Steinberg theory for symmetric pairs

— joint work with Lucas Fresse (IECL, Lorraine University)

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Plan of talk

1. Basics of conormal variety
   A brief introduction & basics
2. Classical Steinberg theory for $G/B \times G/B$
   Explain classical Steinberg variety shortly
3. Mirabolic RSK by Travkin
   Introduce mirablic triple flag variety and explain results obtained by Travkin
4. Double flag variety for symmetric pairs
   Generalize the Steinberg theory to in two ways:
   Generalized Steinberg theory for type A &
   Exotic Steinberg theory for type A

Basics for Conormal variety (cf Chriss-Ginzburg [CG97])

G : algebraic group / $\mathbb{C}$ & $\mathfrak{g} = \mathrm{Lie}\,(G)$ X : smooth variety $\curvearrowleft G$
T^*X : cotangent bdle (symplectic) $\curvearrowleft G$ by Hamiltonian action

$\rightsquigarrow$ hence $\exists$ moment map : $\mu_X : T^*X \longrightarrow \mathfrak{g}^*$

$G\backslash X \ni \mathbb{O}$: G-orbit $\rightsquigarrow$ $T^*_{\mathbb{O}}X$: conormal bdle

Definition–Lemma

$\mathcal{Y} := \mu_X^{-1}(0) = \{(x,\xi) \mid \xi(z_x) = 0 \ (\forall z \in \mathfrak{g})\} \subset T^*X$: conormal variety

$= \bigsqcup_{\mathbb{O} \in X/G} T^*_{\mathbb{O}}X$ (hence the name of conormal variety)

Corollary

Assume $\#X/G < \infty$. (finitely many orbits)

1. $\mathcal{Y}$ *is equi-dimensional of* $\dim X$ *and*
2. $\mathcal{Y} = \bigcup_{\mathbb{O} \in X/G} \overline{T^*_{\mathbb{O}}X}$ *gives irred decomposition as an alg variety*

$$\mathrm{Irr}\,\mathcal{Y} \ni \overline{T^*_{\mathbb{O}}X} \xleftrightarrow{\ \sim\ } \mathbb{O} \in X/G \quad \textit{(bijection)}$$

($\because$ $T^*_{\mathbb{O}}X$: irreducible and $\dim T^*_{\mathbb{O}}X = \dim X$)

Classical Steinberg Theory

G : reductive alg grp, $B \subset G$: Borel subgrp & $\mathscr{B} := G/B$: flag variety

$\mu : T^*\mathscr{B} \to \mathcal{N}$: moment map (Springer resolution)

$$\mathcal{Z} := T^*\mathscr{B} \times_{\mathcal{N}} T^*\mathscr{B} : \text{Steinberg variety (or conormal variety)}$$
$$= \{((\mathfrak{b}_1, x_1), (\mathfrak{b}_2, x_2)) \mid x_i \in \mathfrak{b}_i \cap \mathcal{N}, x_1 + x_2 = 0\}$$

Theorem (Steinberg)

1. $\mathrm{Irr}(\mathcal{Z}) = \{\mathcal{Z}_w \mid w \in W\}$ *so that* $\mathcal{Z} = \bigcup_{w \in W} \mathcal{Z}_w$: *Irred decomp*
2. $\varphi(\mathcal{Z}_w) = \overline{\mathcal{O}_\lambda}$ *for* $\exists! \, \mathcal{O}_\lambda \in \mathcal{N}/G$: *nilpotent orbit*

In this way, we get a map $\Phi : W \to \mathcal{N}/G$

Example ($G = \mathrm{GL}_n$: Robinson-Schensted correspondence)

$\mathcal{N}/G \simeq \mathscr{P}(n)$: partitions via Jordan NF $\mathcal{O}_\lambda \longleftrightarrow \lambda$
For $x \in \mathcal{O}_\lambda \rightsquigarrow \mathrm{Irr}(\mathscr{B}_x) \simeq \mathrm{STab}_\lambda$: std tableaux
$\Phi : W = S_n \to \mathscr{P}(n) = \mathcal{N}/G$ Fiber: $\Phi^{-1}(\lambda) \simeq \mathrm{STab}_\lambda \times \mathrm{STab}_\lambda$

Generalizations 1: mirabolic case

Triple flag variaty (mirabolic case)

by Finkelberg-Ginzburg-Travkin[FGT09] Travkin[Tra09]

