Dirac Cohomology and Unitary Representations

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Dubrovnik, June, 2019

Joint with Chao-Ping-Dong, and Daniel Wong.

Still very much in progress

based on earlier results with Pavle Pandzic

$(\mathfrak{g}, K)$-modules

It is always easier to study representations of the Lie algebra, and then derive properties of the representations of the Lie group.

For real reductive groups, these are the $(\mathfrak{g}, K)-$modules.

Following Harish-Chandra, one associates a $(\mathfrak{g}, K)-$module to each representation of the group. Let V be an admissible representation V of G, i.e., $\dim \mathrm{Hom}(V_\delta, V) < \infty$ for all irreducible K-representations V_δ.

Let V_K be the space of K-finite vectors in V. These vectors are smooth i.e. one can differentiate the group action to get an action of the Lie algebra. $\mathfrak{g} = (\mathfrak{g}_0)_\mathbb{C}$, the complexification of the real Lie algebra acts automatically.

Definition

A $(\mathfrak{g}, K)-$module is a vector space V, with a Lie algebra action of $\mathfrak{g}$ and a locally finite action of K, which are compatible, i.e., induce the same action of $\mathfrak{k}_0 =$ the Lie algebra of K. (If K is disconnected, one requires also that the action $\mathfrak{g} \otimes V \to V$ is $K-$equivariant). Such a V can be decomposed under K as

$$V = \bigoplus_{\delta \in \hat{K}} m_\delta V_\delta.$$

V is called a Harish-Chandra module if it is finitely generated and all $m_\delta < \infty$.

Casimir element

In general, can define $\mathrm{Cas}_{\mathfrak{g}}$ in the center of the enveloping algebra $U(\mathfrak{g})$:

Fix a nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$ (e.g. $\mathrm{tr}\, XY$).

Take dual bases b_i, d_i of $\mathfrak{g}$ with respect to B.

Write

$$\mathrm{Cas}_{\mathfrak{g}} = \sum b_i d_i.$$

Infinitesimal character

The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is a polynomial algebra; one of the generators is $\mathrm{Cas}_{\mathfrak{g}}$.

All elements of $Z(\mathfrak{g})$ act as scalars on irreducible modules.

This defines the infinitesimal character of M, $\chi_M : Z(\mathfrak{g}) \to \mathbb{C}$.

Harish-Chandra proved that $Z(\mathfrak{g}) \cong P(\mathfrak{h}^*)^W$, so infinitesimal characters correspond to $\mathfrak{h}^*/W$.

($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$; in examples, the diagonal matrices. W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$; it is a finite reflection group.)

The Clifford algebra for G

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

($\mathfrak{k}$ and $\mathfrak{p}$ are the ± 1 eigenspaces of the Cartan involution;

$\mathfrak{k}$ is the complexified Lie algebra of the maximal compact subgroup K of G.)

Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B:

the associative algebra with 1, generated by $\mathfrak{p}$, with relations

$$xy + yx + 2B(x, y) = 0.$$

The Dirac operator for G

Let b_i be any basis of $\mathfrak{p}$; let d_i be the dual basis with respect to B.
Dirac operator:

$$D = \sum_i b_i \otimes d_i \qquad \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

D is independent of b_i and K-invariant.

D^2 is the spin Laplacian:

$$D^2 = -\operatorname{Cas}_{\mathfrak{g}} \otimes 1 + \operatorname{Cas}_{\mathfrak{k}_\Delta} + \text{constant}.$$

Here $\operatorname{Cas}_{\mathfrak{g}}$, $\operatorname{Cas}_{\mathfrak{k}_\Delta}$ are the Casimir elements of $U(\mathfrak{g})$, $U(\mathfrak{k}_\Delta)$;

$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g}) \quad \text{and} \quad \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p}).$$

The constant is explicitly computed as $-||\rho||^2 + ||\rho_{\mathfrak{k}}||^2$.

Atiyah-Schmid, Schmid, and Parthasarthy use these ideas to construct Discrete Series.

Dirac cohomology

Motivated by the Dirac inequality and its uses to compute spectral gaps, Vogan introduced the notion of Dirac Cohomology.

Let M be an admissible $(\mathfrak{g}, K)$-module. Let S be a spin module for $C(\mathfrak{p})$; it is constructed as $S = \bigwedge \mathfrak{p}^+$ for $\mathfrak{p}^+ \subset \mathfrak{p}$ max isotropic.

Then D acts on $M \otimes S$.

