Atlas of Lie Groups
and Representations

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Unitary Representations and Hodge Theory

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Summary

Joint with: Peter Trapa and David Vogan

Based on conversations with Wilfried Schmid and Kari Vilonen

Atlas of Lie Groups and Representations:

π: irreducible representation of a real reductive group with real infinitesimal character

Main result: an algorithm to compute the signature of the invariant Hermitian form on an irreducible representation of a real reductive group. Ultimately: determine the unitary dual.

Summary

Conjecture of Schmid and Vilonen:

π admits a canonical Hodge filtration $\dots F_p \subset F_{p+1} \subset \dots$. There is a precise relationship between the signs of the invariant Hermitian form and levels of the Hodge filtration.

This talk:

1) Explain the algorithm to compute Hermitian forms

2) Generalization to compute the Hodge filtration

3) The relationship between signature and the Hodge filtration.

THE C-FORM

G: connected, complex reductive group

$\mathfrak{g}_0 = \text{Lie}(G)$, $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$

θ: algebraic involution of G, $K = G^\theta$ (corresponding to a real form $G(\mathbb{R})$ of G)

$\pi = (\pi, V)$: $(\mathfrak{g}, K)$-module with real infinitesimal character (λ is in the real span of the roots)

An invariant Hermitian form on V:

$$(\pi(X)v, w) + (v, \pi(X)w) = 0 \quad (X \in \mathfrak{g}_0)$$

or equivalently

$$(\pi(X)v, w) + (v, \pi(\sigma(X))w) = 0 \quad (\mathfrak{g}_0 = \mathfrak{g}^\sigma, X \in \mathfrak{g})$$

If π is irreducible, $(\, , \,)$ is unique up to real scalar, and π is unitary if $(\, , \,)$ is (can be chosen to be) positive definite.

Signatures of Hermitian Forms

Problem: Suppose (π, V) supports an invariant Hermitian form $(\,,\,)$. Compute the signature of $(\,,\,)$.

What? $(\,,\,)$ is positive definite if $(v, v) > 0$ for all v

If not, what is the "signature"?

In terms of K-types.

Definition $\mathrm{mult}(\pi) : \widehat{K} \to \mathbb{N}$:

$$\pi|_K = \sum_{\widehat{K}} \mathrm{mult}(\pi)(\mu) \cdot \mu$$

Note: $\widehat{K}$ and $\mathrm{mult}(\pi)(\mu)$ are explicitly computable in Atlas (K may be disconnected)

Signatures of Hermitian Forms

Definition: $\mathbb{W} = \mathbb{Z}[s]$ $(s^2 = 1)$

Definition: $\mathrm{sig}(\pi) : \widehat{K} \to \mathbb{W}$:

$\mathrm{sig}(\pi)(\mu) = a + bs$ means: the invariant form, restricted to the μ-isotypic has signature (a, b) (times the positive definite form on μ).

Note: $\mathrm{sig}(\pi)(\mu)(s = 1) = a + b = \mathrm{mult}(\pi)(\mu)$

The question becomes: how to compute $\mathrm{sig}(\pi)$?

Signatures of Hermitian Forms

Theorem: (Vogan)

$$\mathrm{sig}(\pi) = \sum_{i=1}^{n} w_i \cdot \mathrm{mult}(\pi_i)$$

for some irreducible, tempered representations $\pi_1, \dots, \pi_n$, $w_i \in \mathbb{W}$

The point is this is a finite formula.

In other words

$$\mathrm{sig}(\pi) \in \mathbb{W}\langle \mathrm{mult}(\tau) \mid \tau \text{ tempered } \rangle$$

So we've restated the problem: how do we compute $\{\pi_i\}$ and $\{w_i\}$?

