Nilpotent Orbits in `atlas`

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Representation Theory XVI
Dubrovnik, Croatia, June 24-29, 2019

Collaborators

This is joint work with:

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...and it is in progress.

The `atlas` Software

- Motivating Goal: Compute the Unitary Dual of reductive Lie groups.
- Given an irreducible representation, `atlas` decides whether it is unitarizable.

Example (Finite-dimensional representations of $SL(2,\mathbb{R})$)

```
atlas> set G=SL(2,R)
Variable G: RealForm
atlas> set t=trivial(G)
Variable t: Param
atlas> is_unitary(t)
Value: true
```

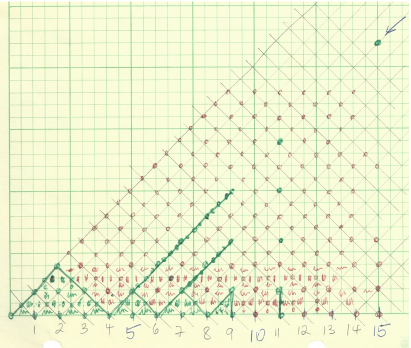
Specify a finite-dimensional representation by its highest weight:

```
atlas> set p=finite_dimensional(G,[2])
Variable p: Param
atlas> dimension(p)
Value: 3
atlas> is_unitary(p)
Value: false
```

Unitarity

- Does `atlas` give the correct answers?
- We checked this on an example for which the unitary dual is known due to Baldoni-Silva and Knapp (1989): $G = Sp(6,2)$.
- We considered two series of representations: spherical and with lowest K-type $triv \otimes (2,2)$.
- Because of the shape of the unitary dual, only a finite number of representations need to be tested. The signature of the invariant Hermitian form can change only at reducibility points.

Example: Spherical Representations of $Sp(6, 2)$



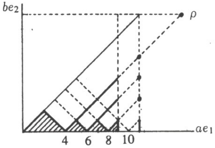
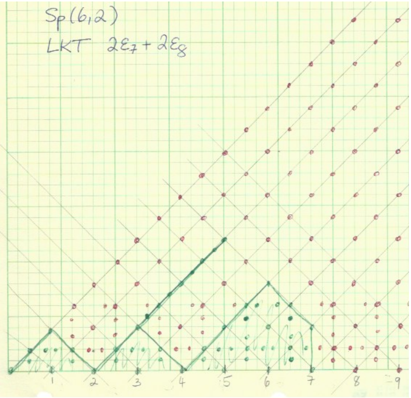


Figure 4: *Known unitary points in $Sp(6,2)$ for σ trivial*

Example: $Sp(6, 2)$ with LKT $triv \otimes (2, 2)$



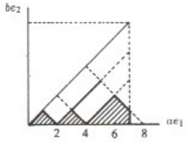


Figure 3: *Unitary points in* $Sp(6, 2)$ *with minimal* K *type* $2e_8 + 2e_7$

Nilpotent Orbits: Questions

Let G be a semisimple Lie group over $\mathbb{C}$ or $\mathbb{R}$ with Lie algebra $\mathfrak{g}$. We are interested in orbits of nilpotent elements in $\mathfrak{g}$ under the adjoint action of G.

Goal: List and describe these orbits.

- Explicitly list the (finite) collection of such orbits: element in $\mathfrak{g}$, weighted Dynkin diagram, Bala-Carter label, etc.
- Calculate properties/invariants: dimension etc.
- Fundamental group $\pi_1(\mathcal{O})$.
- Component group $A(\mathcal{O})$ of the centralizer in G.

Why do we need `atlas` to find this?

Much of this information is available in the literature (e.g., Collingwood & McGovern), especially over algebraically closed fields; however:

- Convenience: All information can be found in one place.
- More general cases, such as if G is not adjoint or simply connected, not simple.
- Real case and K-orbits.
- We can use `atlas` to compute more details.
- Use of orbit information for other `atlas` calculations.

