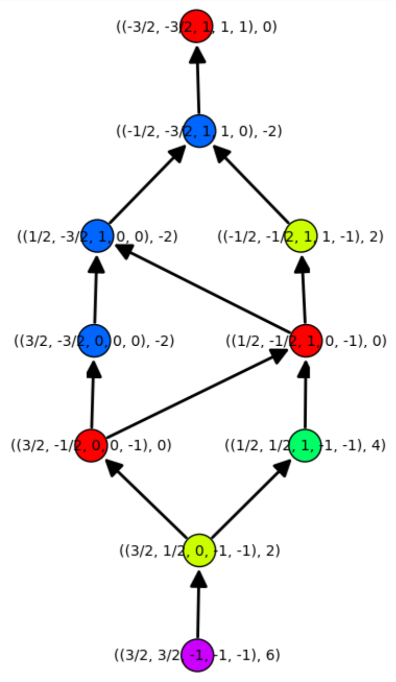
Let's look at *all* PRV components of $F_\lambda \otimes S \subset L(\lambda) \otimes S$

$$SU(2,3)$$

$$\rho_n = (3/2, 3/2, -1, -1, -1)$$

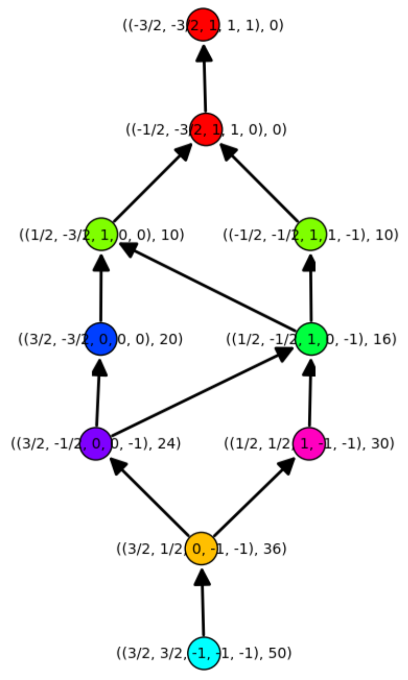
$$\lambda = (0, -1|0, 0, 0)$$



$$SU(2,3)$$

$$\rho_n = (3/2, 3/2, -1, -1, -1)$$

$$\lambda = (-3, -4|1, 1, 0)$$

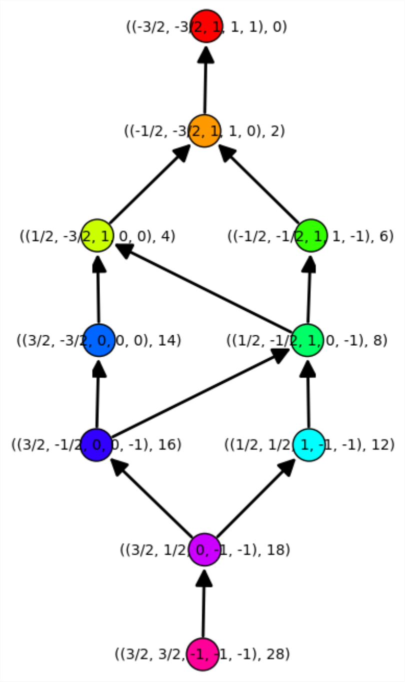


$SU(2,3)$

$\rho_n = (3/2, 3/2, -1, -1, -1)$

$\lambda = (-2, -2|1, 0, 0)$

level 2



Our criterion

We show (modulo some remaining work) that unitarity is in most cases equivalent to just one inequality, involving only the lowest K-type F_λ of $L(\lambda)$ and

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The lowest weight of $\bigwedge^{\dim \mathfrak{p}-1} \mathfrak{p}^+$ is $-\beta + \rho_n$ where β is the highest noncompact root.

So the PRV component of $F_\lambda \otimes \bigwedge^{\dim \mathfrak{p}-1} \mathfrak{p}^+ \subset L(\lambda) \otimes S$ has highest weight $(\lambda - \beta)^+ + \rho_n$ and the corresponding Dirac inequality is

$$\|(\lambda - \beta)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2.$$

(in the remaining cases just one more inequality is needed)

Example: $G = SU(p, q)$

In standard coordinates, $\beta = \epsilon_1 - \epsilon_n$. If $\lambda_1 = \cdots = \lambda_i > \lambda_{i+1}$ and $\lambda_{n-j} > \lambda_{n-j+1} = \cdots = \lambda_n$, then

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Our inequality thus becomes

$$\|(\lambda + \rho) - (\epsilon_i - \epsilon_{n-j+1})\|^2 \geq \|\lambda + \rho\|^2$$

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$$\lambda_1 - \lambda_n \leq -n + i + j$$

(this is equivalent to the final result of [EHW83])

Example - continued

For basic regular case $\Lambda = \rho$ it follows that

$$\lambda = (\underbrace{-q+j, \dots, -q+j}_{i}, -q, \dots, -q \,|\, p, \dots, p, \underbrace{p-i, \dots, p-i}_{j}).$$

Example - continued

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$$\lambda = (\underbrace{-q+j, \ldots, -q+j}_{i}, -q, \ldots, -q \mid p, \ldots, p, \underbrace{p-i, \ldots, p-i}_{j}).$$

We say that $\lambda + \rho$ is the j^{th} point of the i^{th} edge of the Hasse diagram of ρ.

Example - continued

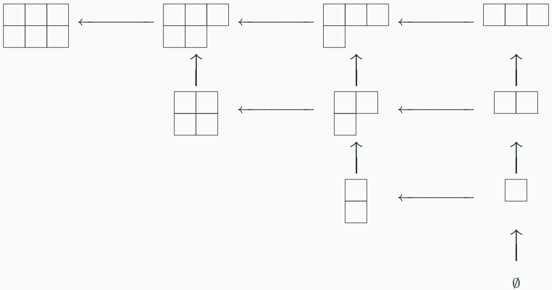
If $p = 2$ and $q = 3$, the Hasse diagram is

$$\begin{tikzcd}
(2,1 \mid 0,-1,-2) & (2,0 \mid 1,-1,-2) \arrow[l] & (2,-1 \mid 1,0,-2) \arrow[l] & (2,-2 \mid 1,0,-1) \arrow[l] \\
 & (1,0 \mid 2,-1,-2) \arrow[u] & (1,-1 \mid 2,0,-2) \arrow[l] \arrow[u] & (1,-2 \mid 2,0,-1) \arrow[l] \arrow[u] \\
 & & (0,-1 \mid 2,1,-2) \arrow[u] & (0,-2 \mid 2,1,-1) \arrow[l] \arrow[u] \\
 & & & (-1,-2 \mid 2,1,0) \arrow[u]
\end{tikzcd}$$

with arrows pointing to larger elements in the Bruhat order.

Example - continued

The corresponding Young diagrams are



The first edge is the last column, and the second edge is the diagonal, both excluding the smallest point $\tilde{\rho} = \rho_{\mathfrak{k}} - \rho_{n}$. The points on each edge are counted in the direction of the arrows.

Unitarity

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1. prove unitarity for "basic cases" (e.g. ladder representations)
2. iterate tensor products of the basic cases and find other unitary modules there. E.g.

$$L((-q+j,\dots,-q+j|1,\dots,1,0\dots,0))$$

sits in

$$L((-1,\dots,-1|1,0,\dots,0))^{\otimes(q-j)}$$

Conclusion

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Cone structure of the set of UHW modules appears more directly and more naturally.

THANK YOU!