Dirac cohomology of M:

$$H_D(M) = \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D$$

$H_D(M)$ is a module for the spin double cover $\widetilde{K}$ of K. It is finite-dimensional if M is of finite length.

If M is unitary, then D is self adjoint wrt an inner product. So

$$H_D(M) = \operatorname{Ker} D = \operatorname{Ker} D^2,$$

and $D^2 \geq 0$ (Dirac inequality).

Vogan's conjecture

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. View $\mathfrak{t}^* \subset \mathfrak{h}^*$ via extension by 0 over $\mathfrak{a}$.

The following was conjectured by Vogan in 1997, and proved by Huang-P. in 2002.

Theorem

Assume M has infinitesimal character and $H_D(M)$ contains a $\widetilde{K}$-type E_γ of highest weight $\gamma \in \mathfrak{t}^*$.

Then the infinitesimal character of M is $\gamma + \rho_{\mathfrak{k}}$ (up to Weyl group $W_{\mathfrak{g}}$).

Vogan's conjecture - structural version

Let $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}) \cong Z(\mathfrak{k}_\Delta)$ be the homomorphism corresponding under Harish-Chandra isomorphism to the restriction map $P(\mathfrak{h}^*)^{W_\mathfrak{g}} \to P(\mathfrak{t}^*)^{W_\mathfrak{k}}$.

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For any $z \in Z(\mathfrak{g})$, there is $a \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ such that

$$z \otimes 1 = \zeta(z) + Da + aD.$$

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This implies the above module version, since $Da + aD$ acts as zero on $H_D(M)$.

Motivation

- unitarity: Dirac inequality and its improvements.
- irreducible unitary M with $H_D \neq 0$ are interesting (discrete series, $A_\mathfrak{q}(\lambda)$ modules, unitary highest weight modules, some unipotent representations...) They should form a nice part of the unitary dual.
- H_D is related to classical topics like generalized Weyl character formula, generalized Bott-Borel-Weil Theorem, construction of discrete series, multiplicities of automorphic forms
- There are nice constructions of representations with $H_D \neq 0$; e.g., Parthasarthy and Atiyah-Schmid constructed the discrete series representations using spin bundles on G/K.

Complex Groups

Let G be a complex reductive group viewed as a real group. Let K be a maximal compact subgroup of G. Let Θ be the corresponding Cartan involution, and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G. Let $H = TA$ be a $\theta-$stable Cartan subgroup of G, with Lie algebra $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$, a $\theta-$stable Cartan subalgebra of $\mathfrak{g}_0$. We assume that $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{s}_0$.

Let $B = HN$ be a Borel subgroup of G. Let $(\lambda_L, \lambda_R) \in \mathfrak{h}_0 \times \mathfrak{h}_0$ be such that $\mu := \lambda_L + \lambda_R$ is integral. Write $\nu := \lambda_L - \lambda_R$. We can view μ as a weight of T and ν as a character of A. Let

$$X(\lambda_L, \lambda_R) := \mathrm{Ind}_B^G[\mathbb{C}_\mu \otimes \mathbb{C}_\nu \otimes \mathbb{1}]_{K-finite}.$$

Then the $K-$type with extremal weight μ occurs in $X(\lambda_L, \lambda_R)$ with multiplicity 1. Let $L(\lambda_L, \lambda_R)$ be the unique irreducible subquotient containing this $K-$type.

Admissible Representations

Theorem ([Zh], [PRV])

1. *Every irreducible admissible* $(\mathfrak{g}, K)$ *module is of the form* $L(\lambda_L, \lambda_R)$.
2. *Two such modules* $L(\lambda_L, \lambda_R)$ *and* $L(\lambda'_L, \lambda'_R)$ *are equivalent if and only if the parameters are conjugate by* $\Delta(W) \subset W_c \cong W \times W$. *In other words, there is* $w \in W$ *such that* $w\mu = \mu'$ *and* $w\nu = \nu'$.
3. $L(\lambda_L, \lambda_R)$ *admits a nondegenerate hermitian form if and only if there is* $w \in W$ *such that* $w\mu = \mu$, $w\nu = -\overline{\nu}$.

This result is a special case of the more general Langlands classification, which can be found for example in [Kn], Theorem 8.54.

Spin Representation

We next describe the spin representation of the group $\widetilde{K}$. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{b},\mathfrak{h})}\alpha$. Let r denote the rank of $\mathfrak{g}$.