Example: $SL(2,\mathbb{R})$

$\pi(\nu)$: spherical principal series with infinitesimal character $\nu \in \mathbb{R}$

$\widehat{K} = \mathbb{Z}$

$\pi(\nu)|_K = 2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$

$\pi(\nu)$ is reducible $\Leftrightarrow \nu \in 2\mathbb{Z}+1$

$\mathrm{sig}(\pi(0)) = \mathrm{mult}(\pi(0))$ (unitary)

in fact
$\mathrm{sig}(\pi(\nu)) = \mathrm{sig}(\pi(0)) = \mathrm{mult}(\pi(0)) \quad \nu < 1$

	...	-6	-4	-2	0	2	4	6	...
sig(I(0))		$+$	$+$	$+$	$+$	$+$	$+$	$+$	
sig(I($1-\epsilon$))		$+$	$+$	$+$	$+$	$+$	$+$	$+$	
sig(I(1))		0	0	0	$+$	0	0	0	
sig(I($1+\epsilon$))		-	-	-	$+$	-	-	-	

Signatures of Hermitian forms

The changes in sign are at the summands $\pi(DS_{\pm})$ (discrete series with infinitesimal character ρ)

Conclusion:

$$\mathrm{sig}(\pi(1+\epsilon)) = \mathrm{mult}(\pi(1-\epsilon)) + (s-1)(\mathrm{mult}(\pi(DS_{+})) + \mathrm{mult}(\pi(DS_{-})))$$

In general the change of sign are on representations with smaller ν.

Idea: Repeat this process, to compute $\mathrm{sig}(\pi)$ in terms of $\mathrm{sig}(\pi_j) = \mathrm{mult}(\pi_j)$ for π_j tempered.

The c-form

Major fly in the ointment:

a) there may be no invariant Hermitian form on (π, V)
b) it may not be unique (up to positive scalar)

Example: odd principal series of $SL(2,\mathbb{R})$ with $\nu \neq 0$

The K-types $1, -1$ have opposite signature

$G(\mathbb{R})$, θ, $\sigma : G(\mathbb{R}) = G(\mathbb{C})^{\sigma} \quad \rightarrow \sigma_c$ compact real form (so $\sigma_c \circ \sigma = \theta$)

Definition The c-form of (π, V) is a Hermitian form on V satisfying:

$$(\pi(X)v, w)_c + (v, \pi(\sigma_c(X))w)_c = 0$$

and $(\ ,\ )_c$ is positive definite on all lowest K-types of π

The c-form

Theorem: (Adams/Trapa/van Leeuwen/Vogan)

(1) The c-form exists and is unique (up to positive scalar)

(2) The c-form determines the invariant Hermitian form (an explicit formula)

Note: if the group is not equal rank we need the c-form on the *extended* group

Definition: $\mathrm{sig}^c(\pi) : \widehat{K} \to \mathbb{W}$:

$\mathrm{sig}^c(\pi)(\mu) = a + bs$ means: the c-form, restricted to the μ-isotypic has signature (a, b) (times the positive definite form on μ).

As before: $\mathrm{sig}^c(\pi) = \sum_i w_i^c \cdot \mathrm{mult}(\pi_i) \quad (\pi_i \text{ tempered})$

Digression: The Langlands classification and the KLV polynomials

Fix infinitesimal character λ

$\mathcal{P}_\lambda$: a set of parameters

$\mathcal{P}_\lambda \ni \gamma \to I(\gamma)$ (standard module)

$\mathrm{J}(\gamma)$ (unique irreducible quotient of $I(\gamma)$)

$\{$irreducible representations with infinitesimal character $\gamma\} \longleftrightarrow \mathcal{P}_\lambda$

$$I(\gamma) = \mathrm{Ind}_{MAN}^{G}(\pi_M \otimes \nu \otimes 1) \quad (\nu \in \mathfrak{a}_0^*)$$

Digression: The Langlands classification and the KLV polynomials

Kazhdan-Lusztig-Vogan polynomials:

$P_{\tau,\gamma} \in \mathbb{Z}[q]$

$\{P_{\tau,\gamma} \mid \tau, \gamma \in \mathcal{P}_\lambda\}$ (upper unitriangular matrix)

Inverse matrix $\{Q_{\tau,\gamma}\}$

$$\mathsf{J}(\gamma) = \sum_\tau (-1)^{\ell(\gamma)-\ell(\tau)} P_{\tau,\gamma}(1)\mathsf{I}(\tau)$$

$$\mathsf{I}(\gamma) = \sum_\tau Q_{\tau,\gamma}(1)\mathsf{J}(\tau)$$

Digression: the Jantzen filtration

$$\mathrm{I}(\gamma) = \sum_{\tau} Q_{\tau,\gamma} J(\tau)$$

The Jantzen filtration is a canonical filtration of $\mathrm{I}(\gamma)$ by $(\mathfrak{g}, K)$-modules. Associated graded: $I(\gamma)_r$ (these are completely reducible).

