Abstract. This paper describes two interesting circles containing intersections of many lines associated to a regular heptagon. These intersections are vertices of regular heptagons. In the proofs we use the complex numbers and the Maple V software.

1. Introduction

Figure 1: Regular heptagon $ABCDEFG$ and one of its heptagonal triangles $ABD$.

The regular heptagon (i.e., the planar regular convex polygon with seven vertices) has not been studied extensively like its cousins the

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equilateral triangle, the square, the regular pentagon, and the regular hexagon. Perhaps the reason is because this is the regular polygon with the smallest number of vertices that cannot be constructed only with compass and straightedge. The few sporadic known results on regular heptagons were reviewed by Leon Bankoff and Jack Garfunkel 30 years ago in the reference [1].

They first recall the following result by Victor Thébault:

*The distance from the midpoint $U$ of side $AB$ of a regular convex heptagon $ABCDEFG$ inscribed in a circle with center $O$ to the midpoint $V$ of the radius perpendicular to $BC$ and cutting this side, is equal to half the side of a square inscribed in the circle.*

In other words, we have $|UV| = \frac{|AO|\sqrt{2}}{2}$. Extending this to diagonals, Hüseyin Demir observed that the circle $k_m$ of radius $UV$, centered at $V$, bisects the segments $AB$, $BG$, $EA$, $GD$, $CE$ and $DC$ in the midpoints $U$, $X$, $Y$, $Y'$, $X'$ and $U'$ (see the left part of Figure 2).

The right part of Figure 2 shows the second result also by Thébault:

*If $W$ is the midpoint of $OF$, $M$ is the point diametrically opposite $F$ and $J$ is the point on $UB$ produced such that $|UJ| = |UM|$, then $|UW| = |UO|\sqrt{2}$, $|OJ| = \frac{|AO|\sqrt{6}}{2}$ and the line $UV$ is tangent to the circle through $U$, $O$ and $W$.*

The rest of [1] is a study of the heptagonal triangle (for example, the triangle $ABD$ in Figure 1) whose angles are $\frac{\pi}{7}$, $\frac{3\pi}{7}$ and $\frac{4\pi}{7}$ radians. We mention only the following four of their properties from an extensive list (see pages 14, 17 and 19 of [1]):
- The sum of cotangents of angles is equal to $\sqrt{7}$.
- The sum of squares of cotangents of angles is equal to 5.
- The triangle formed by joining the feet of the internal angle bisectors of the heptagonal triangle is isosceles.
- The two tangents from the orthocenter to the circumcircle of the heptagonal triangle are mutually perpendicular.

Today we can add new results to the above list with some help from computers. In papers [2], [3] and [4] the author has improved some of the above theorems. We added six more midpoints of segments in Demir observation that also lie on the circle $k_m$ in [2]. Later in [4] we recognized two regular heptagons inscribed in $k_m$ whose vertices are these midpoints. The reference [3] contains the improvement of the second Thébault result above and some new geometric relationships in regular heptagons.

In this paper we show that the intersections of many lines associated to a regular heptagon $ABCDEFG$ lie on its interesting circles determined either by incenters or by the excenters of the triangles $DEB$ and $ABG$. In other words we discover many regular heptagons related to a given regular heptagon which all have easy construction with compass and straightedge.

Recall that every triangle $ABC$ has the incircle and three excircles which touch the lines $BC$, $CA$ and $AB$. Their centers are the incenter $I$ and the excenters $I_a$, $I_b$ and $I_c$. The incenter is inside while the excenters are outside the triangle and in the natural order $I_a$ is called the first excenter since it lies on the first angle bisector $AI$.

In order to simplify our statements we use the following notation. The parallel and the perpendicular to the line $\ell$ through the point $X$ are $X \parallel \ell$ and $X \perp \ell$.

In our proofs we shall use complex numbers because they provide simple expressions and arguments. There are several excellent books, for example [7], [5], [9], [6], [10], and [8], that give introductions into the method which we utilize. In an appendix we implement this approach in Maple V. The reader can see there how the intersection of two lines is computed. This is in fact the only thing to learn.

A point $P$ in the Gauss plane is identified with a complex number $P$ (its affix). The complex conjugate of $P$ is denoted $\bar{P}$. We shall always assume that the complex coordinates of the vertices of the heptagon $ABCDEFG$ are $A = 1$, $B = f^2$, $C = f^4$, $D = f^6$, $E = f^8$, $F = f^{10}$, and $G = f^{12}$, where $f$ is the 14th root of unity. Had we used the 7th roots of unity some important points like the midpoints $P$ and $Q$ of
the shorter arcs $AG$ and $AB$ would have complicated affixes. Hence, all these points are on the unit circle $k$ whose center is the origin $O$.