Complex Orbits

In `atlas`, a complex group G is represented by a root datum:

- Character and cocharacter lattices identified with $\mathbb{Z}^n$ for n the rank, and finite subsets of each to indicate the simple roots and coroots.
- The Cartan subalgebra of the Lie algebra can be identified with $X_* \otimes_{\mathbb{Z}} \mathbb{C}$ (in `atlas`, $X_* \otimes_{\mathbb{Z}} \mathbb{Q}$).

Complex nilpotent orbit: Pair (G, H), where $H \in X_*$ is the semisimple element of a standard $\mathfrak{sl}_2$-triple $\{H, X, Y\}$ (unique up to W).

Example (Orbits in $Sp(4, \mathbb{C})$)

```
atlas> set G=Sp(4)
Variable G: RootDatum
atlas> set orbs=complex_nilpotent_orbits (G)
Variable orbs: [ComplexNilpotent]
atlas> for orb in orbs do prints(orb) od
simply connected root datum of Lie type 'C2'()[ 0, 0 ]
simply connected root datum of Lie type 'C2'()[ 1, 0 ]
simply connected root datum of Lie type 'C2'()[ 1, 1 ]
simply connected root datum of Lie type 'C2'()[ 3, 1 ]
```

Complex Orbits

Some known terminology/facts about complex orbits:

- A nilpotent element X in $\mathfrak{g}$ is *distinguished* if it is not contained in any Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. In that case the corresponding nilpotent orbit is called *distinguished*.
- Every nilpotent element in $\mathfrak{g}$ is distinguished in a unique (up to conjugation) Levi subalgebra $\mathfrak{l}$. We call this Levi subalgebra the "Bala Carter Levi" of the orbit.
- The Lie type of its derived algebra, possibly with one or more pieces of data, is the "Bala Carter label" of the orbit.

Complex Orbits

Algorithm: List all (conjugacy classes of) Levi subalgebras $\mathfrak{l}$ of $\mathfrak{g}$, then find the distinguished orbits in each $\mathfrak{l}$.

- To find all Levi subalgebras, take the subsets of the simple roots.

Proposition

Two Levi subalgebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are conjugate if and only if $\rho(\mathfrak{l}_1)$ and $\rho(\mathfrak{l}_2)$ are W-conjugate.

- The semisimple element H corresponding to X may be taken to be of the form: 2 times the sum of the coweights of some simple roots (in $\mathfrak{l}$).
- X is then distinguished in $\mathfrak{l}$ if $\dim \mathfrak{l}_0 = \dim \mathfrak{l}_2$ (which is computable). These are the 0 and 2 eigenspaces of *ad H* in $\mathfrak{l}$.

Example

Example (One Orbit in $Sp(4, \mathbb{C})$)

```
atlas> orb
Value: (simply connected root datum of Lie type 'C2',(),[ 1, 1 ])
atlas> diagram(orb)
Value: [0,2]
```

This is the weighted Dynkin diagram.

```
atlas> Levi_of_H([1,1],G)
Value: ([0],[  1, -1 ])
atlas> Bala_Carter_Levi (orb)
Value: (root datum of Lie type 'A1.T1',[  1, -1 ])
atlas> set (BC,)=Bala_Carter_Levi (orb)
atlas> fundamental_coweights(BC)
Value: [[  1, -1 ]/2]

atlas> dim_nilpotent (orb)
Value: 6
atlas> minimal_orbits(G)
Value: [(simply connected root datum of Lie type 'C2',(),[ 1, 0 ])]
atlas> principal_orbit (G)
Value: (simply connected root datum of Lie type 'C2',(),[ 3, 1 ])
atlas> subregular_orbits(G)
Value: [(simply connected root datum of Lie type 'C2',(),[ 1, 1 ])]
```

Real Groups in `atlas`

- A real group G may be specified by a complex group $\mathbf{G}$ and a Cartan involution θ. The complexification of the maximal compact subgroup K is then $\mathbf{G}^\theta$. This determines the real form.
- In `atlas`, the complex group $\mathbf{G}$, a maximal torus $\mathbf{T}$, and a Borel $\mathbf{B}$ are fixed (by fixing the root datum). Instead of moving between Cartan subgroups of a fixed real group, we change the Cartan involution, which then changes the real forms of $\mathbf{T}$.
- For a real group G, the Cartan involutions are given by a (finite) set of $\mathbf{K}\backslash\mathbf{G}/\mathbf{B}$ orbits (`kgb` elements), related by Cayley transforms and cross actions.
- In `atlas`, a given `kgb` element `x` specifies both the root datum and the involution; also: which simple roots are real, complex, noncompact imaginary, compact.

