Highest weight vectors in plethysms

Soo Teck Lee

National University of Singapore

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Example. If $V_1 = \mathbb{C}^n$, $V_2 = S^m(\mathbb{C}^n)$, $V_3 = S^k(V_2) = S^k(S^m(\mathbb{C}^n))$, then $\chi_1 = h_m, \quad \chi_2 = h_k$ (complete symmetric polynomials) and $\chi_2 \circ \chi_1 = h_k \circ h_m$.

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Plethysm can be extended to an operation on symmetric functions.

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Equivalent problem in representation theory: Decompose the GL_n representation $S^k(S^m(\mathbb{C}^n))$ into a direct sum of irreducible representations and find the multiplicities.

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Equivalent problem in representation theory: Decompose the GL_n representation $S^k(S^m(\mathbb{C}^n))$ into a direct sum of irreducible representations and find the multiplicities.

The case $k \leq 4$ was done in the paper *"(GL_n, GL_m)-duality and symmetric plethysm"* by Howe in 1988.

Our goal:

Find all the GL_n highest weight vectors in $S^2(S^m(\mathbb{C}^n))$ and in $S^3(S^m(\mathbb{C}^n))$.

The results will give more explicit information on these representations. In particular,

$$\text{highest weight vectors} \Longrightarrow \text{multiplicities.}$$

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Strategy: For fixed k, define a graded algebra which can be decomposed as $\bigoplus_{m=0}^{\infty} S^k(S^m(\mathbb{C}^n))$.

Then the GL_n highest weight vectors in this algebra spans a subalgebra.

Find generators and relations, and a basis for this subalgebra.

Construction of the algebra $\bigoplus_{m=0}^{\infty} S^k(S^m(\mathbb{C}^n))$

We have

$$S^k(S^m(\mathbb{C}^n)) = (\overbrace{S^m(\mathbb{C}^n) \otimes \cdots \otimes S^m(\mathbb{C}^n)}^{k})^{\mathfrak{S}_k}$$

where $\mathfrak{S}_k$ is the symmetric group on $\{1, 2, ..., k\}$.

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Let $\mathcal{P}(\mathbb{C}^n) = \{$polynomial functions on $\mathbb{C}^n\}$.
For $g \in \mathrm{GL}_n$ and $f \in \mathcal{P}(\mathbb{C}^n)$,

$$(g.f)(X) = f(g^t X) \quad X \in \mathbb{C}^n.$$

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Then as a representation of GL_n,

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{m \geq 0} \mathcal{P}^m(\mathbb{C}^n) \cong \bigoplus_{m \geq 0} S^m(\mathbb{C}^n).$$

Next, consider

$$\begin{aligned}
\mathcal{P}(\mathrm{M}_{nk}) &= \mathcal{P}(\mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots \mathbb{C}^n) \cong \mathcal{P}(\mathbb{C}^n) \otimes \mathcal{P}(\mathbb{C}^n) \otimes \cdots \otimes \mathcal{P}(\mathbb{C}^n) \\
&\cong \Big(\bigoplus_{m_1 \geq 0} S^{m_1}(\mathbb{C}^n)\Big) \otimes \Big(\bigoplus_{m_2 \geq 0} S^{m_2}(\mathbb{C}^n)\Big) \otimes \cdots \otimes \Big(\bigoplus_{m_k \geq 0} S^{m_k}(\mathbb{C}^n)\Big) \\
&\cong \bigoplus_{m_1,\ldots,m_k \geq 0} S^{m_1}(\mathbb{C}^n) \otimes S^{m_2}(\mathbb{C}^n) \otimes \cdots \otimes S^{m_k}(\mathbb{C}^n).
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Let $\mathcal{Q} \subset \mathcal{P}(\mathrm{M}_{nk})$ with

$$\mathcal{Q} \cong \bigoplus_{m=0}^{\infty} \mathcal{Q}_m \quad \text{where} \quad \mathcal{Q}_m \cong \overbrace{S^m(\mathbb{C}^n) \otimes \cdots \otimes S^m(\mathbb{C}^n)}^{k}.$$

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Extract $\mathfrak{S}_k$ -invariants

$$\mathcal{Q}^{\mathfrak{S}_k} = \bigoplus_{m=0}^{\infty} \mathcal{Q}_m^{\mathfrak{S}_k}, \quad \text{where} \quad \mathcal{Q}_m^{\mathfrak{S}_k} \cong S^k(S^m(\mathbb{C}^n)).$$

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$\mathcal{Q}^{\mathfrak{S}_k}$ is a graded algebra, and each $S^k(S^m(\mathbb{C}^n))$ is a homogeneous component of $\mathcal{Q}^{\mathfrak{S}_k}$.