2. THE FIRST CIRCLE FROM INCENTERS

We begin our study with the circle $m$ whose center is the incenter $V$ of the triangle $ABG$ and which goes through the incenter $U$ of the triangle $BED$.

Figure 3: The circle $m$ with the center at the incenter of $ABG$ and the radius $\sqrt{2}$ has interesting properties. (Theorems 1–3).

**Theorem 1.** The circle $m$ has the radius $\sqrt{2}$ and it goes through the points $C$ and $F$.

Let $K = IN \cap JM$ where the points $I$, $N$ and $J$, $M$ are intersections of $BC$, $EF$ and $FG$, $CD$ with $G \perp GO$ and $B \perp BO$.

**Theorem 2.** The points $I$, $J$, $M$ and $N$ are on the circle $m$ and the point $V$ is the midpoint of the segment $KO$.

**Theorem 3.** The triangles $BIK$, $GJK$, $BCM$, $FGN$ are heptagonal.

Proof of Theorems 1–3. The points $P$ and $Q$ are $f^{13}$ and $f$. Note that $|BC| = |CD|$ so that $\angle BEC = \angle CED$. It follows that $EC$ is the bisector of the angle $E$ in the triangle $BDE$. In the same way we see
that the line $DG$ is the bisector of the angle $D$ in the triangle $BDE$ and that $BP$, $GQ$ and $AO$ are the bisectors of the angles $B$, $G$ and $A$ in the triangle $ABG$. The incenters $U$ and $V$ are therefore the intersections $CE \cap DG$ and $BP \cap GQ$. Hence, $U = -f_5 + f_4 - 1$ and $V = f_5^2 - f_3 + f_4^2 - f_5^5$. The equation of the circle $m$ with the center $V$ and the radius $\sqrt{2}$ is $(z - V)(z - V) = 2$ or

$$z \bar{z} + f^2 (f - 1)(f^2 + 1)(z + \bar{z}) + f^5 - 2 f^4 + 2 f^3 - f^2 - 1 = 0.$$  

When we substitute the coordinates of the points $C$, $F$, and $U$ for $z$ into this equation we obtain an expression that has the polynomial $p_\pm = f^6 - f^5 + f^4 - f^3 + f^2 - f + 1$ as a factor. Since $f^{14} - 1$ factors as $(f - 1)(f + 1) p_- p_+$, with $p_+ = f^6 + f^5 + f^4 + f^3 + f^2 + f + 1$ and $p_- = 1 + 2 i(1 + 2 \cos \frac{\pi}{7}) \sin \frac{2 \pi}{7}$, it follows that $p_- = 0$ so that the points $C$, $F$, and $U$ are on the circle $m$.

In order to find the affix of the point $I$ (the intersection of the line $BC$ with the perpendicular $G \perp GO$ to the line $GO$ in $G$) notice that $BC$ has the equation $(f^3 - f^3) z - (f^4 - f^2) \bar{z} + f^5 + f^2 = 0$ while the equation of $G \perp GO$ is $f^2 z - f^5 \bar{z} - 2 = 0$. Now we must solve in $z$ and $\bar{z}$ the system formed by these two equations in order to obtain $I = f + f^2 - f^5$. For the points $J$, $M$, and $N$ we get similarly $J = \bar{I}$, $M = f^4 - f^3 + f^2 + f - 1$, and $N = \bar{M}$.

Once we know the points $I$, $J$, $M$, and $N$ the rest of the proof is a routine verification. The substitution of their coordinates into the equation of the circle $m$ always contain the factor $p_-$ which is zero. Notice that the lines $IN$ and $JM$ are tangents of the circle $k$. Finally, solving linear equations we can compute the affix of the intersection $K = 2V$ of these tangents. Clearly, the point $V$ is the midpoint of the segment $KO$. Then we look for conditions (see [5] and the appendix) that the triangles $JKG$ and $FNG$ are directly similar to the heptagonal triangle $DEG$ and that the triangles $IKB$ and $CMB$ are reversely similar to the heptagonal triangle $DEG$. In all four cases the above factor $p_-$ of $f^{14} - 1$ (which is zero) appears.

\[ \square \]

3. Three regular heptagons inscribed in $m$

In the next two theorems we shall describe three regular heptagons inscribed in the circle $m$ whose easy constructions with compass and straightedge depend on the points $I$, $J$, $M$ and $N$.