Lemma

The spinor representation Spin viewed as a $\widetilde{K}$-module is a direct sum of $[\frac{r}{2}]$ copies of the irreducible representation $E(\rho)$ of $\widetilde{K}$ with highest weight ρ.

Lemma 4 implies that in calculating $H_D(\pi)$ for unitary π, one can replace *Spin* by $E(\rho)$ and then in the end simply multiply the result by multiplicity $[\frac{r}{2}]$.

So a unitary representation $L(\lambda_L, \lambda_R)$ has nonzero Dirac cohomology if and only if there is $(w_1, w_2) \in W_c$ such that

$$w_1\lambda_L + w_2\lambda_R = 0, \qquad w_1\lambda_L - w_2\lambda_R = \tau + \rho \tag{1}$$

where τ is the highest weight of a $\widetilde{K}-$type which occurs in $L(\lambda_L, \lambda_R) \otimes E(\rho)$. More precisely

$$[H_D(\pi) : E(\tau)] = \sum_{\mu} [\frac{r}{2}]\,[\pi : E(\mu)]\,[E(\mu)\otimes E(\rho) : E(\tau)], \tag{2}$$

Dirac Cohomology for Unitary Representations

Write $\lambda := \lambda_L$. The first equation in (1) implies that $\lambda_R = -w_2^{-1} w_1 \lambda$. The second one says that $2w_1\lambda = \tau + \rho$, so that $w_1\lambda$ must be regular, and $2w_1\lambda$ regular integral. Replace $w_1\lambda$ by λ. Thus we can write the parameter of π as $(\lambda, -s\lambda)$ with λ dominant, and $s \in W$. Since $L(\lambda, -s\lambda)$ is assumed unitary, it is hermitian. So there is $w \in W$ such that

$$w(\lambda + s\lambda) = \lambda + s\lambda, \quad w(\lambda - s\lambda) = -\lambda + s\lambda. \tag{3}$$

This implies that $w\lambda = s\lambda$, so $w = s$ since λ is regular, and $ws\lambda = s^2\lambda = \lambda$. So s must be an involution.

Thus to compute $H_D(\pi)$ for π that are unitary, we need to know

1. $L(\lambda, -s\lambda)$ that are unitary with

   $$2\lambda = \tau + \rho, \tag{4}$$

   in particular 2λ is regular integral,
2. The multiplicity

   $$\left[\ L(\lambda, -s\lambda) \otimes E(\rho) \ :\ E(\tau)\ \right]. \tag{5}$$

Unitary Dual

Theorem (Classical Groups, [B])

A hermitian module with infinitesimal character $(\lambda, -\lambda)$ *with* 2λ *integral is unitary if and only if it is unitarily induced from a unipotent representation. The unipotent representations for all complex groups are listed in [BP]. For the classical groups, (aside from the trivial representation) they are*

Type A $\lambda = (a, \ldots, -a, b - 1/2, \ldots, -b + 1/2)$, $a, b \in \mathbb{N}$,

Type B $\Theta-$*lifts of the trivial representation of an Sp in the stable range,*
$\lambda = (-K_0 + 1/2, \ldots, -1/2, -N_0, \ldots, -1)$ $K_0 \geq N_0$

Type C *The components of the metaplectic representation,*
$\lambda = (-K_0 + 1/2, \ldots, -1/2)$,

Type D $\Theta-$*lifts of the metaplectic representation,*
$\lambda = (-N_0, \ldots, -1, 0, -K_0 + 1/2, \ldots, -1/2)$
$K_0 \leq N_0$.

Sketch of Proof I

We need to compute the Dirac cohomology of a unitarily induced module. We consider the Dirac cohomology of a representation π which is unitarily induced from a unitary representation of the Levi component M of a parabolic subgroup $P = MN$.

We write $\pi := \mathrm{Ind}_P^G[\mathbb{C}_\xi \otimes \pi_\mathfrak{m}]$ where ξ is a unitary character of M, and $\pi_\mathfrak{m}$ is a unitary representation of M such that the center of M acts trivially. It is straightforward that $\pi_\mathfrak{m}$ has Dirac cohomology if and only if $\mathbb{C}_\xi \otimes \pi_\mathfrak{m}$ has Dirac cohomology.

