Representations of some affine vertex algebras at negative integer levels

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(joint work with Dražen Adamović)

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For a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $k \in \mathbb{C}$, $k \neq -h^\vee$, denote by $N_{\mathfrak{g}}(k, 0)$ the universal affine vertex operator algebra of level $k$ and by $L_{\mathfrak{g}}(k, 0)$ the associated simple quotient.
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We are interested in constructing ideals in $N_\mathfrak{g}(k,0)$ for negative integer $k$ and study the representation theory of $L_\mathfrak{g}(k,0)$ in that case.
In particular, we will consider the case of orthogonal Lie algebra of type $D$. 
Vertex operator algebra associated to $D_{\ell}$ of level $n - \ell + 1$

**Theorem**

Vector

$$v_n = \left( \sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i}(-1)e_{\epsilon_1 + \epsilon_i}(-1) \right)^n 1$$

is a singular vector in $N_{D_{\ell}}(n - \ell + 1, 0)$, for any $n \in \mathbb{Z}_{> 0}$. 
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is a singular vector in $N_{D_\ell}(n - \ell + 1, 0)$, for any $n \in \mathbb{Z}_{>0}$.

In the case $n = 1$, we obtain the singular vector

$$v = \sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i}(-1) e_{\epsilon_1 + \epsilon_i}(-1) 1$$

in $N_{D_\ell}(-\ell + 2, 0)$. 
We will consider representations of quotient vertex operator algebra

\[ \mathcal{V}_{D_\ell}(-\ell + 2, 0) = \frac{N_{D_\ell}(-\ell + 2, 0)}{\langle v \rangle}. \]
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\[ \mathcal{V}_{D\ell}(-\ell + 2, 0) = \frac{\mathcal{N}_{D\ell}(-\ell + 2, 0)}{<v>}. \]

Using Zhu’s theory, we obtain the following classification result:

**Theorem**

The set

\{ L_{D\ell}(-\ell + 2, t\omega_{\ell-1}), L_{D\ell}(-\ell + 2, t\omega_{\ell}) \mid t \in \mathbb{Z}_{\geq 0} \}

provides the complete list of irreducible ordinary \( \mathcal{V}_{D\ell}(-\ell + 2, 0) \)-modules.
Thus, the set of irreducible ordinary $L_{D_\ell}(-\ell + 2, 0)$–modules is a subset of the set

$$\{L_{D_\ell}(-\ell + 2, t\omega_{\ell-1}), L_{D_\ell}(-\ell + 2, t\omega_{\ell}) \mid t \in \mathbb{Z}_{\geq 0}\}.$$
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**Remark:** More generally, we can obtain the classification of irreducible weak $\mathcal{V}_{D_\ell}(-\ell + 2, 0)$–modules from the category $\mathcal{O}$. 
Thus, the set of irreducible ordinary $L_{D_{\ell}}(-\ell + 2, 0)$–modules is a subset of the set

$$\{ L_{D_{\ell}}(-\ell + 2, t\omega_{\ell - 1}), L_{D_{\ell}}(-\ell + 2, t\omega_{\ell}) \mid t \in \mathbb{Z}_{\geq 0} \}.$$  

Remark: More generally, we can obtain the classification of irreducible weak $V_{D_{\ell}}(-\ell + 2, 0)$–modules from the category $\mathcal{O}$.

Natural questions:
1. Is $< v >$ a maximal submodule in $N_{D_{\ell}}(-\ell + 2, 0)$?
2. Representation theory of $L_{D_{\ell}}(-\ell + 2, 0)$?
Thus, the set of irreducible ordinary $L_{D_{\ell}}(-\ell + 2, 0)$–modules is a subset of the set

$$\{ L_{D_{\ell}}(-\ell + 2, t\omega_{-1}), L_{D_{\ell}}(-\ell + 2, t\omega) \mid t \in \mathbb{Z}_{\geq 0} \}.$$ 

**Remark:** More generally, we can obtain the classification of irreducible weak $\mathcal{V}_{D_{\ell}}(-\ell + 2, 0)$–modules from the category $\mathcal{O}$.

**Natural questions:**
1. Is $\langle \nu \rangle$ a maximal submodule in $N_{D_{\ell}}(-\ell + 2, 0)$?
2. Representation theory of $L_{D_{\ell}}(-\ell + 2, 0)$?