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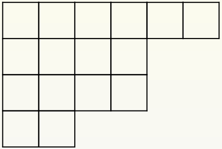
$\mathcal{Q}^{\mathfrak{S}_k}$ is a graded algebra, and each $S^k(S^m(\mathbb{C}^n))$ is a homogeneous component of $\mathcal{Q}^{\mathfrak{S}_k}$.

Next step: Find all the GL_n highest weight vectors in $\mathcal{Q}^{\mathfrak{S}_k}$.

Some notaton:

Young diagrams

$D =$  $= (6, 4, 4, 2)$ or $(6, 4, 4, 2, 0)$ etc.

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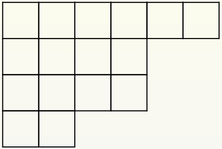
$D =$  $= (6,4,4,2)$ or $(6,4,4,2,0)$ etc.

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Young diagram D with $\ell(D) \leq n$ $\leftrightarrow$ **polynomial representation ρ_n^D of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$**

If $D = (k)$ has only one row, then $\rho_n^{(k)} \cong S^k(\mathbb{C}^n)$.

$$A_n = \left\{ t = \begin{pmatrix} t_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & t_n \end{pmatrix} \in \mathrm{GL}_n \right\}, \quad U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ \mathbf{0} & & 1 \end{pmatrix} \right\}.$$

For $D = (\lambda_1, ..., \lambda_n)$, let

$$\psi_n^D : A_n \to \mathbb{C}, \quad \psi(t) = t_1^{\lambda_1} \cdots t_n^{\lambda_n}.$$

Then

$$\rho_n^D = \text{ irreducible } \mathrm{GL}_n \text{ module with highest weight } \psi_n^D$$

and

$$\left(\rho_n^D\right)^{U_n} = \mathbb{C}\xi_D, \quad t.\xi_D = \left[\psi_n^D(t)\right] \xi_D, \quad \forall t \in A_n$$

So to find GL_n highest weight vectors in $\mathcal{Q}^{\mathfrak{S}_k}$, we consider the algebra

$$(\mathcal{Q}^{\mathfrak{S}_k})^{U_n} = \{U_n \text{ invariants in } \mathcal{Q}^{\mathfrak{S}_k}\}.$$

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Since the actions by $\mathfrak{S}_k$ and U_n commute with each other,

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A nice basis for the algebra $\mathcal{Q}^{U_n}$ may be deduced from SMT (standard monomial theory).

Project the basis to $(\mathcal{Q}^{U_n})^{\mathfrak{S}_k}$ to obtain a spanning set of $(\mathcal{Q}^{U_n})^{\mathfrak{S}_k}$.

Finally get a basis for $(\mathcal{Q}^{U_n})^{\mathfrak{S}_k}$ from this spanning set.

Standard Monomial Theory states that the algebra

$$\mathcal{P}(\mathrm{M}_{nk})^{U_n} \cong \bigoplus_{\mathbf{m}=(m_1,\ldots,m_k)\in\mathbb{Z}^k_{\geq 0}} \left[S^{m_1}(\mathbb{C}^n)\otimes S^{m_2}(\mathbb{C}^n)\otimes\cdots\otimes S^{m_k}(\mathbb{C}^n)\right]^{U_n}$$

is generated by $\{\delta_I : 1 \leq i_1 < i_2 < \cdots < i_p \leq k\}$ where

$$\delta_I = \begin{vmatrix} x_{1i_1} & x_{1i_2} & \cdots & x_{1i_p} \\ x_{2i_1} & x_{2i_2} & \cdots & x_{2i_p} \\ \vdots & \vdots & & \vdots \\ x_{pi_1} & x_{pi_2} & \cdots & x_{pi_p} \end{vmatrix}$$

and a set of products of the form $\delta_{I_1}\delta_{I_2}\cdots\delta_{I_q}$ (called standard monomials) forms a basis for $\mathcal{P}(\mathrm{M}_{nk})^{U_n}$.