**Theorem 4.** Let the points $H$, $J'$, $S$, $U'$, $H'$, $I'$, $S'$ be intersections of $AP$, $AC$, $CG$, $BE$, $BF$, $AF$, $DG$ with $BE$, $N \parallel FG$, $K \parallel AG$, $M \parallel CE$, $K \parallel CG$, $K \parallel BF$, $M \parallel CG$, respectively. Then $FUMHIJ'S$ and $NU'CH'I'JS'$ are regular heptagons inscribed in $m$ (see Fig. 4).
The point $H$ lies also on $N \parallel BF$ and $U'$ is the incenter of the triangle $DEG$.

**Theorem 5.** The midpoints $B_0$, $A_0$, $G_0$, $F_0$, $E_0$, $D_0$, and $C_0$ of the shorter arcs $NF$, $U'U$, $CM$, $HH'$, $I'I$, $JJ'$, and $SS'$ are vertices of a regular heptagon whose sides are parallel to the corresponding sides of $BAGFEDC'$ (see Fig. 5).

**Proof of Theorems 4 and 5.** The equations of the lines $AP$ and $BE$ are

$$(1 - f)z + (f^{13} - 1)\bar{z} + f - f^{13} = 0$$

and

$$(f^{12} - f^6)z + (f^8 - f^2)\bar{z} + f^8 - f^{20} = 0.$$ 

Their intersection $H$ is $-f^5 + 2f^4 - f^3 + 2f^2 - f + 1$. Also,

$J' = -2f^5 + 2f^4 - 2f^3 + f^2 + 1$,  \quad $S = -2f^5 + f^4 - 2f^3 + 2f^2 - f$,

$I' = -f^5 + 2f^4 - 2f^3 + 2f^2 + 1$,  \quad $H' = -f^5 + f^4 + f^2 + f - 1$,

$U' = -f^3 + f^2 - 1$ and $S' = -f^5 - f^3 + f$. Let us define the number $w$ to be $f - f^2 - f^4$. Then $|S'V|^2 = (S' - V)(S' - \bar{V}) = w(1 - w)$ is equal to 2. In the same way we verify that $|HV|^2$, $|J'V|^2$, $|SV|^2$, $|U'V|^2$, $|H'V|^2$, and $|I'V|^2$ are also 2 so that the heptagons $FUMHIJ'S$ and
Figure 5: The regular heptagon on midpoints of shorter arcs $NF, U'U, CM, HH', I'I, JJ'$, and $SS'$ has sides parallel to the corresponding sides of $BAGFEDC$. (Theorem 5).

$NU'CH'IJS'$ are inscribed in $m$. That these are regular heptagons follows from the fact that $|FU|^2, |UM|^2, |MH|^2, |HI|^2, |J'I|^2, |JS'|^2, |SF|^2, |NU'|^2, |U'C|^2, |CH'|^2, |H'I|^2, |I'J|^2, |JS'|^2$, and $|S'N|^2$ all have the same value $2f^5 - 2f^2 + 4$.

In order to find the midpoint $B_0$ of the shorter arc $FN$ we use that it has equal distances from the points $F$ and $N$, that it lies on $m$, that its distance to the point $F$ is less than $\sqrt{2}$ (the radius of $m$) and that it is a polynomial of order at most five in $f$. Hence, $B_0 = -f^5 + f^4 - f^3 + (1 - \sqrt{2})f^2$. Similarly,

$$A_0 = -f^5 + f^4 - f^3 + f^2 - \sqrt{2}, \quad C_0 = -f^5 - (1 - \sqrt{2})f^4 - f^3 + f^2,$$

$$D_0 = (1+\sqrt{2})(1-f)(f^4+f^2+2-\sqrt{2}), \quad E_0 = \overline{D_0}, \quad F_0 = \overline{C_0}, \quad G_0 = \overline{B_0}.$$

It is now easy to check that the regular heptagons $B_0A_0G_0F_0E_0D_0C_0$ and $BAGFEDC$ have parallel corresponding sides. \qed
4. Four inscribed regular heptagons

In this section we describe four regular heptagons inscribed in the circumcircles of the triangles $BIK$, $GJK$, $FGN$ and $BCM$ and show that their centers are vertices of a rectangle.

