The Racah algebra and multivariate Racah polynomials

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Outline

Discrete orthogonal polynomials

What?

- family of polynomials $\phi_n(x)$, $n = 0, 1, \ldots$
- $\deg \phi_n(x) = n$
- orthogonal w.r.t. discrete measure

$$\sum_{x \in S} w(x)\phi_m(x)\phi_n(x) = \gamma_n \delta_{mn}$$

Example: Krawtchouk polynomials

Put, for $n = 0, \ldots, N$

$$\boxed{K_n(x; p, N) = {}_2F_1 \left[\begin{array}{c} -n, -x \\ -N \end{array} ; \frac{1}{p} \right]}$$

- x variable
- p parameter, $0 < p < 1$
- N grid length
- of hypergeometric type

Orthogonality:

$$\sum_{x=0}^{N} \underbrace{\binom{N}{x} p^x (1-p)^{N-x}}_{\text{weight } w(x)} K_m(x; p, N) K_n(x; p, N) = \gamma_n \delta_{mn}$$

The Askey scheme

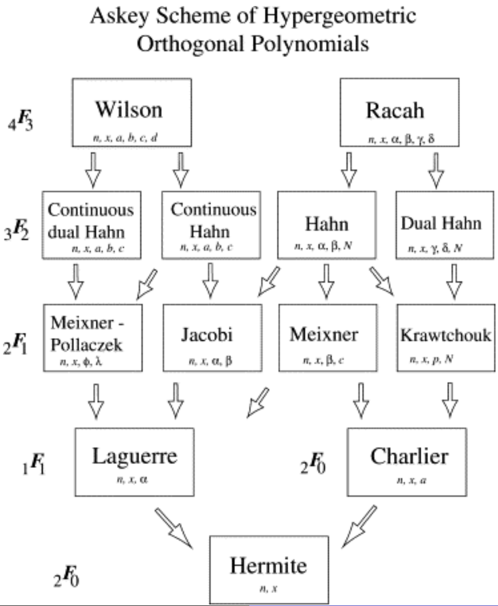
Orthogonal polynomials:

- univariate
- of hypergeometric type
- continuous or discrete orthogonality
- satisfying some generalization of Bochner's theorem

have been classified in the so-called Askey scheme

R. Askey, J. Wilson,
Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials,
Memoirs of the American Mathematical Society, 54 (1985): iv+55

R. Koekoek, P. A. Lesky, and R. F. Swarttouw.
Hypergeometric Orthogonal Polynomials and Their q-Analogues.
Springer, 2010.


Askey Scheme of Hypergeometric
Orthogonal Polynomials
$_4F_3$
Wilson
n, x, a, b, c, d
Racah
n, x, α, β, γ, δ
$_3F_2$
Continuous dual Hahn
n, x, a, b, c
Continuous Hahn
n, x, a, b, c
Hahn
n, x, α, β, N
Dual Hahn
n, x, γ, δ, N
$_2F_1$
Meixner - Pollaczek
n, x, ϕ, λ
Jacobi
n, x, α, β
Meixner
n, x, β, c
Krawtchouk
n, x, p, N
$_1F_1$
Laguerre
n, x, α
$_2F_0$
Charlier
n, x, a
$_2F_0$
Hermite
n, x

Racah polynomials

Definition

The Racah polynomials are defined as

$$r_n(\alpha,\beta,\gamma,\delta;x) := (\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n \times {}_4F_3\left[{-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1 \atop \alpha+1,\beta+\delta+1,\gamma+1};1\right]$$

- most complicated discrete OPs in Askey scheme
- appear in many different contexts
- highly complicated
- rather unpleasant to work with
- no need to remember definition!

Racah algebra

Algebra with 2 generators satisfying:

$$[K_1, K_2] = K_3$$
$$[K_2, K_3] = K_2^2 + \{K_1, K_2\} + dK_2 + e_1$$
$$[K_3, K_1] = K_1^2 + \{K_1, K_2\} + dK_1 + e_2$$

d, e_1 and e_2 structure constants

Y.A. Granovskii, A.S. Zhedanov,
Nature of the symmetry group of the $6j$-symbol.
Sov. Phys. JETP 67:1982-1985, 1988.