The representation $\pi_\mathfrak{m} = L_\mathfrak{m}(\lambda_\mathfrak{m}, -s\lambda_\mathfrak{m})$ satisfies

$$\lambda_\mathfrak{m} + s\lambda_\mathfrak{m} = \mu_\mathfrak{m}, \qquad 2\lambda_\mathfrak{m} = \mu_\mathfrak{m} + \nu_\mathfrak{m}, \tag{6}$$

$$\lambda_\mathfrak{m} - s\lambda_\mathfrak{m} = \nu_\mathfrak{m}, \qquad 2s\lambda_\mathfrak{m} = \mu_\mathfrak{m} - \nu_\mathfrak{m}, \tag{7}$$

with $s \in W_\mathfrak{m}$.

Assume that $\pi_\mathfrak{m}$ has Dirac cohomology. So

$$2\lambda_\mathfrak{m} = \mu_\mathfrak{m} + \nu_\mathfrak{m} = \tau_\mathfrak{m} + \rho_\mathfrak{m} \tag{8}$$

Sketch of Proof II

is regular integral for a positive system $\Delta_{\mathfrak{m}}$. Here $\tau_{\mathfrak{m}}$ is dominant with respect to $\Delta_{\mathfrak{m}}$, and $\rho_{\mathfrak{m}}$ is the half sum of the roots in $\Delta_{\mathfrak{m}}$. Also,

$$\left[\pi_{\mathfrak{m}} \otimes F(\rho_{\mathfrak{m}}) \ : \ F(\tau_{\mathfrak{m}})\right] \neq 0. \tag{9}$$

For a dominant $\mathfrak{m}$-weight χ, let $F(\chi)$ denote the finite-dimensional $\mathfrak{m}$-module with highest weight χ. For a dominant $\mathfrak{g}$-weight η, let $E(\eta)$ denote the finite-dimensional $\mathfrak{g}$-module with highest weight η. We are also going to use analogous notation when χ and η are not necessarily dominant, but any extremal weights of the corresponding modules.

The lowest $K-$type subquotient of π is $L(\lambda, -s\lambda)$. It has parameters

$$\begin{aligned} \lambda &= \xi/2 + \lambda_{\mathfrak{m}}, & \mu &= \xi + \mu_{\mathfrak{m}}, \\ s\lambda &= \xi/2 + s\lambda_{\mathfrak{m}}, & \nu &= \nu_{\mathfrak{m}}. \end{aligned} \tag{10}$$

Sketch of Proof III

We assume that ξ is dominant for $\Delta(\mathfrak{n})$ the roots of N. This is justified in view of the results in [V1] and [B] which say that any unitary representation is unitarily induced irreducible from a representation $\pi_{\mathfrak{m}}$ on a Levi component with these properties. In order to have Dirac cohomology, 2λ must be regular integral; so assume this is the case. Let Δ' be the positive system such that λ is dominant. Then

$$2\lambda = \xi + \mu_{\mathfrak{m}} + \nu_{\mathfrak{m}} = \tau' + \rho'. \tag{11}$$

Here ρ' is the half sum of the roots in Δ', and τ' is dominant with respect to Δ'. In order to see that π has nonzero Dirac cohomology, we need the following lemma.

Lemma

Sketch of Proof IV

The restriction of the $\mathfrak{g}$-module $E(\rho)$ to $\mathfrak{m}$ is isomorphic to $F(\rho_{\mathfrak{m}}) \otimes \mathbb{C}_{-\rho_{\mathfrak{n}}} \otimes \bigwedge^* \mathfrak{n}$, where $F(\rho_{\mathfrak{m}})$ denotes the irreducible $\mathfrak{m}$-module with highest weight $\rho_{\mathfrak{m}}$ and $\rho_{\mathfrak{n}}$ denotes the half sum of roots in $\Delta(\mathfrak{n})$.

Proof.