Real Orbits

- A real nilpotent orbit is a real form of a complex nilpotent orbit $\mathcal{O}$; or a **K**-orbit of nilpotent elements in the -1 eigenspace of the Cartan involution θ in the Lie algebra of **G**. Here $\mathbf{K} = \mathbf{G}^{\theta}$.
- In `atlas`, a real nilpotent orbit in a real Lie algebra $\mathfrak{g}$ is given by a pair (H, x), where H is the semisimple element determining $\mathcal{O}$, and `x` is a `kgb` element satisfying certain compatibility conditions.

Example (Real orbits in $Sp(4, \mathbb{R})$)

```
atlas> set G=Sp(4,R)
Variable G: RealForm
atlas> for orb in real_nilpotent_orbits(G) do prints(orb) od
[ 0, 0 ]KGB element #0()
[ 1, 0 ]KGB element #1()
[ 1, 0 ]KGB element #2()
[ 1, 1 ]KGB element #2()
[ 1, 1 ]KGB element #3()
[ 1, 1 ]KGB element #0()
[ 3, 1 ]KGB element #0()
[ 3, 1 ]KGB element #1()
```

Listing the Real Forms of an Orbit $\mathcal{O}$

Algorithm: Given a complex nilpotent orbit $\mathcal{O}$ with semisimple element H and distinguished in the Bala-Carter Levi $\mathbf{L}$, and a real form G of $\mathbf{G}$,

- Find the real forms L of $\mathbf{L}$ in G: Given a `kgb` element `x`, check whether θ_x preserves $\mathbf{L}$. Several `kgb` elements may define the same real form of $\mathbf{L}$. This is easy to do in `atlas`, using code written for other calculations.
- For each L, check whether H defines a real orbit in $\mathfrak{l}_0$.
- If we had the element X (which `atlas` doesn't)*, this would be easy: Check that θ_x fixes H and takes X to $-X$.
- One can also write down a condition in terms of roots and weights suitable for `atlas`.
- Check for conjugacy: $(H_1, \mathtt{x}_1)$ and $(H_2, \mathtt{x}_2)$ may specify the same orbit.

Real Orbits in F_4 (split)

```
[ 0, 0, 0, 0 ]KGB element #0()
[ 2, 3, 2, 1 ]KGB element #7()
[ 2, 4, 3, 2 ]KGB element #10()
[ 2, 4, 3, 2 ]KGB element #1()
[ 3, 6, 4, 2 ]KGB element #11()
[ 3, 6, 4, 2 ]KGB element #0()
[ 4, 6, 4, 2 ]KGB element #5()
[ 4, 6, 4, 2 ]KGB element #11()
[ 4, 6, 4, 2 ]KGB element #0()
[ 4, 8, 6, 4 ]KGB element #3()
[ 4, 8, 6, 3 ]KGB element #0()
[  6, 10,  7,  4 ]KGB element #0()
[  6, 10,  7,  4 ]KGB element #7()
[  5, 10,  7,  4 ]KGB element #1()
[  6, 11,  8,  4 ]KGB element #0()
[  6, 11,  8,  4 ]KGB element #8()
[  6, 12,  8,  4 ]KGB element #0()
[  6, 12,  8,  4 ]KGB element #2()
[  6, 12,  8,  4 ]KGB element #8()
[ 10, 18, 12,  6 ]KGB element #10()
[ 10, 18, 12,  6 ]KGB element #0()
[ 10, 19, 14,  8 ]KGB element #0()
[ 10, 20, 14,  8 ]KGB element #0()
[ 10, 20, 14,  8 ]KGB element #2()
[ 14, 26, 18, 10 ]KGB element #0()
[ 14, 26, 18, 10 ]KGB element #4()
[ 22, 42, 30, 16 ]KGB element #0()
```

Component Groups

- If $\mathcal{O}$ is a complex or real nilpotent orbit, $X \in \mathcal{O}$, then the component group $A(\mathcal{O}) := C_G(X)/C_G^0(X)$ is of interest.
- Characters of $A(\mathcal{O})$ give information about the representation theory of G.
- This group depends on the isogeny of G, and is in general quite small.