The standard realization

$$K_1 := x(x+\gamma+\delta+1)$$
$$K_2 := B(x)E_x - (B(x)+D(x))\mathbb{I} + D(x)E_x^{-1}$$

with the shift operator $E_x(x) = x+1$ and

$$B(x) := \frac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}$$
$$D(x) := \frac{x(x-\alpha+\gamma+\delta)(x-\beta+\gamma)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}$$

Racah polynomials

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$$r_n(\alpha,\beta,\gamma,\delta;x) := (\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n \times {}_4F_3\left[\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}; 1\right]$$

K_2 has Racah polynomials as eigenvectors:

$$\boxed{K_2 r_n(\alpha,\beta,\gamma,\delta;x) = n(n+\alpha+\beta+1) r_n(\alpha,\beta,\gamma,\delta;x)}$$

Racah polynomials

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$$\boxed{K_2 r_n(\alpha,\beta,\gamma,\delta;x) = n(n+\alpha+\beta+1) r_n(\alpha,\beta,\gamma,\delta;x)}$$

- algebra simpler than polynomials
- can be made even simpler, by taking linear combinations of generators K_1, K_2 and K_3

The centrally extended Racah algebra

Rewrite Racah algebra as:

$$C_{123} = C_{12} + C_{23} + C_{13} - C_1 - C_2 - C_3$$
$$[C_{12}, C_{23}] =: 2F$$
$$[C_{23}, C_{13}] = 2F$$
$$[C_{13}, C_{12}] = 2F$$
$$[C_{12}, F] = C_{23}C_{12} - C_{12}C_{13} + (C_2 - C_1)(C_3 - C_{123})$$
$$[C_{23}, F] = C_{13}C_{23} - C_{23}C_{12} + (C_3 - C_2)(C_1 - C_{123})$$
$$[C_{13}, F] = C_{12}C_{13} - C_{13}C_{23} + (C_1 - C_3)(C_2 - C_{123})$$

with C_1, C_2, C_3 and C_{123} central elements

S. Gao, Y. Wang, and B. Hou.
The classification of Leonard triples of Racah type.
Linear Algebra and Appl., 439:1834–1861, jan 2013.

V. X. Genest, L. Vinet, and A. Zhedanov.
The equitable Racah algebra from three $\mathfrak{su}(1,1)$ algebras.
J. Phys. A, 47:025203, 2014.

Outline

Multivariate discrete orthogonal polynomials

Two possible generalizations:

1. Macdonald-Koornwinder polynomials related to root systems
2. Tratnik-Gasper-Rahman polynomials

we are concerned with type 2

Tratnik-Gasper-Rahman polynomials

Multivariate Racah (or XX) polynomials

- are a product of univariate Racah (or XX) polynomials
- entangled: variable of first polynomial appears as parameter of subsequent one etc.
- explicit formulas cumbersome
- discrete orthogonality on subset of $\mathbb{R}^n$

M.V. Tratnik,
Some multivariable orthogonal polynomials of the Askey tableau-discrete families. *J. Math. Phys.* **32** (1991), 2337–2342.

G. Gasper and M. Rahman,
Some systems of multivariable orthogonal Askey-Wilson polynomials. In:
Theory and applications of special functions, p. 209–219, *Dev. Math.* **13**, Springer, New York, 2005.

G. Gasper and M. Rahman,
Some systems of multivariable orthogonal q-Racah polynomials.
Ramanujan J. **13** (2007), 389–405.

Can we understand these multivariate polynomials algebraically?