$G = \mathrm{GL}(V) \supset B$: Std Borel subgrp

$\mathscr{B} = G/B = \mathscr{F}(V)$: variety of complete flags

$X := \mathscr{B} \times \mathscr{B} \times V$ (note that $V \fallingdotseq \mathbb{P}(V) \simeq G/P_{\mathrm{mir}}$ & $\#X/G < \infty$)

Theorem (Zelevinsky)

$$X/G \simeq \widetilde{W} := \{\tilde{w} = (w, \sigma) \mid w \in S_n, \sigma\text{: decreasing seq in } w\}$$

$T^*X = T^*\mathscr{B} \times T^*\mathscr{B} \times T^*V \subset (\mathscr{B} \times \mathcal{N})^2 \times V \times V^*$: cotangent bundle

Proposition (Steinberg variety = conormal variety)

$$\mathcal{Z} = \{(\mathfrak{b}_1, \mathfrak{b}_2, u_1, u_2, v, v^*) \mid u_i \in \mathfrak{b}_i \cap \mathcal{N},\ v \in V,\ v^* \in V^* \ s.t.\ (\star)\}$$

$$= \bigcup_{(w,\sigma) \in \widetilde{W}} \mathcal{Z}_{(w,\sigma)} \text{ : irred decomp}$$

$$(\star) \quad u_1 + u_2 + v \otimes v^* = 0$$

mirabolic RSK correspondence

$\mathcal{N} \subset \mathfrak{g}$: nilpotent variety (as above)

Theorem (Enhanced nilpotent cone)

$$(\mathcal{N} \times V)/G \simeq \{(\lambda, \mu) \mid |\lambda| + |\mu| = n\} \qquad \textit{Achar-Henderson [AH08]}$$

$$\simeq \mathfrak{P} := \{(\nu, \theta) \mid \nu_i \geqslant \theta_i \geqslant \nu_{i+1}\} \qquad \textit{Travkin}$$

Theorem (Mirabolic RSK (Travkin))

1. $\mathcal{Z} = \bigcup_{\tilde{w} \in RB} \mathcal{Z}_{\tilde{w}}$: *irred decomp* $\quad (\tilde{w} = (w, \sigma) \in \widetilde{W})$
2. $\varphi(\mathcal{Z}_{\tilde{w}}) = \{(u_1, u_2, v) \mid (u_1, v) \leftrightarrow (\nu, \theta), (u_2, v) \leftrightarrow (\nu', \theta)\}$
   $\longleftrightarrow (\nu, \theta, \nu') \in \mathfrak{P} \times_{\{\theta\}} \mathfrak{P}$
3. $\Phi : \widetilde{W} \to \mathfrak{P} \times_{\{\theta\}} \mathfrak{P} \quad \& \quad \Phi^{-1}(\nu, \theta, \nu') \simeq \mathrm{STab}(\nu) \times \mathrm{STab}(\nu')$
   : *mirabolic RSK correspondence* (*$\exists$ combinatorial algorithm*)

1. Henderson-Trapa[HT12] $(G, K) = (\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$
$$X = V \times G/B \quad \longleftrightarrow \quad K/Q_{\mathrm{mir}} \times G/B$$
$Q_{\mathrm{mir}} = \mathsf{Stab}_K(\text{line})$ so that $K/Q_{\mathrm{mir}} \simeq \mathbb{P}(V)$

2. Fresse-N[FN16] $(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$ $(n = p + q)$
$V = \mathbb{C}^n = V^+ \oplus V^-$ $(V^+ = \mathbb{C}^p, V^- = \mathbb{C}^q)$
$$X = K/Q_{\mathrm{mir}} \times G/P \leftrightarrow V^+ \times \mathsf{Gr}_k(V)$$
$Q_{\mathrm{mir}} = \mathsf{Stab}_K(\text{line in } V^+)$ so that $K/Q_{\mathrm{mir}} \simeq \mathbb{P}(V^+)$

3. Fresse-N [FN19] $(G, K) = (\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$ $\cdots$ this talk
$V = \mathbb{C}^{2n} = V^+ \oplus V^-$ $(V^{\pm} \simeq \mathbb{C}^n)$
$$X = K/B_K \times G/P \simeq \mathscr{F}\ell(V^+) \times \mathscr{F}\ell(V^-) \times \mathsf{Gr}_n(V)$$
$B_K = B_n \times B_n \subset K$: Borel subgroup
$P = \mathsf{Stab}_G(V^+) \subset G$: maximal psg