Example (Component Groups in $Sp(4, \mathbb{C})$)

It is not difficult to calculate by hand that for the non-zero orbits in $\mathfrak{sp}(4, \mathbb{C})$ the centralizers in $Sp(4, \mathbb{C})$ have two components, those in the adjoint group $PSp(4, \mathbb{C})$ are connected, except for the subregular orbit, which has two components.

How to Compute Component Groups over $\mathbb{C}$

Our algorithm is based on the following result from 2002:

Theorem (G. J. McNinch, E. Sommers)

Let G be connected and reductive (over an alg. closed field of good characteristic). There is a bijection between G-conjugacy classes of:

$$(L, sZ_L^0, u) \longleftrightarrow (u, sC_G^0(u)),$$

where L is a pseudo-Levi subgroup of G with center Z_L, $sZ_L^0 \in Z_L/Z_L^0$ a coset such that $L = C_G^0(sZ_L^0)$, $u \in L$ is a unipotent element, distinguished in L, and $sC_G^0(u)$ an element in $A(u)$.

- Over $\mathbb{C}$, we can replace the element u by $X \in \mathfrak{g}$.
- This provides an algorithm to calculate *representatives for the conjugacy classes* in $A(\mathcal{O})$.
- The case of G simple adjoint complex is somewhat easier (and was analyzed by Eric Sommers in 1998).

What is a Pseudo-Levi Subgroup?

Definition

A **pseudo-Levi** subgroup L in G is the identity component of the centralizer in G of a semisimple element t in G.

- Every Levi subgroup of G is a pseudo-Levi subgroup.
- For example, in $Sp(2(p+q),\mathbb{C})$, $L = Sp(2p,\mathbb{C}) \times Sp(2q,\mathbb{C})$ is a pseudo-Levi that is not a Levi subgroup.
- While Levis may be given by subsets of the simple roots, pseudo-Levis are given by subsets of the set of simple roots and the highest root (i. e., remove vertices from the *extended* Dynkin diagram).
- We need to find them up to conjugacy. This is more difficult than for Levi subgroups: If the $\rho(L)$ are conjugate, we need to check whether the sets of simple roots are (simultaneously) conjugate.
- This slows down the algorithm.

The Algorithm for Simple Adjoint G

Why is this situation easier?

- The center of Levi subgroups is connected, so the Bala Carter Levi contributes only the identity element to $A(\mathcal{O})$.
- The center of each pseudo-Levi has cyclic component group, and all non-identity elements are conjugate. So each pseudo-Levi contributes precisely one conjugacy class to $A(\mathcal{O})$.
- The order of an element in this conjugacy class is easy to compute: It is the g.c.d. d_L of the coefficients of the simple roots NOT occurring in the highest root.

Algorithm: Given G and a complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$,

- Find all pseudo-Levi subgroups L (up to G-conjugacy) in which $\mathcal{O}$ is distinguished.
- For each L, calculate the number d_L.
- The result is a list of integers representing the conjugacy classes in $A(\mathcal{O})$.

The General Algorithm

Given a (complex) nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ and the complex group G,

- Find all pseudo-Levi subgroups L (up to G-conjugacy) in which $\mathcal{O}$ is distinguished.
- For each such L, find generators t for the center of Z_L/T_L, where $T_L = Z_L^0$ is the central (connected) torus of L.
- Keep only the *regular* t, i. e., those for which $L = Cent_G(tT_L)$.
- Check for conjugacy of $t_1 T_L$ and $t_2 T_L$ by the centralizer of X in G.
- For each t, calculate the order modulo $C_G^0(X)$.
- The result is a list of conjugacy classes in the group $A(\mathcal{O})$, with the orders, as well as a representative t.