Multivariate Krawtchouk:

- understood in terms of representation theory of $\mathfrak{sl}_{n+1}$

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- 2 abelian algebras of (complicated) difference operators that diagonalize these polynomials
- bispectral (in sense of Duistermaat and Grünbaum)
- further algebraic foundation unclear

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Multivariate Racah:

- 2 abelian algebras of (complicated) difference operators that diagonalize these polynomials
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- further algebraic foundation unclear

P. Iliev,
A Lie-theoretic interpretation of multivariate hypergeometric polynomials,
Compositio Math. 148 (2012), no. 3, 991-1002.

J.S. Geronimo, P. Iliev,
Bispectrality of multivariable Racah-Wilson polynomials.
Constr. Approx. 31: 417-457, 2010.

Goals:

- Generalize Racah algebra to higher rank ✓
- Establish connection with multivariate Racah polynomials ✓
- Initiate algebraic study of Racah algebra: in progress

Outline

Construction:

Start from $\mathfrak{su}(1,1)$ generated by $J_\pm$ and A_0:

$$[J_-, J_+] = 2A_0, \qquad [A_0, J_\pm] = \pm J_\pm.$$

$\mathcal{U}(\mathfrak{su}(1,1))$ contains the Casimir element:

$$C := A_0^2 - A_0 - J_+ J_-$$

- Consider in the following algebra

$$\bigotimes_{i=1}^{n} \mathcal{U}(\mathfrak{su}(1,1))$$

the element

$$\Delta = \sum_{j=1}^{n} \underbrace{1 \otimes \ldots \otimes 1}_{j-1 \text{ times}} \otimes J_- \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-j \text{ times}}$$

- The following elements immediately commute with this operator:

$$C_{\{\ell\}} := \underbrace{1 \otimes \ldots \otimes 1}_{\ell-1 \text{ times}} \otimes C \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-\ell \text{ times}}$$

- we want more commuting elements

Definition

The comultiplication μ^* is an algebra morphism

$$\mu^* : \mathcal{U}(\mathfrak{su}(1,1)) \to \mathcal{U}(\mathfrak{su}(1,1)) \otimes \mathcal{U}(\mathfrak{su}(1,1))$$

acting as follows on the generators

$$\mu^*(J_\pm) = J_\pm \otimes 1 + 1 \otimes J_\pm,$$
$$\mu^*(A_0) = A_0 \otimes 1 + 1 \otimes A_0.$$

The comultiplication is coassociative:

$$(1 \otimes \mu^*)\mu^* = (\mu^* \otimes 1)\mu^*$$

- We have the following operators

$$\mathbf{C}_1 := C, \qquad \mathbf{C}_k := (\underbrace{1 \otimes \ldots \otimes 1}_{k-2 \text{ times}} \otimes \mu^*)(\mathbf{C}_{k-1})$$

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- Let $A \subset [n] := \{1, \ldots, n\}$. Using the τ map we define the generators

$$C_A := \left(\overrightarrow{\prod_{k \in [n] \setminus A}} \tau_k \right) (\mathbf{C}_{|A|})$$

$$\tau_k(A_1 \otimes \ldots \otimes A_l) := A_1 \otimes \ldots \otimes A_{k-1} \otimes 1 \otimes A_k \otimes \ldots \otimes A_l$$

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- An example

$$C_{24} = \tau_3(\tau_1(\mathbf{C}_2)) \text{ with } \mathbf{C}_2 = \mu^*(C)$$

For ease of notation we write C_{24} instead of $C_{\{2,4\}}$

Using the comultiplication we find Casimirs that commute with Δ in

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- The lower Casimirs:

$$C_1 := C \otimes 1 \otimes 1 \qquad C_2 := 1 \otimes C \otimes 1 \qquad C_3 := 1 \otimes 1 \otimes C$$

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- The lower Casimirs:

$$C_1 := C \otimes 1 \otimes 1 \qquad C_2 := 1 \otimes C \otimes 1 \qquad C_3 := 1 \otimes 1 \otimes C$$