Jantzen conjecture: if $Q_{\tau,\gamma} = \sum a_j q^j$, then

a_r is the multiplicity of $\mathrm{J}(\tau)$ in $I(\gamma)_j$, $j = \frac{1}{2}(\ell(\gamma) - \ell(\tau) + r)$.

Note: $Q_{\tau,\gamma}(1) = \sum_j a_j$ is the multiplicity of $\mathrm{J}(\tau)$ in $\mathrm{I}(\gamma)$.

Deformation of the c-form

Deformation: $I(t\gamma) = \mathrm{Ind}_{MAN}^{G}(\pi_M \otimes t\nu \otimes 1)$

$\mathrm{I}_t = \mathrm{I}(t\gamma)$: a continuous family of standard modules ($t \in \mathbb{R}$)

Fact: $\mathrm{sig}^c(I_t)$ is constant (in t) as long as I_t is irreducible.

Assume: I_t irreducible for $0 < |1 - t| < \epsilon$.

$$\mathrm{I}_{1-\epsilon} \to \mathrm{I}_1 \to \mathrm{I}_{1+\epsilon}$$

Problem: how does the c-form change as you deform from $\mathrm{I}(\gamma_{1-\epsilon})$ to $\mathrm{I}(\gamma_{1+\epsilon})$ through the reducible point $I(\gamma)$?

Deformation of the c-form

The signature of $I(\gamma)$, restricted to and irreducible factor $J(\delta)$ in the r^{th} graded level of the Jantzen filtration, agrees with the signature of $J(\delta)$ up to a shift. Precisely:

Lemma:

$$\mathrm{sig}^c(I(\gamma), r)|_{J(\delta)} = s^{(\ell_0(\gamma)-\ell_0(\delta))/2} s^{\ell(\gamma)-\ell(\delta)-r} \mathrm{sig}^c(J(\delta))$$

Key fact: the c-form changes sign on odd levels of the Jantzen filtration at I(π)

(Comes down to: $f(x) = x^n$ changes sign at $x = 0$ if and only if n is odd.)

That is:

$$\mathrm{sig}^c(I_{1+\epsilon}) = \mathrm{sig}^c(I_{1-\epsilon}) + (1-s) \sum_{r \text{ odd}} \mathrm{sig}^c(J(\Gamma), r)$$

Computing the c-form

Deformation plus KLV implies:

Algorithm (Deformation of the c-form):

$$\begin{aligned}\mathrm{sig}(I_{1+\epsilon}) = \mathrm{sig}(I_{1-\epsilon}) + \\ (1-s) \sum_{\substack{\phi,\tau \\ \phi<\tau<\gamma \\ \ell(\gamma)-\ell(\tau) \text{ odd}}} s^{(\ell_0(\gamma)-\ell_0(\tau))/2} P_{\phi,\tau}(s) Q_{\tau,\gamma}(s) \mathrm{sig}(\mathrm{I}(\tau))\end{aligned}$$

Corollary: the Atlas algorithm There is an inductive algorithm to compute $\mathrm{sig}(\mathrm{I}(\gamma))$, in terms of $\mathrm{sig}(\mathrm{I}(\tau))$ where $\mathrm{I}(\tau) = \mathrm{J}(\tau)$ is and (irreducible) tempered representation .

Digression: Orientation Numbers

For each parameter γ there is an orientation number $\ell_0(\gamma) \in \mathbb{Z}$.

These satisfy:

1) $\ell_0(\gamma) = 0$ if the infinitesimal character of γ is integral.

2) if δ, γ are in the same block then $\ell_0(\gamma) \equiv \ell_0(\delta) \pmod 2$

This is a technical point in the theory which I'll mostly ignore.