Exotic moment maps for Double flag var: General setting

G: reductive alg grp $/\mathbb{C}$ $\quad K \subset G$: a symmetric subgroup

By definition, $\exists \theta \in \operatorname{Aut} G$: involution s.t. $\mathfrak{k} = \mathfrak{g}^{\theta}$ (θ-fixed pts)

$\rightsquigarrow$ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$: Cartan decomp $\quad (\mathfrak{s} = \mathfrak{g}^{-\theta} = \{x \in \mathfrak{g} \mid \theta(x) = -x\})$

- $P \subset G$: parabolic in G
- $Q \subset K$: parabolic in K $\quad$ (afterwards $Q = B_K$: Borel subgrp)

$\mathfrak{X} := K/Q \times G/P$: double flag variety for symmetric pair ([NO11])

Assumption (Always assume below)

Assume $\#\mathfrak{X}/K < \infty$ $\quad (\iff \#Q\backslash G/P < \infty)$ $\overset{\text{def}}{\iff}$ $\mathfrak{X}$ *is of finite type*

Many examples: N-Ochiai [NO11] &

Even classification (Q or P: Borel) by N-He-Ochiai-Oshima [HNOO13] moment map: $$\begin{array}{ccc} \mu_{\mathfrak{X}} : T^*\mathfrak{X} \simeq T^*(K/Q) \times T^*(G/P) & \longrightarrow & \mathfrak{k} \\ \Uparrow & & \Uparrow \\ (x, \mathfrak{q}_0; y, \mathfrak{p}_0) & \longmapsto & x + y^\theta \end{array}$$

Conormal variety: $$\mathcal{Y} = \mu_{\mathfrak{X}}^{-1}(0) = \coprod_{\mathbb{O} \in \mathfrak{X}/K} T^*_{\mathbb{O}}\mathfrak{X}$$
$$= \{(x, \mathfrak{q}_0; y, \mathfrak{p}_0) \mid x \in \mathfrak{q}_0^\perp, \ \ y \in \mathfrak{p}_0^\perp, \ \ x + y^\theta = 0\}$$

Here $\mathfrak{q}_0^\perp = \mathfrak{u}(\mathfrak{q}_0)$ & $\mathfrak{p}_0^\perp = \mathfrak{u}(\mathfrak{p}_0)$ are nilpotent radicals

Definition

1. generalized Steinberg map
$$\varphi^\theta : \mathcal{Y} \longrightarrow \mathcal{N}_{\mathfrak{k}}, \qquad \varphi^\theta((x, \mathfrak{q}_0; y, \mathfrak{p}_0)) = y^\theta = -x,$$
2. exotic moment map
$$\varphi^{-\theta} : \mathcal{Y} \longrightarrow \mathfrak{s}, \qquad \varphi^\theta((x, \mathfrak{q}_0; y, \mathfrak{p}_0)) = y^{-\theta} = x + y.$$

The image of $\varphi^{-\theta}$ need not be in a nilpotent variety. But if $\mathfrak{u}(\mathfrak{p})$ is abelian & $\mathfrak{g}$ is of type A, the image is in $\mathcal{N}_{\mathfrak{s}}$, the nilpotent variety in $\mathfrak{s}$ (see [FN16]).

Symmetric pair of type AIII

From now on, concentrate on the symmetric pair

$$(G, K) = (\mathrm{GL}_{2n}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \xleftarrow{\text{complexify}} (\mathrm{U}(n,n), \mathrm{U}(n) \times \mathrm{U}(n))$$

Complexification of Hermitian symmetric of tube type

- $Q = B_K = B_n^+ \times B_n^-$: Borel subgrp in K,
- $P_S = K \ltimes \mathrm{M}_n = \{ \begin{pmatrix} a & z \\ 0 & b \end{pmatrix} \mid a, b \in \mathrm{GL}_n, z \in \mathrm{M}_n \}$: Siegel psg in G

$\rightsquigarrow$ double flag variety:

$$\mathfrak{X} = K/B_K \times G/P \simeq (\mathscr{F}\ell(\mathbb{C}^n) \times \mathscr{F}\ell(\mathbb{C}^n)) \times \mathrm{Gr}_n(\mathbb{C}^{2n})$$

$\mathscr{F}\ell(V)$: set of complete flags in V

$\mathrm{Gr}_r(V)$: Grassmannian of r-subsp in V

Lemma ([FN16])