- The intermediate Casimirs

$$C_{12} := \mu^*(C) \otimes 1 \qquad C_{23} := 1 \otimes \mu^*(C) \qquad C_{13} := \tau_2(\mu^*(C))$$

with $\tau_2(a \otimes b) := a \otimes 1 \otimes b$

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  with $\tau_2(a \otimes b) := a \otimes 1 \otimes b$
- The total Casimir

$$C_{123} := (1 \otimes \mu^*)(\mu^*(C))$$

The centrally extended Racah algebra $R(3)$

$$C_{123} = C_{12} + C_{23} + C_{13} - C_1 - C_2 - C_3$$

$$[C_{12}, C_{23}] =: 2F$$

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$$[C_{13}, C_{12}] = 2F$$

$$[C_{12}, F] = C_{23}C_{12} - C_{12}C_{13} + (C_2 - C_1)(C_3 - C_{123})$$

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with C_1, C_2, C_3 and C_{123} central operators

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V. X. Genest, L. Vinet, and A. Zhedanov.
The equitable racah algebra from three $\mathfrak{su}(1,1)$ algebras.
J. Phys. A, 47:025203, 2014.

The higher rank Racah algebra

- The Racah algebra $R(n)$ is the subalgebra of $(\mathcal{U}(\mathfrak{su}(1,1)))^{\otimes n}$ generated by the set $\{C_A | A \subset [n]\}$
- $\{C_{jk} | 1 \leq j < k \leq n\} \cup \{C_j | 1 \leq j \leq n\}$ is a generating set as

$$C_A = \sum_{\{i,j\} \subset A} C_{ij} - (|A| - 2) \sum_{i \in A} C_i$$

- If either $A \subset B$ or $B \subset A$ or $A \cap B = \emptyset$ then C_A and C_B commute.
- rank is $n-2$ as it contains abelian subalgebra of dimension $n-2$. E.g.

$$\mathcal{Y}_1 = \langle C_{12}, C_{123}, C_{1234}, \dots, C_{[n-1]} \rangle$$

or

$$\mathcal{Y}_2 = \langle C_{23}, C_{234}, C_{2345}, \dots, C_{[2\dots n]} \rangle$$

Relations:

- If either $A \subset B$ or $B \subset A$ or $A \cap B = \emptyset$ then $[C_A, C_B] = 0$
- Let K, L and M be three disjoint subsets of $[n]$. The subalgebra generated by the set

$$\{C_K, C_L, C_M, C_{K\cup L}, C_{K\cup M}, C_{L\cup M}, C_{K\cup L\cup M}\}$$

is isomorphic to the rank 1 algebra $R(3)$. Introduce the operator F:

$$2F := [C_{KL}, C_{LM}] = [C_{KM}, C_{KL}] = [C_{LM}, C_{KM}].$$

Then

$$[C_{KL}, F] = C_{LM}C_{KL} - C_{KL}C_{KM} + (C_L - C_K)(C_M - C_{KLM}),$$
$$[C_{LM}, F] = C_{KM}C_{LM} - C_{LM}C_{KL} + (C_M - C_L)(C_K - C_{KLM}),$$
$$[C_{KM}, F] = C_{KL}C_{KM} - C_{KM}C_{LM} + (C_K - C_M)(C_L - C_{KLM}).$$

How to find connection with multivariate Racah polynomials?

- rank one Racah $R(3)$ algebra acts on univariate Racah polynomials
- rank $n-2$ Racah algebra $R(n)$ should act naturally on multivariate $(n-2)$ Racah polynomials (defined by Tratnik)

How to find connection with multivariate Racah polynomials?

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Our approach:

- we work with specific realization of $R(n)$
- constructed using Dunkl operators
- module of Dunkl harmonics of fixed homogeneity

M. V. Tratnik.
Some multivariable orthogonal polynomials of the Askey tableau-discrete families.
J. Math. Phys., 32:2337–2342, 1991.