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The basis elements which belong to $\mathcal{Q}^{U_n}$ forms a basis for $\mathcal{Q}^{U_n}$.

The case $k = 2$
Find the highest weight vectors in $S^2(S^m(\mathbb{C}^n))$

$\mathcal{P} = \mathcal{P}(\mathrm{M}_{n2}(\mathbb{C})) =$ polynomial algebra on $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ \vdots & \vdots \\ x_{n1} & x_{n2} \end{pmatrix}$.

$$\begin{aligned} \mathcal{P} &= \mathcal{P}(\mathrm{M}_{n2}) \cong \mathcal{P}(\mathbb{C}^n \oplus \mathbb{C}^n) \cong \mathcal{P}(\mathbb{C}^n) \otimes \mathcal{P}(\mathbb{C}^n) \\ &\cong \bigoplus_{m_1, m_2 \geq 0} S^{m_1}(\mathbb{C}^n) \otimes S^{m_2}(\mathbb{C}^n) \end{aligned}$$

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$$\mathcal{Q} \subset \mathcal{P} \quad \text{with} \quad \mathcal{Q} \cong \bigoplus_{m \geq 0} S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n).$$

$$\mathcal{Q}^{\mathfrak{S}_2} \cong \bigoplus_{m \geq 0} \left(S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n)\right)^{\mathfrak{S}_2} \cong \bigoplus_{m \geq 0} S^2\left(S^m(\mathbb{C}^n)\right).$$

By STM, the algebra $\mathcal{P}^{U_n}$ is generated by

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Proposition. The algebra $\mathcal{Q}^{U_n} \cong \bigoplus_{m \geq 0}(S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n))^{U_n}$ is a polynomial algebra generated freely by

$$\alpha = x_{11}x_{12} \quad \text{and} \quad \gamma = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix},$$

and for each m, the set

$$\{\alpha^{m-a}\gamma^a : 0 \leq a \leq m\}$$

is a basis for $(S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n))^{U_n}$.

GL_n weight of $\alpha = x_{11}x_{12}$ is $(2, 0, 0, ..., 0)$.

GL_n weight of $\gamma = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$ is $(1, 1, 0, ..., 0)$.

So GL_n weight of $\alpha^{m-a}\gamma^a$ is $(2m - a, a, 0, ..., 0)$,

$$S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n) \cong \bigoplus_{0 \leq a \leq m} \rho_n^{(2m-a,a)},$$

and $\alpha^{m-a}\gamma^a$ is the unique GL_n highest weight vector in $\rho_n^{(2m-a,a)}$.

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Now $S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n) \cong S^2(S^m(\mathbb{C}^n)) \oplus \Lambda^2(S^m(\mathbb{C}^n))$.

Which of the representations $\rho_n^{(2m-a,a)}$ are in $S^2(S^m(\mathbb{C}^n))$?

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Which of the representations $\rho_n^{(2m-a,a)}$ are in $S^2(S^m(\mathbb{C}^n))$?
Just check how the transposition $(1\ 2) \in \mathfrak{S}_2$ transforms the highest weight vectors.

The transposition $(1\ 2) \in \mathfrak{S}_2$ acts by $(1\ 2).x_{i1} = x_{i2}$ and $(1\ 2).x_{i2} = x_{i1}$.

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So $(1\ 2).\alpha = x_{12}x_{11} = \alpha, \quad (1\ 2).\gamma = \begin{vmatrix} x_{12} & x_{11} \\ x_{22} & x_{21} \end{vmatrix} = -\gamma$,

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So $\alpha^{m-a}\gamma^a \in S^2(S^m(\mathbb{C}^n))$ if and only if a is even, and this gives

$$S^2(S^m(\mathbb{C}^n)) \cong \bigoplus_{\substack{0 \le a \le m \\ a \text{ even}}} \rho_n^{(2m-a,a)}$$

$$\Lambda^2(S^m(\mathbb{C}^n)) \cong \bigoplus_{\substack{1 \le a \le m \\ a \text{ odd}}} \rho_n^{(2m-a,a)}.$$

The case $k = 3$: Find highest weight vectors in $S^3(S^m(\mathbb{C}^n))$

$\mathcal{P} = \mathcal{P}(\mathrm{M}_{n3}(\mathbb{C})) =$ polynomial algebra on $\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} \end{pmatrix}$.