Since $\mathfrak{g}$ and $\mathfrak{m}$ have the same rank, we can use Lemma 4 to replace $E(\rho)$ and $F(\rho_{\mathfrak{m}})$ by the corresponding spin modules. Recall that the spin module $Spin_{\mathfrak{m}}$ can be constructed as $\bigwedge^* \mathfrak{m}^+$, where $\mathfrak{m}^+$ is a maximal isotropic subspace of $\mathfrak{m}$. We can choose $\mathfrak{m}^+$ so that it contains all the positive root subspaces for $\mathfrak{m}$, as well as a maximal isotropic subspace $\mathfrak{h}^+$ of the Cartan subalgebra $\mathfrak{h}$. To construct $Spin_{\mathfrak{g}}$, we can use the maximal isotropic subspace $\mathfrak{g}^+ = \mathfrak{m}^+ \oplus \mathfrak{n}$ of $\mathfrak{g}$. It follows that $Spin_{\mathfrak{g}} = Spin_{\mathfrak{m}} \otimes \mathbb{C}_{-\rho_{\mathfrak{n}}} \otimes \bigwedge^* \mathfrak{n}$. The ρ-shift comes from the fact that the highest weight of $Spin_{\mathfrak{m}}$ is $\rho_{\mathfrak{m}}$ and the highest weight of $Spin_{\mathfrak{g}}$ is ρ, while the highest weight of $Spin_{\mathfrak{m}} \otimes \bigwedge^* \mathfrak{n}$ is $\rho_{\mathfrak{m}} + 2\rho_{\mathfrak{n}} = \rho + \rho_{\mathfrak{n}}$. $\square$

Sketch of Proof V

Since π is unitary, the computation for its Dirac cohomology is

$$\begin{aligned}&\left[\pi \otimes E(\rho)\ :\ E(\tau')\right] = \left[\pi_{\mathfrak{m}} \otimes \mathbb{C}_\xi \otimes E(-\tau')\mid_{\mathfrak{m}} :\ E(\rho)\mid_{\mathfrak{m}}\right] = \\ &\left[\pi_{\mathfrak{m}} \otimes \mathbb{C}_\xi \otimes E(-\tau')\mid_{\mathfrak{m}} :\ F(\rho_{\mathfrak{m}}) \otimes \mathbb{C}_{-\rho_{\mathfrak{n}}} \otimes \bigwedge \mathfrak{n}^\star\right] = \\ &\left[\mathbb{C}_{\xi+\rho_{\mathfrak{n}}} \otimes \pi_{\mathfrak{m}} \otimes F(\rho_{\mathfrak{m}}) \otimes E(-\tau')\mid_{\mathfrak{m}} :\ \bigwedge \mathfrak{n}^\star\right].\end{aligned} \tag{12}$$

Here the first equality used Frobenius reciprocity, while the second equality used Lemma 6. Note that the dual of $E(\tau')$ is the module $E(-\tau')$ which has lowest weight $-\tau'$ with respect to Δ'.
Using (11) and (8), we can write

$$-\tau' = -2\lambda + \rho' = -\xi - \mu_{\mathfrak{m}} - \nu_{\mathfrak{m}} + \rho' = -\xi - \tau_{\mathfrak{m}} - \rho_{\mathfrak{m}} + \rho'. \tag{13}$$

We have assumed ξ to be dominant for $\Delta(\mathfrak{n})$, and $2\lambda_{\mathfrak{m}}$ is dominant for $\Delta(\mathfrak{m})$. Thus $\Delta_{\mathfrak{m}} \subset \Delta, \Delta'$. Because of (9), the LHS of the last line of (12) contains the representation

$$\mathbb{C}_{\xi+\rho_{\mathfrak{n}}} \otimes F(\tau_{\mathfrak{m}}) \otimes E(-\tau')\mid_{\mathfrak{m}} \supseteq \mathbb{C}_{\xi+\rho_{\mathfrak{n}}} \otimes F(\tau_{\mathfrak{m}} - \tau').$$

Sketch of Proof VI

Namely, $F(\tau_{\mathfrak{m}} - \tau')$ is the PRV component of $F(\tau_{\mathfrak{m}}) \otimes F(-\tau') \subseteq F(\tau_{\mathfrak{m}}) \otimes E(-\tau') |_{\mathfrak{m}}$. By (11) and (8), $\tau_{\mathfrak{m}} - \tau' = -\xi - \rho_{\mathfrak{m}} + \rho'$, so

$$\mathbb{C}_{\xi+\rho_{\mathfrak{n}}} \otimes F(\tau_{\mathfrak{m}} - \tau') \supseteq F(\rho_{\mathfrak{n}} - \rho_{\mathfrak{m}} + \rho') = F(w_{\mathfrak{m}}\rho + \rho'),$$

where $w_{\mathfrak{m}}$ is the longest element of the Weyl group of $\mathfrak{m}$. Namely, $w_{\mathfrak{m}}$ sends all roots in $\Delta_{\mathfrak{m}}$ to negative roots for $\mathfrak{m}$, while permuting the roots in $\Delta(\mathfrak{n})$, so $w_{\mathfrak{m}}\rho = -\rho_{\mathfrak{m}} + \rho_{\mathfrak{n}}$.
So we see that the LHS of the last line of (12) contains the $\mathfrak{m}$-module $F(w_{\mathfrak{m}}\rho + \rho') = F(w_{\mathfrak{m}}\rho' + \rho)$. Namely, both $w_{\mathfrak{m}}\rho + \rho'$ and $w_{\mathfrak{m}}\rho' + \rho = w_{\mathfrak{m}}(w_{\mathfrak{m}}\rho + \rho')$ are extremal weights for the same module.
We will show that

$$\left[F(w_{\mathfrak{m}}\rho' + \rho) : \bigwedge \mathfrak{n}\right] \neq 0. \qquad (14)$$