Dunkl-operators

Definition

The Dunkl operators, corresponding to the abelian group $\mathbb{Z}_2^n$, are defined as follows:

$$T_i := \partial_{x_i} + \frac{\mu_i}{x_i}(1 - R_i)$$

with real parameters $\mu_i > 0$ and reflection operators: $R_i f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, -x_i, \ldots, x_n)$.

Definition

The $\mathbb{Z}_n^2$ Laplace-Dunkl operator

$$\Delta = \sum_{i=1}^{n} T_i^2$$

Fits in our abstract framework

- $\mathfrak{su}(1,1) = \text{span}(x^2, T^2, E+\gamma)$

$$[T^2, x^2] = 4(E+\gamma), \quad [E+\gamma, x^2] = 2x^2, \quad [E+\gamma, T^2] = -2T^2$$

- The Casimir

$$C := \frac{1}{4}\left((E+\gamma)^2 - 2\,(E+\gamma) - x^2 T^2\right)$$

As $R(n)$ commutes with Δ it acts on

$$\boxed{\mathcal{H}_k(\mathbb{R}^n) = \mathcal{P}_k(\mathbb{R}^n) \cap \ker \Delta,}$$

the space of Dunkl harmonics of degree k

Strategy of proof:

- spaces $\mathcal{H}_k(\mathbb{R}^n)$ of Dunkl harmonics of fixed degree carry representation of $R(n)$
- construct explicitly 2 ON bases for Dunkl harmonics
- first ON basis diagonalizes

$$\mathcal{Y}_1 = \langle C_{12}, C_{123}, C_{1234}, \ldots, C_{[n-1]} \rangle$$

  second ON basis diagonalizes

$$\mathcal{Y}_2 = \langle C_{23}, C_{234}, C_{2345}, \ldots, C_{[2\ldots n]} \rangle$$

- connection coefficients between 2 bases can be written as multivariate Racah polynomials
- action of $R(n)$ on $\mathcal{H}_k(\mathbb{R}^n)$ can be lifted to action on the connection coefficients

Hendrik De Bie, Wouter van de Vijver
A discrete realization of the higher rank Racah algebra.
To appear, Constr. Approx., 24 pages.

Conclusions:

- Generalized Racah algebra to higher rank ✓
- Established connection with multivariate Racah polynomials ✓

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- Generalized Racah algebra to higher rank ✓
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Next steps:

- Compare with approach of Iliev (superintegrable quantum systems)
- Initiate algebraic study of Racah algebra

H. De Bie, V.X. Genest, L. Vinet, W. van de Vijver,
A higher rank Racah algebra and the $(\mathbb{Z}_2)^n$ Laplace-Dunkl operator.
J. Phys. A: Math. Theor. 51 025203 (20pp), 2018.

P. Iliev,
The generic quantum superintegrable system on the sphere and Racah operators.
Lett. Math. Phys. 107 no. 11: 2029-2045, 2017.

P. Iliev,
Symmetry algebra for the generic superintegrable system on the sphere,
J. High Energy Phys., 44 no. 2, front matter+22 pp, 2018.

Further algebraic study:

$R(n)$ is defined as subalgebra of $\mathcal{A} = \bigotimes_{i=1}^{n} \mathcal{U}(\mathfrak{su}(1,1))$

- fairly complicated algebra $\mathcal{A}$
- need n-fold tensor product for rank $n-2$ algebra
- so two factors/variables 'too many'?

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Claim

$R(n)$ can also be embedded as a subalgebra of $\mathcal{U}(\mathfrak{sl}_{n-1})$

No proof yet!

Evidence

- $R(3)$ can be embedded in a single $\mathcal{U}(\mathfrak{sl}_2)$
- statement is true for $R(4)$ (computer computation)
- statement is true for two specific realizations of $R(n)$ and $\mathcal{U}(\mathfrak{sl}_{n-1})$

I'll focus on the last step

V. X. Genest, L. Vinet, and A. Zhedanov.
The equitable racah algebra from three $\mathfrak{su}(1,1)$ algebras.
J. Phys. A, 47:025203, 2014.