In the above setting, exotic moment map takes $\mathcal{Y}$ (conormal variety) to $\mathcal{N}_{\mathfrak{s}}$ (nilpotent variety for G/K) so that it induces $\Phi^{-\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}_{\mathfrak{s}}/\operatorname{Ad} K$

Two moment maps (Steinberg & exotic) $\varphi^{\pm\theta}$ induce orbit maps

$$\Phi^{\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}_{\mathfrak{k}}/\operatorname{Ad} K, \qquad \Phi^{-\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}_{\mathfrak{s}}/\operatorname{Ad} K$$

Call them generalized Steinberg map & exotic moment map

by abuse of terminology

Key point

$$G/P_S \simeq \operatorname{Gr}_n(\mathbb{C}^{2n}) \simeq \mathrm{M}^{\circ}_{2n,n}/\mathrm{GL}_n \qquad (\mathrm{M}^{\circ}_{2n,n}\text{: full rank matrices})$$

Identification via

$$A \in \mathrm{M}^{\circ}_{2n,n}/\mathrm{GL}_n \longleftrightarrow [A] := \operatorname{Im} A \in \operatorname{Gr}_n(\mathbb{C}^{2n})$$

maximal parabolic subgroup $P_A = \operatorname{Stab}_G([A]) \in G/P_S$

$\rightsquigarrow$ rewrite the conormal fiber using

$$A \in \mathrm{M}^{\circ}_{2n,n} \leftrightarrow (\mathfrak{b}_K, \mathfrak{p}_A) \in K/B_K \times G/P_S = \mathfrak{X}$$

Fiber at $(\mathfrak{b}_K, \mathfrak{p}_A) \in \mathbb{O}_A$ $\quad$ ($\mathbb{O}_A$: orbit through $(\mathfrak{b}_K, \mathfrak{p}_A)$):

$$\mathcal{Y}_A = \{ y = \begin{pmatrix} a & z \\ w & b \end{pmatrix} \in \mathrm{M}_{2n} \mid a \in \mathfrak{u}_n^+, b \in \mathfrak{u}_n^-, \ \operatorname{Im} y \subset [A] \subset \operatorname{Ker} y \}$$

$\exists$ two moment maps (Steinberg & exotic)

$$\Phi^{\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}_{\mathfrak{k}}/\operatorname{Ad} K, \qquad \Phi^{-\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}_{\mathfrak{s}}/\operatorname{Ad} K$$

Recall: $A \in \mathrm{M}^{\circ}_{2n,n} \leftrightarrow [A] = \operatorname{Im} A \in \mathrm{Gr}_n(\mathbb{C}^{2n}) \leftrightarrow \mathfrak{p}_A = \mathrm{Stab}([A])$

Lemma

A complete representative of the orbits for the action

$$B_n^+ \times B_n^- \curvearrowright \mathrm{M}^{\circ}_{2n,n} \curvearrowleft \mathrm{GL}_n(\mathbb{C}) \quad \textit{is given by}$$

$$(B_n^+ \times B_n^-)\backslash \mathrm{M}^{\circ}_{2n,n}/\mathrm{GL}_n(\mathbb{C}) \simeq \{ \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathrm{M}^{\circ}_{2n,n} \mid \tau_1, \tau_2 \in T_n \}/S_n,$$

T_n: the set of partial permutations and S_n acts by right

$$\rightsquigarrow \quad \mathfrak{X}/K \simeq (T_n \times T_n)^{\circ}/S_n,$$

where $^{\circ}$ denotes "full rank" & S_n acts diagonally

Thus two moment maps induces combinatorial maps:

$$\Phi^{\theta} : (T_n \times T_n)^{\circ}/S_n \to \mathscr{P}(n)^2 \quad \& \quad \Phi^{-\theta} : (T_n \times T_n)^{\circ}/S_n \to \mathscr{P}(2n)^{\pm}$$

Generic orbits in $\mathfrak{X}/K \simeq \mathrm{Gr}_n(\mathbb{C}^{2n})/B_K$

Below, we will concentrate on the orbit through $A = \begin{pmatrix} \tau \\ 1_n \end{pmatrix}$
for a partial permutation $\tau \in T_n$