Theorem 6. Let $D_1, C_1, B_1, G_1, G_2, D_2, C_2, B_2, B_3, G_3, F_3, E_3, A_4, E_4, D_4, C_4$ be intersections of $EG$, $BF$, $CG$, $AG$, $AG$, $EG$, $DG$, $CG$, $AB$, $BF$, $BE$, $BD$, $AB$, $BD$, $BE$, $BF$ with $CF$, $DG$, $AF$, $F \parallel BC$, $K \parallel FG$, $J \parallel DE$, $J \parallel BF$, $N \parallel BC$, $K \parallel BC$, $I \parallel EF$, $I \parallel AB$, $I \parallel BG$, $C \parallel FG$, $CF$, $CG$, $AC$. Then $NFD_1C_1B_1GG_1, GG_2KJD_2C_2B_2, IKB_3BG_3F_3E_3$, and $BA_4MCE_4D_4C_4$ are regular heptagons inscribed in the circumcircles of $FGN$, $GJK$, $BIK$, and $BCM$ whose sides are parallel to the corresponding sides of $FEDCBAG$, $AGFEDCB$, $DCBAGFE$, and $BAGFEDC$ (see Fig. 6).

Proof. The circumcenter $O_1$ of the triangle $FGN$ is $-f^5 + f^4 - f^3$ and the equation of its circumcircle $m_1$ is

$$(f^4 + f^2 + 1)z \bar{z} - f^4(f^2 + 1)(z + f^8 \bar{z}) + f^{16} = 0.$$ 

The points $D_1, C_1, B_1, G_1$ are $-f^5 + f^4 - f^2, -f^5 + f^4 - f^3 + f - 1, -f^5 + 2f^4 - 2f^3 + f^2 - f + 1, -f^5 + f^4 - f^3 - f^2 + f$, respectively.
As the expression $f^{2n}(N - O_1) + O_1$, for $n = 1, \ldots, 6$ is $G_1$, $G$, $B_1$, $C_1$, $D_1$, and $F$, we infer that $NFD_1C_1B_1GG_1$ is a regular heptagon inscribed in $m_1$. That its sides are parallel with the corresponding sides of the heptagon $FEDCBAG$ is now easy to verify. The remaining three circumcircles of the triangles $GJK$, $BIK$, and $BCM$ are treated similarly.

Figure 7: The circumcenters of the triangles $FGN$, $GJK$, $BIK$, $BCM$ are vertices of a rectangle. (Theorem 7).

**Theorem 7.** The circumcenters $O_1, O_2, O_3, O_4$ of the triangles $FGN$, $GJK$, $BIK$, and $BCM$ are vertices of a rectangle – the translation for the vector $\overrightarrow{OV}$ of the rectangle $P_1GBQ_1$ where $P_1$ and $Q_1$ are the midpoints of the shorter arcs $EF$ and $CD$ (see Fig. 7).

Proof. Notice that $P_1 = f^9$, $Q_1 = f^5$, $O_2 = -2f^5 + f^4 - f^3 + f^2$, $O_3 = -f^5 + f^4 - f^3 + 2f^2$ and $O_4 = f^4 - f^2 + f^2$. The claim follows from $P_1 + V = O_1$, $G + V = O_2$, $B + V = O_3$ and $Q_1 + V = O_4$. \qed

5. The second circle from excenters

Since the circle $m$ is determined by the incenters of the triangles $DEB$ and $ABG$, we can ask if the excenters of these triangles give a circle containing intersections of some lines related to the regular
heptagon ABCDEFG. The answer to this natural question is given in the following theorems.

Figure 8: The circle \( n \) determined by excenters \( U_0 \) and \( V_0 \) and two regular heptagons inscribed in it. (Theorems 8 and 9).

**Theorem 8.** Let \( U_0 \) and \( V_0 \) be the first excenters of the triangles \( DEB \) and \( ABG \) in the regular heptagon \( ABCDEFG \) inscribed to the circle \( k \) with the center \( O \) and the radius \( R \). Then the circle \( n \) with the center \( V_0 \) and the radius \( R \frac{\cos 3 \pi}{\sin \frac{3 \pi}{7}} \) goes through the points \( U_0 \), \( I \) and \( J \) (see Fig. 8).