H. De Bie, P. Iliev, L. Vinet,
Bargmann and Barut-Girardello models for the Racah algebra.
J. Math. Phys. **60**, 011701 (2019).

Introduce, for $\nu > 0$, the following operators

$$\begin{aligned} K_0 &= x\partial_x + \nu \\ K_- &= \partial_x \\ K_+ &= x^2\partial_x + 2\nu x. \end{aligned}$$

It is easy to verify that they satisfy the $\mathfrak{su}(1,1)$ relations:

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0.$$

Consider in n-fold tensor product

$$\Delta = K_-^{[n]} = \sum_{j=1}^{n} \partial_{x_j}$$

$$K_+^{[n]} = \sum_{j=1}^{n} (x_j^2 \partial_{x_j} + 2\nu_j x_j)$$

$$K_0^{[n]} = \sum_{j=1}^{n} x_j \partial_{x_j} + \sum_{j=1}^{n} \nu_j$$

which again generate $\mathfrak{su}(1,1)$.

We define the space of Bargmann harmonics

$$\boxed{\mathcal{H}_k(\mathbb{R}^n) = \mathcal{P}_k(\mathbb{R}^n) \cap \ker K_-^{[n]}}$$

with $\mathcal{P}_k(\mathbb{R}^n)$ the space of homogeneous polynomials of degree k.

Proposition

The space $\mathcal{P}_k(\mathbb{R}^n)$ decomposes as

$$\mathcal{P}_k(\mathbb{R}^n) = \bigoplus_{j=0}^{k} \left(K_+^{[n]}\right)^j \mathcal{H}_{k-j}(\mathbb{R}^n).$$

here $K_+^{[n]} = \sum_{j=1}^n (x_j^2 \partial_{x_j} + 2\nu_j x_j)$

This is decomposition into irreps of $\mathcal{P}_k(\mathbb{R}^n)$ under $\mathfrak{su}(1,1) \times R(n)$

Indeed, consider for a subset $B \subset [n]$, $\mathfrak{su}(1,1)$ generated by

$$K_-^B = \sum_{j\in B} \partial_{x_j}$$

$$K_+^B = \sum_{j\in B} (x_j^2 \partial_{x_j} + 2\nu_j x_j)$$

$$K_0^B = \sum_{j\in B} x_j \partial_{x_j} + \sum_{j\in B} \nu_j$$

Its Casimir is given by

$$C_B := \left(K_0^B\right)^2 - K_0^B - K_+^B K_-^B.$$

- The collection of C_B, for all $B \subset [n]$, generate $R(n)$
- $R(n)$ acts naturally on $\mathcal{H}_k(\mathbb{R}^n)$

- The operators C_B are defined using n variables.
- However, the algebra is only of rank $n-2$.

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The space $\mathcal{H}_k(\mathbb{R}^n)$ has a basis

$$(x_1 - x_2)^{k-j_1-j_2-\dots-j_{n-2}}(x_3 - x_2)^{j_1}(x_4 - x_3)^{j_2}\dots(x_n - x_{n-1})^{j_{n-2}}$$
$$= (x_1 - x_2)^k u_1^{j_1} u_2^{j_2} \dots u_{n-2}^{j_{n-2}}$$

with j_ℓ positive integers with $\sum_{\ell=1}^{n-2} j_\ell \leq k$ and $n-2$ new variables $\{u_1, u_2, \dots, u_{n-2}\}$ given by

$$u_j := \frac{x_{j+2} - x_{j+1}}{x_1 - x_2}, \qquad j \in \{1, \dots, n-2\}.$$

Gauging the operators C_B by $(x_1 - x_2)^k$ to

$$\widetilde{C_B} = (x_1 - x_2)^{-k} C_B (x_1 - x_2)^k$$

then yields a realization of $\mathcal{R}_n$ on Π_k^{n-2} with

$$\Pi_k^{n-2} = \oplus_{\ell=0}^{k} \mathcal{P}_\ell(u_1, \ldots, u_{n-2})$$

Theorem

The space Π_k^{n-2} of all polynomials of degree k in $n-2$ variables carries a realization of the rank $n-2$ Racah algebra $R(n)$. This realization is given explicitly by