Let us denote it by $\mathbb{O}_\tau := \mathbb{O}_A$

$\rightsquigarrow$ $\exists$ combinatorial algorithm which gives

$$(\lambda, \mu) \in \mathscr{P}(n)^2 \ \& \ \Lambda \in \mathscr{P}_n^{\pm}$$

from the above generic $A \in \mathrm{M}^{\circ}_{2n,n}$

Thus we have arrived at the problem to give

$$\Phi^{\theta} : T_n \longrightarrow \mathscr{P}(n)^2 \quad \& \quad \Phi^{-\theta} : T_n \longrightarrow \mathscr{P}(2n)^{\pm}$$

explicitly

generalized Steinberg map $\Phi^\theta : \mathfrak{X}/K \longrightarrow \mathcal{N}_{\mathfrak{k}}/\operatorname{Ad} K$

Take $\tau \in T_n$ and recall $A = \begin{pmatrix} \tau \\ 1_n \end{pmatrix} \in \mathrm{M}^\circ_{2n,n}$ & orbit through A: $\mathbb{O}_\tau := \mathbb{O}_A$

$\rightsquigarrow$ gen Steinberg map: $\Phi^\theta(\mathbb{O}_\tau) = \mathrm{O}_\lambda \times \mathrm{O}_\mu \quad (\exists \lambda, \mu \in \mathscr{P}(n))$

In this setting, Image of conormal fiber is

$$\varphi^\theta(\mathcal{Y}_A) = \{(\tau w, -w\tau) \in \mathfrak{u}_n^+ \times \mathfrak{u}_n^- \mid w \in \mathrm{M}_n\}$$

Interesting coincidence

This is the very same as the conormal variety for the action:

$$B_n^+ \times B_n^- \curvearrowright \mathrm{M}_n$$

Notice the resembrance with the Steinberg's setting:

$$(G/B_n^+ \times G/B_n^-)/G \simeq B_n^+ \backslash G / B_n^- \quad \rightsquigarrow \quad B_n^+ \times B_n^- \curvearrowright \mathrm{GL}_n \subset \mathrm{M}_n$$

Repalce GL_n by the full matrix space M_n!!

Proposition

Consider the action: $B_n^+ \curvearrowright \mathrm{M}_n \curvearrowleft B_n^-$

1. *orbit decomposition:* $B_n^+ \backslash \mathrm{M}_n / B_n^- \simeq T_n$
2. *For* $T^* \mathrm{M}_n \simeq \mathrm{M}_n^2 \ni (x, y)$, *moment map :* $\mu(x, y) = (xy, -yx)$
3. *The moment map image of irred comp* $\mathcal{Y}_\tau$ *generates a pair of nilpotent* GL_n*-orbits* $\mathcal{O}_\lambda \times \mathcal{O}_\mu$ *&*
   The correspondence $\tau \mapsto (\lambda, \mu)$ *agrees with* Φ^θ

Theorem

There is a combinatorial algorithm to give the correspondence:

$$\tau \mapsto (\lambda, \mu) \qquad \text{or} \qquad \Phi^\theta : T_n \to \mathscr{P}(n)^2$$

Let us explain the algorithm...

Algorithm for gen RS correspondence I

Take $\tau \in T_n \rightsquigarrow$ permutation notation:

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ \tau(1) & \tau(2) & \cdots & \tau(n) \end{pmatrix} \qquad (0 \leqslant \tau(i) \leqslant n)$$

$$= \begin{pmatrix} j_1 & j_2 & \cdots & j_r & m_1 & m_2 & \cdots & m_{n-r} \\ i_1 & i_2 & \cdots & i_r & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \text{where}$$

- $r = \operatorname{rank} \tau$ (the rank of the matrix τ);
- $j_1 < j_2 < \cdots < j_r$ & $m_1 < m_2 < \cdots < m_{n-r}$: increasing sequence
- $i_k \neq 0$ $(1 \leqslant k \leqslant r)$: permutations
- $\{\ell_1, \ldots, \ell_{n-r}\} := [n] \setminus \{i_1, \ldots, i_r\}$ & $\ell_1 < \ell_2 < \cdots < \ell_{n-r}$: increasing

We put $\sigma = \begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ i_1 & i_2 & \cdots & i_r \end{pmatrix}$: the non-degenerate part of τ