Sketch of Proof VII

This will prove that (12) is nonzero, and consequently that π has nonzero Dirac cohomology.
Note that $w_{\mathfrak{m}}\rho' + \rho$ is a sum of roots in $\Delta(\mathfrak{n})$, and antidominant for $\Delta_{\mathfrak{m}}$, because for any simple $\gamma \in \Delta_{\mathfrak{m}}$, $\langle \rho', \check{\gamma} \rangle \in \mathbb{N}^+$ and $\langle \rho, \check{\gamma} \rangle = 1$. Moreover,

$$w_{\mathfrak{m}}\rho' + \rho = \sum_{\langle \alpha, w_{\mathfrak{m}}\rho' \rangle > 0,\ \langle \alpha, \rho \rangle > 0} \alpha. \tag{15}$$

To show that (14) holds, it is enough to show that

$$v := \bigwedge_{\langle \alpha, \rho \rangle > 0,\ \langle \alpha, w_{\mathfrak{m}}\rho' \rangle > 0} e_\alpha \quad \in \bigwedge \mathfrak{n}^\star \tag{16}$$

is a lowest weight vector for $\Delta_{\mathfrak{m}}$. Here e_α denotes a root vector for the root α.

Sketch of Proof VIII

Let $\gamma \in \Delta_{\mathfrak{m}}$. Then, up to constant factors,

$$\operatorname{ad} e_{-\gamma} e_\alpha = \begin{cases} 0 & \text{if } \alpha - \gamma \text{ is not a root,} \\ e_{-\gamma+\alpha} & \text{if } \alpha - \gamma \text{ is a root.} \end{cases} \tag{17}$$

But $\langle -\gamma, w_{\mathfrak{m}}\rho' \rangle > 0$, and $\langle \alpha, w_{\mathfrak{m}}\rho' \rangle > 0$ by assumption, so

$$\langle -\gamma + \alpha, w_{\mathfrak{m}}\rho' \rangle > 0 + 0 = 0. \tag{18}$$

Also, if $-\gamma + \alpha$ is a root, then it is in $\Delta(\mathfrak{n})$, since $\alpha \in \Delta(\mathfrak{n})$ and $\mathfrak{n}$ is an $\mathfrak{m}$-module. So $\langle -\gamma + \alpha, \rho \rangle > 0$. Thus every $e_{-\gamma+\alpha}$ appearing in (17) is one of the factors in (16).

The claim now follows from the formula

$$\operatorname{ad} e_{-\gamma} \bigwedge e_\alpha = \sum e_{\alpha_1} \wedge \cdots \wedge \operatorname{ad} e_{-\gamma} e_{\alpha_i} \wedge \ldots. \tag{19}$$

Sketch of Proof IX

In each summand either $\mathrm{ad}\, e_{-\gamma} e_{\alpha_i}$ equals 0, or is a multiple of one of the root vectors already occurring in the same summand. So $\mathrm{ad}\, e_{-\gamma} v = 0$. We have proved the following theorem.

Theorem

Let $P = MN$ be a parabolic subalgebra of G and let $\Delta = \Delta_{\mathfrak{m}} \cup \Delta(\mathfrak{n})$ be the corresponding system of positive roots. Let $\pi_{\mathfrak{m}} := L_{\mathfrak{m}}(\lambda, -s\lambda)$ be an irreducible unitary representation of M with nonzero Dirac cohomology such that its parameter is zero on the center of $\mathfrak{m}$. Let ξ be a unitary character of M which is dominant with respect to Δ. Suppose that twice the infinitesimal character of $\pi = Ind_P^G[\pi_{\mathfrak{m}} \otimes \xi]$ is regular and integral. Then π has nonzero Dirac cohomology. $\square$

The multiplcity ≤ 1 requires a finer analysis of equation (12).

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