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$\mathcal{Q} \subset \mathcal{P}$ with $\mathcal{Q} \cong \bigoplus_{m \geq 0} S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n)$.

$$\mathcal{Q}^{\mathfrak{S}_3} \cong \bigoplus_{m \geq 0} S^3\left(S^m(\mathbb{C}^n)\right).$$

$$\left(\mathcal{Q}^{\mathfrak{S}_3}\right)^{U_n} = \left(\mathcal{Q}^{U_n}\right)^{\mathfrak{S}_3} \cong \bigoplus_{m\geq 0} S^3\left(S^m(\mathbb{C}^n)\right)^{U_n}.$$

Proposition. (SMT + some work) For $1 \leq i,j \leq 3$, let

$$\delta_{ij} = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix} \quad \text{and} \quad \gamma_1 = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}.$$

Then the algebra $\mathcal{Q}^{U_n} \cong \bigoplus_{m\geq 0}(S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n) \otimes S^m(\mathbb{C}^n))^{U_n}$ is generated by

$$\alpha_1 = x_{11}x_{12}x_{13}, \quad \beta_2 = x_{13}\delta_{12}, \quad \beta_3 = x_{12}\delta_{13},$$

$$\gamma_1, \quad \gamma_2 = \delta_{12}\delta_{13}\delta_{23}.$$

MainTheorem. Let $\alpha_2 = \beta_2^2 + \beta_3^2 - \beta_2\beta_3$ and $\alpha_3 = 2(\beta_2^3 + \beta_3^3) - 3(\beta_2^2\beta_3 + \beta_2\beta_3^2)$. Then the set

$$\mathbf{B} = \{\xi_{(a,b,c,d,e,f)} = \alpha_1^a\alpha_2^b\alpha_3^c\gamma_1^{2d+f}\gamma_2^{2e+f} \,:\, a,b,d,e \in \mathbb{Z}_{\geq 0}, c,f \in \{0,1\}\}$$

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	m	GL_n weight
α_1	1	(3)
γ_1	1	$(1,1,1)$
α_2	2	$(4,2)$
γ_2	2	$(3,3)$
α_3	3	$(6,3)$

The GL_n weight of $\xi_{(a,b,c,d,e,f)}$ is $D = (\lambda_1, \lambda_2, \lambda_3)$ where

$$\lambda_1 = 3a + 4b + 6c + 2d + 6e + 4f$$
$$\lambda_2 = 2b + 3c + 2d + 6e + 4f$$
$$\lambda_3 = 2d + f$$

and $\xi_{(a,b,c,d,e,f)} \in S^3(S^m(\mathbb{C}^n))$ where $m = a + 2b + 3c + 2d + 4e + 3f$.

Examples

Highest weight vectors in $S^3(S^1(\mathbb{C}^n))$

$S^3(S^1(\mathbb{C}^n)) = S^3(\mathbb{C}^n) = \rho_n^{(3)}$ is irreducible and has only one highest weight vector

$$\alpha_1 = x_{11}x_{12}x_{13}.$$

Highest weight vectors in $S^3(S^2(\mathbb{C}^n))$

	m	GL_n weight
α_1^2	$1+1=2$	$(3)+(3)=(6)$
α_2	2	$(4,2)$
γ_1^2	2	$(2,2,2)$

Thus,

$$S^3(S^2(\mathbb{C}^n)) \cong \rho_n^{(6)} \oplus \rho_n^{(4,2)} \oplus \rho_n^{(2,2,2)}.$$