1. Robinson-Schensted-Knuth correspondence tells ([Ful97])

$$\mathrm{RSK}(\sigma) = (T_1, T_2) \in \mathrm{STab}(\nu') \times \mathrm{STab}(\nu') \quad (\exists \nu' \in \mathscr{P}(r))$$

$$\text{s.t.} \quad \mathrm{contents}(T_1) = \{i_1, \ldots, i_r\}$$

$$\mathrm{contents}(T_2) = \{j_1, \ldots, j_r\}$$

2. Put $\nu := {}^t\nu'$ (transpose is more convenient)
3. Perform row insertions (row bumping) ([Ful97]):

$$\widehat{T_1} = {}^tT_1 \leftarrow \ell_{n-r} \leftarrow \ell_{n-r-1} \leftarrow \cdots \leftarrow \ell_2 \leftarrow \ell_1,$$

$$\widehat{T_2} = {}^tT_2 \leftarrow m_{n-r} \leftarrow m_{n-r-1} \leftarrow \cdots \leftarrow m_2 \leftarrow m_1,$$

Theorem (gen RS correspondence)

Assume that $\Phi(\tau) = (\lambda, \mu)$, *i.e.,* $\overline{(\mathrm{Ad}\, G)^2 \cdot \varphi(T^*_{\mathbb{O}_\tau} M_n)} = \overline{\mathrm{O}_\lambda} \times \overline{\mathrm{O}_\mu}$

Then (λ, μ) *is given by* $(\mathrm{shape}(\widehat{T_1}), \mathrm{shape}(\widehat{T_2}))$ *above*

Example

1 $\tau=\begin{pmatrix}1&2&3&4&5\\0&4&5&0&1\end{pmatrix}$ $\quad\sigma=\begin{pmatrix}2&3&5\\4&5&1\end{pmatrix}$

$\mathrm{RSK}(\sigma)=(T_1,T_2)\quad T_1=\begin{array}{|c|c|}\hline 1&5\\\hline 4\\\cline{1-1}\end{array}\quad T_2=\begin{array}{|c|c|}\hline 2&3\\\hline 5\\\cline{1-1}\end{array}\quad \nu=\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}$

$\widehat{T_1}=\begin{array}{|c|c|}\hline 1&4\\\hline 5\\\cline{1-1}\end{array}\leftarrow 3\leftarrow 2=\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}\quad \widehat{T_2}=\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1}\end{array}\leftarrow 4\leftarrow 1=\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}\quad \lambda=\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1} \ \\\cline{1-1} \ \\\cline{1-1}\end{array}\quad \mu=\begin{array}{|c|c|}\hline \ &\ \\\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}$

2 $\tau=\begin{pmatrix}1&2&3&4&5\\0&4&0&1&0\end{pmatrix}$ $\quad\sigma=\begin{pmatrix}2&4\\4&1\end{pmatrix}$

$\mathrm{RSK}(\sigma)=(T_1,T_2)\quad T_1=\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\quad T_2=\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\quad \nu=\begin{array}{|c|c|}\hline \ &\ \\\hline\end{array}$

$\widehat{T_1}=\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}\leftarrow 5\leftarrow 3\leftarrow 2=\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\quad \widehat{T_2}=\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}\leftarrow 5\leftarrow 3\leftarrow 1=\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}\quad \lambda=\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ \\\cline{1-1} \ \\\cline{1-1}\end{array}=\mu$

$\exists$ partial permutations with $\lambda=\mu$ ($\leftarrow\!\!\sim\!\!\sim$ premutations always have $\lambda=\mu$)

Note: $\lambda\backslash\nu$ & $\mu\backslash\nu$ are column strips

i.e., there is at most one box in each row

Theorem

1. *Assume that* $\Phi(\tau) = (\lambda, \mu)$, *i.e.*, $\overline{(\mathrm{Ad}\, G)^2 \cdot \varphi(T^*_{\mathbb{O}_\tau} M_n)} = \overline{\mathrm{O}_\lambda} \times \overline{\mathrm{O}_\mu}$
   Then $(\lambda, \mu) = (\mathrm{shape}(\widehat{T_1}), \mathrm{shape}(\widehat{T_2}))$ *holds.* (*already stated*)
2. *Conversely, for* $\forall(\widehat{T_1}, \widehat{T_2}) \in \mathrm{STab}(\lambda) \times \mathrm{STab}(\mu)$,
   if $\exists \nu \in \mathscr{P}_r$ $(0 \leqslant r \leqslant n)$ *s.t.* $\lambda \backslash \nu$ & $\mu \backslash \nu$ *are column strips,*
   $\exists!\ \tau \in T_n$ *of rank* r *s.t. the corresponding tableaux is* $(\widehat{T_1}, \widehat{T_2})$
3. $\exists$ *bijection between* T_n *& the set*

$$\coprod_{(\lambda,\mu) \in \mathscr{P}_n^2,\ 0 \leqslant r \leqslant n} \left\{ \begin{array}{r} (\widehat{T_1}, \widehat{T_2}; \nu) \in \mathrm{STab}(\lambda) \times \mathrm{STab}(\mu) \times \mathscr{P}_r \mid \\ \lambda \backslash \nu, \mu \backslash \nu : \textit{column strips} \end{array} \right\}$$