$$\begin{aligned}\widetilde{C_{ij}} = &- \left(\sum_{\ell=j-1}^{i-2} u_\ell\right)^2 \left(\partial_{u_{i-2}} - \partial_{u_{i-1}}\right)\left(\partial_{u_{j-2}} - \partial_{u_{j-1}}\right) \\ &+ 2\nu_j \left(\sum_{\ell=j-1}^{i-2} u_\ell\right)\left(\partial_{u_{i-2}} - \partial_{u_{i-1}}\right) - 2\nu_i \left(\sum_{\ell=j-1}^{i-2} u_\ell\right)\left(\partial_{u_{j-2}} - \partial_{u_{j-1}}\right. \\ &+ (\nu_i+\nu_j)(\nu_i+\nu_j-1)\end{aligned}$$

and similar formulas for limiting cases such as C_{1j} and C_{2j}.

Next steps:

- there also exists realization of $\mathfrak{sl}_{n-1}$ by differential operators in the variables u_j
- it is relatively straightforward to write the generators C_{ij} of $R(n)$ as algebraic expressions of the generators of $\mathfrak{sl}_{n-1}$
- this gives us a guess for what should be the abstract embedding of $R(n)$ into $\mathcal{U}(\mathfrak{sl}_{n-1})$

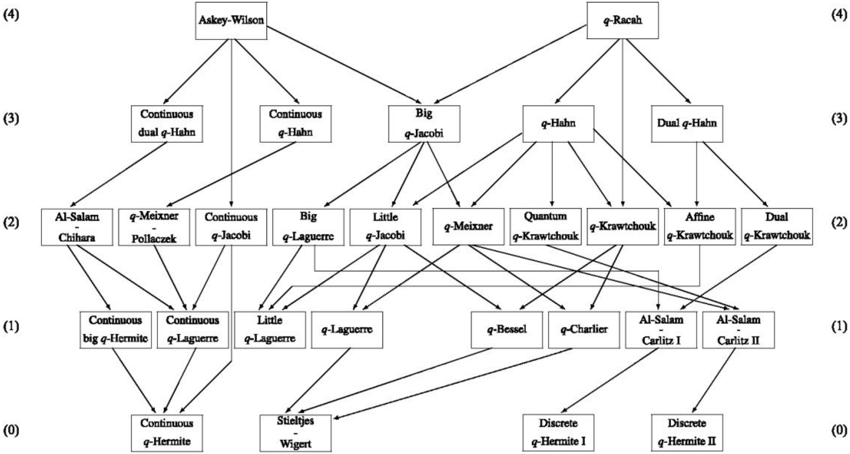
Generalizations:

What about q-deformed orthogonal polynomials?

Generalizations:

What about q-deformed orthogonal polynomials?

- classified in q Askey scheme
- most complicated: q-Racah or Askey-Wilson
- multivariate counterparts exist (Tratnik-Gasper-Rahman)


(4)
Askey-Wilson
q-Racah
(3)
Continuous dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
q-Hahn
Dual q-Hahn
(2)
Al-Salam - Chihara
q-Meixner - Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
(1)
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam - Carlitz I
Al-Salam - Carlitz II
(0)
Continuous q-Hermite
Stieltjes - Wigert
Discrete q-Hermite I
Discrete q-Hermite II

Similar construction can be made for Askey-Wilson algebra

- now starting from quantum algebra $\mathfrak{osp}_q(1|2)$
- much more complicated to obtain relations
- algebraic structure underlying multivariate Askey-Wilson (or q-Racah) polynomials
- $q = 1$ limit gives Racah case
- $q = -1$ limit gives other interesting case (Bannai-Ito algebra, Dirac operator)

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