Example: The Subregular Orbit in G_2

```
atlas> set G=adjoint(G2)
Variable G: RootDatum
atlas> set orb=subregular_orbits(G)[0]
Variable orb: ComplexNilpotent
atlas> orb
Value: (simply connected adjoint root datum of Lie type 'G2',(),[ 0, 2

atlas> print_component_info(orb)

Component info for orbit:
H=[ 0, 2 ] diagram:[0,2] dim:10
orders:[1,2,3]
pseudo_Levi  Generators
2A1          [[  0, -1 ]/2]
A2           [[ -1,  0 ]/3]
G2           [[ 0, 0 ]/1]
```

The component group is isomorphic to S_3. The elements listed are elements in the Lie algebra which exponentiate to t.

Orbits for $Sp(4, \mathbb{C})$

$PSp(4, \mathbb{C})$

```
Component info for orbit:
H=[ 0, 0 ] diagram:[0,0] dim:0
orders:[1]
pseudo_Levi   Generators
              [[ 0, 0 ]/1]

Component info for orbit:
H=[ 1, 0 ] diagram:[1,0] dim:4
orders:[1]
pseudo_Levi   Generators
A1            [[ 0, 0 ]/1]

Component info for orbit:
H=[ 0, 2 ] diagram:[0,2] dim:6
orders:[1,2]
pseudo_Levi   Generators
A1            [[ 0, 0 ]/1]
2A1           [[ -1,  0 ]/2]

Component info for orbit:
H=[ 2, 2 ] diagram:[2,2] dim:8
orders:[1]
pseudo_Levi   Generators
C2            [[ 0, 0 ]/1]
```

$Sp(4, \mathbb{C})$

```
Component info for orbit:
H=[ 0, 0 ] diagram:[0,0] dim:0
orders:[1]
pseudo_Levi   Generators
              [[ 0, 0 ]/1]

Component info for orbit:
H=[ 1, 0 ] diagram:[1,0] dim:4
orders:[1,2]
pseudo_Levi   Generators
A1            [[ 0, 0 ]/1,[ 0, 1 ]/2]

Component info for orbit:
H=[ 1, 1 ] diagram:[0,2] dim:6
orders:[1,2]
pseudo_Levi   Generators
A1            [[ 0, 0 ]/1]
2A1           [[ 0, 1 ]/2]

Component info for orbit:
H=[ 3, 1 ] diagram:[2,2] dim:8
orders:[1,2]
pseudo_Levi   Generators
C2            [[ 0, 0 ]/1,[ 1, 1 ]/2]
```

Component Groups: *SO*(9) and *Spin*(9)

$SO(9)$

i	diag	dim	A(O)
0	[0,0,0,0]	0	[1]
1	[0,1,0,0]	12	[1]
2	[2,0,0,0]	14	[1,2]
3	[0,0,0,1]	16	[1]
4	[1,0,1,0]	20	[1,2]
5	[0,2,0,0]	22	[1,2]
6	[2,2,0,0]	24	[1,2]
7	[0,0,2,0]	24	[1]
8	[0,2,0,1]	26	[1]
9	[2,1,0,1]	26	[1]
10	[2,0,2,0]	28	[1,2,2,2]
11	[2,2,2,0]	30	[1,2]
12	[2,2,2,2]	32	[1]

$Spin(9)$

i	diag	dim	A(O)
0	[0,0,0,0]	0	[1]
1	[0,1,0,0]	12	[1]
2	[2,0,0,0]	14	[1,2]
3	[0,0,0,1]	16	[1,2]
4	[1,0,1,0]	20	[1,2]
5	[0,2,0,0]	22	[1,2]
6	[2,2,0,0]	24	[1,2]
7	[0,0,2,0]	24	[1]
8	[0,2,0,1]	26	[1,2]
9	[2,1,0,1]	26	[1,2]
10	[2,0,2,0]	28	[1,2,2,2,4]
11	[2,2,2,0]	30	[1,2]
12	[2,2,2,2]	32	[1,2]