Corollary

$$\# T_n = \dim \Big(\bigoplus_{r=0}^{n} \mathrm{Ind}_{S_r^2 \times S_{n-r}^2}^{S_n^2} (\mathbb{C} S_r \otimes \mathrm{sgn}^{\otimes 2}) \Big)$$

where $\mathbb{C} S_r \simeq \bigoplus_\nu \rho_\nu^{(r)} \otimes (\sigma_\nu^{(r)})^*$: *regular repr of* $S_r \times S_r$

fiber $\Phi^{-1}(\lambda, \mu)$ $((\lambda, \mu) \in \mathscr{P}_n^2) \leftrightarrow$ *isotypic comp* $\rho_\lambda^{(n)} \otimes (\rho_\mu^{(n)})^*$

$\exists$ Astonishing coincidence with mirabolic RS (private comm by Henderson)

Relation to mirabolic RS by Travkin

$V = \mathbb{C}^n$ tentatively & put $X = \mathscr{F}\ell(V) \times \mathscr{F}\ell(V) \times V$

Recall Travkin:

$X/\mathrm{GL}_n \simeq \widetilde{W} = \{\tilde{w} = (w, \sigma) \mid w \in S_n, \sigma\colon \text{decreasing in } w\}$

$\xleftrightarrow{\text{bijection}}$

$\{((\nu, T_1), (\nu', T_2); \theta) \mid T_1 \in \mathrm{STab}(\nu),\, T_2 \in \mathrm{STab}(\nu'),\, (\nu, \nu'; \theta) \in \mathfrak{P} \times_{\{\theta\}} \mathfrak{P}\}$

Our formula tells:

$B_n^+ \backslash \mathrm{M}_n / B_n^- \simeq T_n = \text{(partial permitations)}$

$\xleftrightarrow{\text{bijection}}$

$\{((\lambda, T_1), (\mu, T_2); \nu) \mid T_1 \in \mathrm{STab}(\lambda),\, T_2 \in \mathrm{STab}(\mu),\, (\lambda, \mu; \nu) \in \Upsilon_n\}$

where $\Upsilon_n := \{(\lambda, \mu; \nu) \mid \lambda \backslash \nu, \mu \backslash \nu : \text{ column strip}\}$

Coincidence

These are all in bijection!! $\exists$? geometric interpretation

Exotic moment map $\Phi^{-\theta} : \mathfrak{X}/K \longrightarrow \mathcal{N}_{\mathfrak{s}}/\operatorname{Ad} K$

Conormal fiber $\mathcal{Y}_A$ at $A = \begin{pmatrix} \tau \\ 1_n \end{pmatrix}$ $\rightsquigarrow$ the image by $\varphi^{-\theta}$ is:

$$\varphi^{-\theta}(\mathcal{Y}_A) = \{\begin{pmatrix} & z \\ w & \end{pmatrix} \mid z = \tau w \tau, \ \ \tau w \in \mathfrak{u}_n^+, \ w\tau \in \mathfrak{u}_n^- \}$$

Then $\Phi^{-\theta}(\mathbb{O}_\tau) = \mathcal{O}_\Lambda \in \mathcal{N}_{\mathfrak{s}}/\operatorname{Ad} K$ densely intersects with RHS
$\rightsquigarrow$ determines the signed Young diagram $\Lambda \in \mathscr{P}_{2n}^{\pm}$

$\exists$ A recipe how to calculate Λ from τ $\cdots$ but before that we need

Notations & terminology

- For $\lambda \in \mathscr{P}(n)$, $\lambda_{\leqslant k}$: the first k columns of λ
- $\Lambda \in \mathscr{P}_{2n}^{\pm}$: signed Young diagram (left & upper adjusted) of $2n$ boxes filled with $\pm$ in alternating manner in each rows; $\#+$'s $= \#-$'s $= n$
- $\Lambda_{\leqslant k}$: the first k columns of Λ (including $\pm$'s)
  $\#\lambda_{\leqslant k}$: # of boxes in $\lambda_{\leqslant k}$ $\#\Lambda_{\leqslant k}(\pm)$: # of $\pm$'s in $\Lambda_{\leqslant k}$
- Then $\#\lambda_{\leqslant k}$ determines λ & $\#\Lambda_{\leqslant k}(\pm)$ determines Λ completely