We give an answer in some special cases.
Case $\ell = 4$

Denote by $\theta$ the automorphism of $N_{D_4}(-2,0)$ induced by the automorphism of the Dynkin diagram of $D_4$ of order three.
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$$(v =)$$

\begin{align*}
(e_{\epsilon_1-\epsilon_2}(-1))e_{\epsilon_1+\epsilon_2}(-1)+e_{\epsilon_1-\epsilon_3}(-1)\epsilon_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1)1
\end{align*}

is a singular vector in $N_{D_4}(-2,0)$,
Denote by $\theta$ the automorphism of $N_{D_4}(-2,0)$ induced by the automorphism of the Dynkin diagram of $D_4$ of order three. Since 

\[(v =) (e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1)+e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1))1\]

is a singular vector in $N_{D_4}(-2,0)$, it follows that 

\[(\theta(v) =) (e_{\epsilon_3-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1)-e_{\epsilon_2-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_1-\epsilon_4}(-1))1,\]
Case $\ell = 4$

Denote by $\theta$ the automorphism of $N_{D_4}(-2,0)$ induced by the automorphism of the Dynkin diagram of $D_4$ of order three. Since $(v =)$

$$(e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1) + e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1) + e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1))1$$

is a singular vector in $N_{D_4}(-2,0)$, it follows that $(\theta(v) =)$

$$(e_{\epsilon_3-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1) - e_{\epsilon_2-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1) + e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_1-\epsilon_4}(-1))1,$$

and $(\theta^2(v) =)$

$$(e_{\epsilon_3+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1) - e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1) + e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))1$$

are also singular vectors in $N_{D_4}(-2,0)$. 
Case $\ell = 4$

Denote by $\theta$ the automorphism of $N_{D_4}(-2, 0)$ induced by the automorphism of the Dynkin diagram of $D_4$ of order three. Since $(v =)$
\[
e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1)+e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1)1
\]
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\[
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\]
are also singular vectors in $N_{D_4}(-2, 0)$. We consider the associated quotient vertex operator algebra
\[
\tilde{L}_{D_4}(-2, 0) = \frac{N_{D_4}(-2, 0)}{\langle v, \theta(v), \theta^2(v) \rangle}.
\]
Using Zhu’s theory, we obtain the following classification result:

\[
\{ L^0_{\lambda}(-2,0), L^0_{\lambda}(-2,-2\omega_1), L^0_{\lambda}(-2,-2\omega_3), L^0_{\lambda}(-2,-2\omega_4) \}\]
provides a complete list of irreducible weak \( \hat{\lambda} \)-modules from the category \( O \).
In particular, \( L^0_{\lambda}(-2,0) \) is the unique irreducible ordinary module for \( \hat{\lambda} \).
Case $\ell = 4$

Using Zhu’s theory, we obtain the following classification result:

Theorem

The set

$$\{L_{D_4}(-2, 0), L_{D_4}(-2, -2\omega_1), L_{D_4}(-2, -2\omega_3), L_{D_4}(-2, -2\omega_4), L_{D_4}(-2, -\omega_2)\}$$

provides a complete list of irreducible weak $\widetilde{L}_{D_4}(-2, 0)$–modules from the category $\mathcal{O}$. In particular, $L_{D_4}(-2, 0)$ is the unique irreducible ordinary module for $\widetilde{L}_{D_4}(-2, 0)$. 
Case $\ell = 4$

It follows immediately that:
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**Theorem**

Vertex operator algebra $\tilde{L}_{D_4}(-2,0)$ is simple, i.e.

$$L_{D_4}(-2,0) = \frac{N_{D_4}(-2,0)}{\langle v, \theta(v), \theta^2(v) \rangle}.$$ 

These results were recently generalized by Arakawa and Moreau.
Top components of irreducible $\mathcal{V}_{D_\ell}(-\ell + 2, 0)$–modules are irreducible modules for the simple Lie algebra of type $D$. 
Case $\ell$ odd

Top components of irreducible $V_{D_{\ell}}(-\ell + 2, 0)$–modules are irreducible modules for the simple Lie algebra of type $D$. So we get an interesting series of modules:

$$V_{D_{\ell}}(t \omega_{\ell - 1}), \; V_{D_{\ell}}(t \omega_{\ell}) \quad (t \in \mathbb{Z}_{\geq 0}).$$
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Let

$$U_{D_{\ell}}(t) := \begin{cases} 
V_{D_{\ell}}(t\omega_{\ell-1}), & \text{for } t \geq 0 \\
V_{D_{\ell}}(-t\omega_{\ell}), & \text{for } t < 0. 
\end{cases}$$
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$$U_{D_\ell}(t) := \left\{ \begin{array}{ll} V_{D_\ell}(t\omega_{\ell-1}), & \text{for } t \geq 0 \\ V_{D_\ell}(-t\omega_\ell), & \text{for } t < 0. \end{array} \right.$$ 