Summary of the c-form calculation

Fix $\gamma, I(\gamma) = \mathrm{Ind}_{MAN}^{G}(\pi_M \otimes \nu \otimes 1)$

$\pi = (\pi, V)$: $(\mathfrak{g}, K)$-module with real infinitesimal character, standard or irreducible

$\mathrm{sig}^c(\pi) : \widehat{K} \to \mathbb{Z}[s]$

$\mathrm{sig}^c(\pi)(\mu) = a + bs \Leftrightarrow (\ ,\ )$ has signature (a, b) on the μ-isotypic (times the positive definite form on μ)

1) $\mathrm{sig}^c(I(t\gamma))$ only changes at reducibility points

2) relation with the Jantzen filtration:

$$\mathrm{sig}^c(I(\gamma), r)|_{J(\delta)} = s^{(\ell_0(\gamma)-\ell_0(\delta))/2} s^{(\ell(\gamma)-\ell(\delta)-r)/2} \mathrm{sig}^c(J(\delta))$$

Summary of the c-form calculation

These imply:

3) $\mathrm{sig}(J(\gamma)) = \sum_\delta s^{(\ell_0(\gamma)-\ell_0(\delta))/2}(-1)^{\ell(\gamma)-\ell(\delta)}P_{\gamma,\delta}(s)\mathrm{sig}(I(\delta))$

4) Deformation:

$$\mathrm{sig}(\gamma_{1+\epsilon}) = \mathrm{sig}(\gamma_{1-\epsilon}) + (1-s) \sum_{\substack{\phi,\tau \\ \phi<\tau<\gamma \\ \ell(\gamma)-\ell(\tau) \text{ odd}}} s^{(\ell_0(\gamma)-\ell_0(\tau))/2} P_{\phi,\tau}(s) Q_{\tau,\gamma}(s) \mathrm{sig}(\mathrm{I}(\phi))$$

This gives a formula for $\mathrm{sig}^c(J(\gamma))$ in terms of tempered representations ($\nu = 0$), and:

5) $I(\gamma)$ tempered implies $(\, , \,)$ is positive definite, and $(\, , \,)_c$ is computable. (White lie: $(\, , \,)$ is computable)

The Hodge filtration

Saito's theory of mixed Hodge modules.

Beilinson-Bernstein theory of $\mathcal{D}$-modules, $\mathcal{D}_\lambda$-modules

Global section functor: equivalence of categories $\mathcal{D}_\lambda$-modules and $(\mathfrak{g}, K)$-modules with infinitesimal character λ.

Reference: Wilfried Schmid's talk the last day of this conference.

The Hodge filtration

Proceed formally

Assume we are given a K-equivariant filtration:

$(\pi, V) \quad 0 \subset F_0 \subset F_1 \subset \dots$

$\mathrm{gr}(\pi) = F_p/F_{p-1} \quad$ (a finite dimensional representation of K)

Definition: $\mathrm{h}(\pi) : \widehat{K} \to \mathbb{Z}[v]$

$\mathrm{h}(\pi)(\mu) = a_0 + a_1 v + \dots + a_n v^n: \quad a_i = \mathrm{mult}(\mathrm{gr}_i(\pi))(\mu)$

$\mathrm{h}(\pi)(\mu)(v = 1) = \mathrm{mult}(\pi)(\mu)$

Properties of the filtration

Recall (basic c-form properties:)
1) $\mathrm{sig}^c(I(t\gamma))$ only changes at reducibility points

2) relation with the Jantzen filtration:

$$\mathrm{sig}^c(I(\gamma),r)|_{J(\delta)} = s^{(\ell_0(\gamma)-\ell_0(\delta))/2} s^{(\ell(\gamma)-\ell(\delta)-r)/2} \mathrm{sig}^c(J(\delta))$$

Assume the filtration F_p satisfies:

1) $\mathrm{sig}^c(F_p(I(t\gamma)))$ only changes at reducibility points

2) relation with the Jantzen filtration:

$$\mathrm{h}(I(\gamma),r)|_{J(\delta)} = v^{(\ell_0(\gamma)-\ell_0(\delta))/2} v^{((\ell(\gamma)-\ell(\delta)-r))/2} \mathrm{h}(J(\delta))$$

Computing the Hodge filtration

These imply:

3) Formula of $h(J(\gamma)$ in terms of standard representations:

$$\mathsf{h}(J(\gamma)) = \sum_{\delta} v^{(\ell_0(\gamma)-\ell_0(\delta))/2} v^{\ell(\gamma)-\ell(\delta)} P_{\gamma,\delta}(v)\mathsf{h}(I(\delta))$$

$$\mathsf{h}(\gamma_{1+\epsilon}) = \mathsf{h}(\gamma_{1-\epsilon}) + (1-v) \sum_{\substack{\phi,\tau \\ \phi<\tau<\gamma \\ \ell(\gamma)-\ell(\tau)\ \mathrm{odd}}} v^{(\ell_0(\gamma)-\ell_0(\tau))/2} P_{\phi,\tau}(v) Q_{\tau,\gamma}(v)\mathsf{h}(\mathrm{I}(\phi))$$

This gives a formula for $\mathsf{h}(J(\gamma))$ in terms of tempered representations ($\nu = 0$).