Highest weight vectors in $S^3(S^3(\mathbb{C}^n))$

	m	GL_n weight
α_1^3	$1+1+1=3$	$(3)+(3)+(3)=(9)$
$\alpha_1\alpha_2$	$1+2=3$	$(3)+(4,2)=(7,2)$
α_3	3	$(6,3)$
$\alpha_1\gamma_1^2$	$1+2=3$	$(3)+(2,2,2)=(5,2,2)$
$\gamma_1\gamma_2$	3	$(4,4,1)$

Thus,

$$S^3(S^3(\mathbb{C}^n)) = \rho_n^{(9)} \oplus \rho_n^{(7,2)} \oplus \rho_n^{(6,3)} \oplus \rho_n^{(5,2,2)} \oplus \rho_n^{(4,4,1)}.$$

Highest weight vectors in $S^3(S^4(\mathbb{C}^n))$

	m	GL_n weight
α_1^4	$1+1+1+1=4$	$4(3)=(12)$
$\alpha_1^2\alpha_2$	$2+2=4$	$(6)+(4,2)=(10,2)$
$\alpha_1\alpha_3$	$1+3=4$	$(3)+(6,3)=(9,3)$
$\alpha_1^2\gamma_1^2$	$2+2=4$	$(6)+(2,2,2)=(8,2,2)$
$\alpha_1(\gamma_1\gamma_2)$	$1+3=4$	$(3)+(4,4,1)=(7,4,1)$
α_2^2	$2+2=4$	$2(4,2)=(8,4)$
$\alpha_2\gamma_1^2$	$2+2=4$	$(4,2)+(2,2,2)=(6,4,2)$
$(\gamma_1^2)^2$	$2(2)=4$	$2(2,2,2)=(4,4,4)$
γ_2^2	4	$(6,6)$

$$S^3(S^4(\mathbb{C}^n)) = \rho_n^{(12)} \oplus \rho_n^{(10,2)} \oplus \rho_n^{(9,3)} \oplus \rho_n^{(8,4)} \oplus \rho_n^{(8,2,2)} \oplus \rho_n^{(7,4,1)} \oplus \rho_n^{(6,6)} \oplus \rho_n^{(6,4,2)} \oplus \rho_n^{(4,4,4)}.$$

Highest weight vectors in $S^3(S^5(\mathbb{C}^n))$

	m	GL_n weight
α_1^5	$5(1)=5$	$5(3)=(15)$
$\alpha_1^3\alpha_2$	$3+2=5$	$3(3)+(4,2)=(13,2)$
$\alpha_1\alpha_2^2$	$1+2(2)=5$	$(3)+2(4,2)=(11,4)$
$\alpha_1^2\alpha_3$	$2+3=5$	$2(3)+(6,3)=(12,3)$
$\alpha_1^3\gamma_1^2$	$3(1)+2=5$	$3(3)+(2,2,2)=(11,2,2)$
$\alpha_1(\gamma_1^2)^2$	$1+2(2)=5$	$(3)+2(2,2,2)=(7,4,4)$
$\alpha_1^2(\gamma_1\gamma_2)$	$2(1)+3=5$	$2(3)+(4,4,1)=(10,4,1)$
$\alpha_1\gamma_2^2$	$1+4=5$	$(3)+(6,6)=(9,6)$
$\alpha_1\alpha_2\gamma_1^2$	$1+2+2$	$(3)+(4,2)+(2,2,2)=(9,4,2)$
$\alpha_2\alpha_3$	$2+3=5$	$(4,2)+(6,3)=(10,5)$
$\alpha_2(\gamma_1\gamma_2)$	$2+3=5$	$(4,2)+(4,4,1)=(8,6,1)$
$\alpha_3\gamma_1^2$	$3+2=5$	$(6,3)+(2,2,2)=(8,5,2)$
$\gamma_1^2(\gamma_1\gamma_2)$	$2+3=5$	$(2,2,2)+(4,4,1)=(6,6,3)$

Thus,

$$S^3(S^5(\mathbb{C}^n)) = \rho_n^{(15)} \oplus \rho_n^{(13,2)} \oplus \rho_n^{(12,3)} \oplus \rho_n^{(11,4)} \oplus \rho_n^{(11,2,2)} \oplus \rho_n^{(10,5)} \oplus \rho_n^{(10,4,1)} \oplus \rho_n^{(9,6)} \oplus \rho_n^{(9,4,2)} \oplus \rho_n^{(8,6,1)} \oplus \rho_n^{(8,5,2)} \oplus \rho_n^{(7,4,4)} \oplus \rho_n^{(6,6,3)}.$$