The Recipe $\quad r = \operatorname{rank}\tau, \quad s := n-r = \operatorname{null}\tau$

Already knew gen RS correspondence: $\operatorname{genRS}(\tau) = (\lambda, \mu) \in \mathscr{P}(n)^2$
obtained an auxirially partition $\nu \in \mathscr{P}(r)$ from nondeg part σ

$$\tau = \begin{pmatrix} j_1 & j_2 & \cdots & j_r & m_1 & m_2 & \cdots & m_s \\ i_1 & i_2 & \cdots & i_r & 0 & 0 & \cdots & 0 \end{pmatrix} \xrightarrow[]{\text{complete to a permutation}}$$

$$\widehat{\tau} = \begin{pmatrix} j_1 & j_2 & \cdots & j_r & m_1 & m_2 & \cdots & m_s & \overline{1} & \overline{2} & \cdots & \overline{s} \\ i_1 & i_2 & \cdots & i_r & \overline{1} & \overline{2} & \cdots & \overline{s} & \ell_1 & \ell_2 & \cdots & \ell_s \end{pmatrix} \in S_{2n-r}$$

where $\overline{1} < \overline{2} < \cdots < \overline{s} < 1 < 2 < \cdots < n$

Put $\eta := {}^t\operatorname{shape}(\operatorname{RS}(\widehat{\tau})) \in \mathscr{P}_{2n-r}$ (note the transpose)

Theorem (Exotic RS correspondence)

For any $0 \leqslant k \leqslant 2n$, the signed Young diagram Λ satisfies :

$$\#\Lambda_{\leqslant k}(+) = \begin{cases} \#\lambda_{\leqslant k} & k \text{ even} \\ \#\eta_{\leqslant k} - s & k \text{ odd} \end{cases} \qquad \#\Lambda_{\leqslant k}(-) = \begin{cases} \#\mu_{\leqslant k} & k \text{ even} \\ \#\nu_{\leqslant k} + s & k \text{ odd} \end{cases}$$

Conversely these conditions determine Λ uniquely

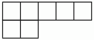
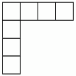
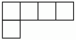
Corollary

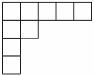
If $\tau \in S_n$ is a permutation, then Λ is a union of $\begin{array}{|c|c|c|c|c|}\hline + & - & + & - & \cdots \\ \hline - & + & - & + & \cdots \\ \hline \end{array}$ *(the length of the rows are both λ_i) for $1 \leqslant i \leqslant \ell(\lambda)$.*

($\because$ If τ is a permutation, then $\lambda = \mu = \nu = \eta$ and $n - r = 0$)

Example

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 0 & 5 & 0 & 2 & 1 & 3 \end{pmatrix} \qquad \hat{\tau} = \begin{pmatrix} \bar{1} & \bar{2} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 6 & \bar{1} & 5 & \bar{2} & 2 & 1 & 3 \end{pmatrix}$$

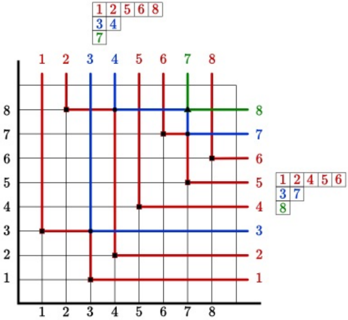
$\mathrm{genRS}(\tau) = (\lambda, \mu)$; $\lambda =$  $\mu =$  $\nu =$ 

$\eta =$ 

k	1	2	3	4	5
$\#+$	2	4	5	6	7
$\#-$	4	5	6	7	7

$$\Lambda = \begin{array}{|c|c|c|c|c|}\hline + & - & + & - & + \\ \hline - & + & - & + \\ \cline{1-4} - & + \\ \cline{1-2} + \\ \cline{1-1} - \\ \cline{1-1} - \\ \cline{1-1} \end{array}$$

Thank you for your attention!!



From:

`https://en.wikipedia.org/wiki/Viennot%27s_geometric_construction`

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