Computing the Hodge filtration

Conclusion: assuming the filtration $\{F_p\}$ satisfies (1) and (2). Then there is an algorithm to compute $\mathrm{h}(J(\gamma))$ in terms of $\mathrm{h}(I(\delta))$ with δ tempered.

Recall in the case of sig^c we are essentially done: the invariant Hermitian form on an irreducible tempered representation is positive definite.

However, the Hodge filtration on a tempered representation is a non-trivial object.

Computing the Hodge filtration

Assume:

1) $\mathrm{sig}^c(F_p(I(t\gamma)))$ only changes at reducibility points

2) relation with the Jantzen filtration:

$$\mathrm{h}(I(\gamma), r)|_{J(\delta)} = v^{(\ell_0(\gamma)-\ell_0(\delta))/2} v^{((\ell(\gamma)-\ell(\delta)-r))/2} \mathrm{h}(J(\delta))$$

3) Suppose G is split and $I(\gamma)$ is irreducible, spherical, and tempered (i.e. $\nu = 0$). Then

$$I(\gamma)|_K \simeq \mathcal{R}(\mathcal{N}_\theta)$$

Here $\mathcal{N}_\theta$ is the nilpotent cone in $\mathfrak{p}$ ($\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$), and this is an isomorphism on the level of gradings.

Schmid and Vilonen: such a family of filtrations should come from Hodge theory on the flag variety (see Schmid's talk on Friday)

The Main Result

Theorem:

Suppose every irreducible $(\mathfrak{g}, K)$ module π with real infinitesimal character comes equipped with a filtration $\{F_p(\pi)\}$, satisfying conditions (1-3).

1) Assume the real group $G(\mathbb{R})$ corresponding to K is complex. Then

$$\boxed{\mathrm{h}(\pi)(v = s) = \mathrm{sig}^c(\pi)}$$

That is:

The signature is the reduction of the Hodge filtration modulo 2

2) In general we need an additional technical (and difficult) assertion about the Hodge filtration for $\mathcal{D}_\lambda$ modules where λ is not dominant. Assuming this the same result holds.

Proof of the Theorem

Assuming the stated properties, $\mathrm{h}(\pi)$ is given by a (long, complicated) algorithm. At each step of this algorithm set $v = s$. The resulting algorithm is precisely the one to compute $\mathrm{sig}^c(\pi)$.

Proof of the Main Theorem

Main Step:

Algorithm (Deformation of the c-form):

$$\operatorname{sig}^c(I(\gamma_{1+\epsilon})) = \operatorname{sig}^c(I(\gamma_{1-\epsilon})) - \sum_{\tau<\gamma} s^{(\ell_0(\gamma)-\ell_0(\tau)/2}$$

$$\Big[\sum_{\substack{\delta \\ \tau\leq\delta\leq\gamma}} (-1)^{\ell(\delta)-\ell(\tau)} s^{\ell(\gamma)-\ell(\delta)} P_{\tau,\delta}(s) Q_{\delta,\gamma}(s)\Big] \operatorname{sig}^c(I(\delta))$$

Algorithm (Deformation of the Hodge filtration):

$$\mathrm{h}(I(\gamma_{1+\epsilon})) = \mathrm{h}(I(\gamma_{1-\epsilon})) - \sum_{\tau<\gamma} v^{(\ell_0(\gamma)-\ell_0(\tau)/2}$$

$$\Big[\sum_{\substack{\delta \\ \tau\leq\delta\leq\gamma}} (-1)^{\ell(\delta)-\ell(\tau)} v^{\ell(\gamma)-\ell(\delta)} P_{\tau,\delta}(v) Q_{\delta,\gamma}(v^{-1})\Big] \mathrm{h}(I(\delta))$$

Remark: This proof is not computer-dependent - one does not need to acutally *compute* either algorithm, but just observe they are formally related as indicated. Nevertheless we probably would never have formulated this result without the Atlas software.

In particular: the notion of c-form, in principle, could have been invented (discovered?) 30 years ago. We found it in our effort to formula the Atlas algorithm.

Thank You