Highest weight vectors in $S^3(S^6(\mathbb{C}^n))$

	m	GL_n weight
α_1^6	$6(1) = 6$	$6(3) = (18)$
$\alpha_1^4\alpha_2$	$4(1) + 2 = 6$	$4(3) + (4,2) = (16,2)$
$\alpha_1^2\alpha_2^2$	$2(1) + 2(2) = 6$	$2(3) + 2(4,2) = (14,4)$
$\alpha_1^3\alpha_3$	$3(1) + 3 = 6$	$3(3) + (6,3) = (15,3)$
$\alpha_1^4\gamma_1^2$	$4(1) + 2 = 6$	$4(3) + (2,2,2) = (14,2,2)$
$\alpha_1^2(\gamma_1^2)^2$	$2(1) + 2(2) = 6$	$2(3) + 2(2,2,2) = (10,4,4)$
$\alpha_1^3(\gamma_1\gamma_2)$	$3(1) + 3 = 6$	$3(3) + (4,4,1) = (13,4,1)$
$\alpha_1^2\gamma_2^2$	$2(1) + 4 = 6$	$2(3) + (6,6) = (12,6)$
$\alpha_1^2\alpha_2\gamma_1^2$	$2(1) + 2 + 2 = 6$	$2(3) + (4,2) + (2,2,2) = (12,4,2)$
$\alpha_1\alpha_2\alpha_3$	$1 + 2 + 3 = 6$	$(3) + (4,2) + (6,3) = (13,5)$
$\alpha_1\alpha_2(\gamma_1\gamma_2)$	$1 + 2 + 3 = 6$	$(3) + (4,2) + (4,4,1) = (11,6,1)$
$\alpha_1\alpha_3\gamma_1^2$	$1 + 3 + 2 = 6$	$(3) + (6,3) + (2,2,2) = (11,5,2)$
$\alpha_1\gamma_1^2(\gamma_1\gamma_2)$	$1 + 2 + 3 = 6$	$(3) + (2,2,2) + (4,4,1) = (9,6,3)$
α_2^3	$3(2) = 6$	$3(4,2) = (12,6)$
$\alpha_2^2\gamma_1^2$	$2(2) + 2 = 6$	$2(4,2) + (2,2,2) = (10,6,2)$
$\alpha_2(\gamma_1^2)^2$	$2 + 2(2) = 6$	$(4,2) + 2(2,2,2) = (8,6,4)$
$\alpha_2\gamma_2^2$	$2(1) + 4 = 6$	$(4,2) + (6,6) = (10,8)$
$\alpha_3(\gamma_1\gamma_2)$	$3 + 3 = 6$	$(6,3) + (4,4,1) = (10,7,1)$
$(\gamma_1^2)^3$	$3(2) = 6$	$3(2,2,2) = (6,6,6)$
$\gamma_1^2\gamma_2^2$	$2 + 4 = 6$	$(2,2,2) + (6,6) = (8,8,2)$

$$S^3(S^6(\mathbb{C}^n)) = \rho_n^{(18)} \oplus \rho_n^{(16,2)} \oplus \rho_n^{(15,3)} \oplus \rho_n^{(14,4)} \oplus \rho_n^{(14,2,2)} \oplus \rho_n^{(13,5)} \oplus \rho_n^{(13,4,1)}$$
$$\oplus\, 2\rho_n^{(12,6)} \oplus \rho_n^{(12,4,2)} \oplus \rho_n^{(11,6,1)} \oplus \rho_n^{(11,5,2)} \oplus \rho_n^{(10,8)} \oplus \rho_n^{(10,7,1)}$$
$$\oplus\, \rho_n^{(10,6,2)} \oplus \rho_n^{(10,4,4)} \oplus \rho_n^{(9,6,3)} \oplus \rho_n^{(8,8,2)} \oplus \rho_n^{(8,6,4)} \oplus \rho_n^{(6,6,6)}.$$

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Thank you