Tensor products of these modules have been described by S. Okada:

**Theorem**

Assume that $\ell \geq 3$ is an odd natural number. Assume that $r, s \in \mathbb{Z}$. Then $U_{D_\ell}(t)$ appears in the tensor product $U_{D_\ell}(r) \otimes U_{D_\ell}(s)$ if and only if $t = r + s$. The multiplicity is one.
Case $\ell$ odd

These tensor product decompositions give upper bounds for the associated fusion rules for vertex operator algebra:
Case $\ell$ odd

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**Theorem**

Assume that $\ell \geq 3$ is an odd natural number and that $\tilde{\pi}_r$, $r \in \mathbb{Z}$ are $\mathbb{Z}_{\geq 0}$–graded $\mathcal{V}_{D_\ell}((-\ell + 2)\Lambda_0)$-modules such that top component of $\tilde{\pi}_r$ is isomorphic to the irreducible $\mathfrak{g}_{D_\ell}$–module $U_{D_\ell}(r)$. Let $\pi_r$ denote the associated simple quotient. Assume that there is a non-trivial intertwining operator of type

$$
\begin{pmatrix}
\pi_t \\
\tilde{\pi}_r & \pi_s
\end{pmatrix}.
$$

Then $t = r + s$. 
Case $\ell$ odd

It follows immediately that:

Assume that $\ell \geq 3$ is an odd natural number. The set

\[ \{ L_D^{\ell}(-\ell + 2, t\omega - 1), L_D^{\ell}(-\ell + 2, t\omega) \mid t \in \mathbb{Z} \geq 0 \} \]

provides the complete list of irreducible ordinary $L_D^{\ell}(-\ell + 2, 0)$–modules.

Question: Can we construct nontrivial intertwining operators? We give an answer in the case $\ell = 5$. 
Case $\ell$ odd

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**Question:** Can we construct nontrivial intertwining operators? We give an answer in the case $\ell = 5$. 
Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

Let $\mathfrak{g}_{E_6}$ be the simple Lie algebra of type $E_6$. 
Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

Let $\mathfrak{g}_{E_6}$ be the simple Lie algebra of type $E_6$. The subalgebra of $\mathfrak{g}_{E_6}$ generated by positive root vectors

$$e(5) = e_1^{1/2}(\epsilon_8 - \epsilon_7 - \epsilon_6 + \epsilon_5 - \epsilon_4 - \epsilon_3 - \epsilon_2 - \epsilon_1), \quad e_{\alpha_2} = e_{\epsilon_2 + \epsilon_1},$$

$$e_{\alpha_4} = e_{\epsilon_3 - \epsilon_2}, \quad e_{\alpha_3} = e_{\epsilon_2 - \epsilon_1}, \quad e_{\alpha_5} = e_{\epsilon_4 - \epsilon_3}$$

and associated negative root vectors is a simple Lie algebra $\mathfrak{g}_{D_5}$ of type $D_5$. 
Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

Let $\mathfrak{g}_{E_6}$ be the simple Lie algebra of type $E_6$. The subalgebra of $\mathfrak{g}_{E_6}$ generated by positive root vectors

\begin{align*}
    e(5) &= e_1^{\frac{1}{2}}(\epsilon_8-\epsilon_7-\epsilon_6+\epsilon_5-\epsilon_4-\epsilon_3-\epsilon_2-\epsilon_1), \\
    e_{\alpha_2} &= e_{\epsilon_2+\epsilon_1}, \\
    e_{\alpha_4} &= e_{\epsilon_3-\epsilon_2}, \\
    e_{\alpha_3} &= e_{\epsilon_2-\epsilon_1}, \\
    e_{\alpha_5} &= e_{\epsilon_4-\epsilon_3}
\end{align*}

and associated negative root vectors is a simple Lie algebra $\mathfrak{g}_{D_5}$ of type $D_5$. Thus, $\mathfrak{g}_{E_6}$ has a reductive subalgebra $\mathfrak{g}_{D_5} \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathbb{C}H$, and

\[ H = \frac{1}{3}(h_8 - h_7 - h_6 - 3h_5) \]

(where $h_i$ are determined by $\epsilon_i(h_j) = \delta_{ij}$).
Conformal embedding of $L_{D_5}(-3, 0) \otimes M(1)$ into $L_{E_6}(-3, 0)$