A Non-Simple Reductive Example

```
atlas> set rd=GL(2)*SL(2)
atlas> print_component_info(rd)

Component info for orbit:
H=[ 0, 0, 0 ] diagram:[0,0] dim:0
orders:[1]
pseudo_Levi  Generators
             [[ 0, 0, 0 ]/1]

Component info for orbit:
H=[  1, -1,  0 ] diagram:[2,0] dim:2
orders:[1]
pseudo_Levi  Generators
A1           [[ 0, 0, 0 ]/1]

Component info for orbit:
H=[ 0, 0, 1 ] diagram:[0,2] dim:2
orders:[1,2]
pseudo_Levi  Generators
A1           [[ 0, 0, 0 ]/1,[ 0, 0, 1 ]/2]

Component info for orbit:
H=[  1, -1,  1 ] diagram:[2,2] dim:4
orders:[1,2]
pseudo_Levi  Generators
2A1          [[ 0, 0, 0 ]/1,[ 0, 0, 1 ]/2]
```

Are the Results Correct?

- We compared the `atlas` results with tables and results in Collingwood & McGovern.
- This enabled us to make some corrections...
- Now everything we checked matches, except one component group in simply-connected E_7. (Eric Sommers tells us that this mistake is known.)
- We will continue to find ways to check the calculations.
- Please use `atlas` and let us know if there are any errors, or if you have questions!

Remark: Eric Sommers also has written software implementing his results.

Recent Improvements: The Centralizer

In addition to conjugacy classes in the component group of the centralizer, it is of interest to know the reductive part C of the centralizer of the orbit itself. (It has the same component group as the full centralizer.)

- C is the centralizer of the $\mathfrak{sl}(2)$ containing the nilpotent element X.
- The identity component of the center of the Bala-Carter Levi L is a maximal torus T_C of C.
- The roots are certain restrictions of roots of G to T_C: The set of roots with a given restriction α_C to T_C has an action of the orbit-$SL(2)$, so its span carries an $SL(2)$ representation. If that representation contains the trivial representation, then α_C is a root of C.
- Then find the coroots.

Example $Sp(4,\mathbb{C})$

Example

```
atlas> show_nilpotent_orbits_long(Sp(4))

Nilpotent orbits for C2
i   H       diagram   dim   Cent   A(O)
0   [0,0]   [0,0]     0     B2     [1]
1   [1,0]   [1,0]     4     A1     [1,2]
2   [1,1]   [0,2]     6     e      [1,2]
3   [3,1]   [2,2]     8     e      [1,2]
```

`atlas` has more information than the Lie type: Root Datum hence the isogeny.

One More Example

Example (F_4)

```
atlas> show_nilpotent_orbits_long(simply_connected(F4))

Nilpotent orbits for F4
i   H              diagram    dim  Cent  A(O)
0   [0,0,0,0]      [0,0,0,0]  0    F4    [1]
1   [2,3,2,1]      [1,0,0,0]  16   C3    [1]
2   [2,4,3,2]      [0,0,0,1]  22   A3    [1,2]
3   [3,6,4,2]      [0,1,0,0]  28   2A1   [1]
4   [4,6,4,2]      [2,0,0,0]  30   A2    [1,2]
5   [4,8,6,4]      [0,0,0,2]  30   G2    [1]
6   [4,8,6,3]      [0,0,1,0]  34   A1    [1]
7   [6,10,7,4]     [2,0,0,1]  36   2A1   [1,2]
8   [5,10,7,4]     [0,1,0,1]  36   A1    [1]
9   [6,11,8,4]     [1,0,1,0]  38   A1    [1,2]
10  [6,12,8,4]     [0,2,0,0]  40   e     [1,2,2,3,4]
11  [10,18,12,6]   [2,2,0,0]  42   A1    [1]
12  [10,19,14,8]   [1,0,1,2]  42   A1    [1]
13  [10,20,14,8]   [0,2,0,2]  44   e     [1,2]
14  [14,26,18,10]  [2,2,0,2]  46   e     [1,2]
15  [22,42,30,16]  [2,2,2,2]  48   e     [1]
```

What To Do Next

Things to do:

- Compute/display the actual component group (not just conjugacy classes)
- Component groups for real orbits.
- New and interesting questions keep coming up as we work....

The End

THANK YOU!