It follows that we have an embedding $N_{D_5}(-3, 0) \otimes M(1)$ into $N_{E_6}(-3, 0)$, where $M(1)$ denotes the Heisenberg vertex subalgebra generated by $H$. 
Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

It follows that we have an embedding $N_{D_5}(-3,0) \otimes M(1)$ into $N_{E_6}(-3,0)$, where $M(1)$ denotes the Heisenberg vertex subalgebra generated by $H$. Moreover, the singular vector in this copy of $N_{D_5}(-3,0)$:

$$v = (e_{(5)}(-1)e_{(12345)}(-1) + e_{(125)}(-1)e_{(345)}(-1) + e_{(135)}(-1)e_{(245)}(-1) + e_{(235)}(-1)e_{(145)}(-1))1$$

is also a singular vector for $\hat{g}_{E_6}$ in $N_{E_6}(-3,0)$. 
Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

It follows that we have an embedding $N_{D_5}(-3,0) \otimes M(1)$ into $N_{E_6}(-3,0)$, where $M(1)$ denotes the Heisenberg vertex subalgebra generated by $H$. Moreover, the singular vector in this copy of $N_{D_5}(-3,0)$:

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is also a singular vector for $\hat{\mathfrak{g}}_{E_6}$ in $N_{E_6}(-3,0)$.

Denote by $\tilde{L}_{D_5}(-3,0)$ the subalgebra of $L_{E_6}(-3,0)$ generated by $\mathfrak{g}_{D_5}$. 

**Theorem**

Vertex operator algebra $\tilde{L}_{D_5(1)}(-3, 0) \otimes M(1)$ is conformally embedded in $L_{E_6(1)}(-3, 0)$.
Conformal embedding of $L_{D_5}(-3, 0) \otimes M(1)$ into $L_{E_6}(-3, 0)$

Now, the results on fusion rules give the following decomposition:

**Theorem**

We have:

$$L_{E_6}(-3, 0) \cong \bigoplus_{t \in \mathbb{Z}_{\geq 0}} L_{D_5}(-3, t\omega_4) \otimes M(1, t)$$

$$\bigoplus_{t \in \mathbb{Z}_{< 0}} \bigoplus_{t \in \mathbb{Z}_{< 0}} L_{D_5}(-3, -t\omega_5) \otimes M(1, t).$$
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Modules appearing in the decomposition are generated by the following singular vectors for $\hat{g}_{D_5}$:
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Modules appearing in the decomposition are generated by the following singular vectors for $\hat{g}_{D_5}$: $e_{(234)}(-1)^t \mathbf{1}$ generates $L_{D_5}(-3, t \omega_4) \otimes M(1, t)$, and $e_{\epsilon_5 + \epsilon_4}(-1)^{-t} \mathbf{1}$ generates $L_{D_5}(-3, -t \omega_5) \otimes M(1, t)$. 
Corollary

As a consequence of this decomposition, one also obtains the following result.
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As a consequence of this decomposition, one also obtains the following result. Using conformal embeddings of $L_{F_4}(-3,0)$ into $L_{E_6}(-3,0)$ and $L_{B_4}(-3,0)$ into $L_{D_5}(-3,0)$ from ART (2013), one can easily obtain that $L_{B_4}(-3,0) \otimes M(1)^+$ is a vertex subalgebra of $L_{F_4}(-3,0)$ with the same conformal vector. Here, $M(1)^+$ denotes the $\mathbb{Z}_2$-orbifold of $M(1)$ (Dong-Nagatomo).
As a consequence of this decomposition, one also obtains the following result. Using conformal embeddings of $L_{F_4}(-3, 0)$ into $L_{E_6}(-3, 0)$ and $L_{B_4}(-3, 0)$ into $L_{D_5}(-3, 0)$ from ART (2013), one can easily obtain that $L_{B_4}(-3, 0) \otimes M(1)^+$ is a vertex subalgebra of $L_{F_4}(-3, 0)$ with the same conformal vector. Here, $M(1)^+$ denotes the $\mathbb{Z}_2$-orbifold of $M(1)$ (Dong-Nagatomo).

We obtain the following decomposition:

$$L_{F_4}(-3, 0) \cong L_{B_4}(-3, 0) \otimes M(1)^+ \oplus L_{B_4}(-3, \omega_1) \otimes M(1)^- \oplus \bigoplus_{t \in \mathbb{Z}_{>0}} L_{B_4}(-3, t\omega_4) \otimes M(1, t).$$