**Proof.** Since the excenter \( V_0 \) is the intersection of lines \( AO \) and \( G \perp GV \) we get \( V_0 = f^5 - f^4 + f^3 - f^2 - 2 \). Similarly, the excenter \( U_0 \) is the intersection of the lines \( DG \) and \( E \perp EU \) so that \( U_0 = -f^5 + f^4 + 1 \). The equation of the circle \( n \) with the center at the point \( V_0 \) through the point \( U_0 \) is \( z \bar{z} - V_0(z + \bar{z}) + 7 f^5 - 2 f^4 + 2 f^3 - 7 f^2 - 9 = 0 \). Its radius is \( \sqrt{14 - 10 f^5 + 4 f^4 - 4 f^3 + 10 f^2} \) which reduces to \( \frac{\sqrt{2} \cos \frac{3 \pi}{7}}{\sin \frac{3 \pi}{7}} \). By substitution of coordinates of the points \( I \) and \( J \) in the above equation we can verify that they lie on the circle \( n \). \( \square \)

**Theorem 9.** Let the points \( G_5, F_5, E_5, D_5, C_5, B_5, G_6, F_6, E_6, D_6, C_6, B_6 \) be intersections of \( BG, D \perp DO, FG, JU, JM, I \perp FI, J \perp CJ, IN, CV_0, JV, E \perp EO, BG \) with \( FU, V_0 \parallel FM, D \parallel BU, \)
Proof. Solving linear equations we get $G_5 = -2f^5 - 2f^3 + f^2 - 2f + 1$, $F_5 = -3f^4 - 2f^2 - f - 2$, $E_5 = 2f^5 - 2f^3 + f^2 - 4$, $D_5 = 3f^5 - f^4 + 2f^3 - 2f^2 - 5$, $C_5 = 4f^5 - f^4 + 3f^3 - f^2 + f - 3$, $B_5 = f^5 + 3f^3 + f - 1$, $G_6 = B_5$, $F_6 = C_5$, $E_6 = D_5$, $D_6 = E_5$, $C_6 = F_5$ and $B_6 = G_5$.

Since the expressions $f^{2k}(1 - V_0) + V_0$ and $f^{2k}(I - V_0) + V_0$ for $k$ from 1 to 6 are $B_5$, $C_5$, $D_5$, $E_5$, $F_5$, $G_5$ and $B_6$, $C_6$, $D_6$, $E_6$, $F_6$, $G_6$ we conclude that $IG_5F_5E_5D_5C_5B_5$ and $JG_6F_6E_6D_6C_6B_6$ are regular heptagons inscribed in $n$.

Let $\eta = \sqrt{\frac{3}{7}}$. As in the proof of Theorem 5 we find

$$A_7 = (1 - 3\eta) \left( f^5 + \frac{1 + 14\eta}{11} f^3(f - 1) - f^2 - 2 \right),$$

$$B_7 = f^5 - (1 - 2\eta)f^4 + (1 + \eta)f^3 - (1 - 5\eta)f^2 + \eta f - 2(1 - \eta).$$

The condition for the lines $AB$ and $A_7B_7$ to be parallel (which must be zero) holds because it contains $p_-$ as a factor. Since $A_7B_2C_7D_7E_7F_7G_7$ is obviously a regular heptagon it follows that its sides are parallel with the corresponding sides of $ABCDEFG$. \hfill \Box

Theorem 10. Let the points $C_8$, $B_8$, $A_8$, $G_8$, $F_8$, $E_8$, $D_9$, $C_9$, $B_9$, $A_9$, $G_9$, $F_9$, $E_9$ be intersections of $AG$ with $BE$, $EJ$, $IM$, $CG$, $BF$, $CG$, $BJ$, $JN$, $DI$, $DG$, $AB$, $BE$ with $J \perp JM$, $FI$, $IV_0$, $J \perp CQ$, $FU$, $E \perp EF$, $J \perp JV_0$, $D \perp FO$, $U_0 \perp BD$, $JV_0$, $CJ$, $N \perp CO$, $CV_0$. Then $U_0C_8B_8A_8G_8F_8E_8$ and $D_9C_9B_9A_9G_9F_9E_9$ are regular heptagons inscribed in $n$ (see Fig. 9). The midpoints $D_{10}$, $C_{10}$, $B_{10}$, $A_{10}$, $G_{10}$, $F_{10}$, $E_{10}$ of the shorter arcs $U_0D_9$, $C_8C_9$, $B_8B_9$, $A_8A_9$, $G_8G_9$, $F_8F_9$, $E_8E_9$ are vertices of a regular heptagon whose sides are parallel to the corresponding sides of $DCBAGFE$ (see Fig. 10).

Proof. From linear equations we get $C_8 = -2f^5 - f^4 - f^3 - f^2 - f - 1$, $B_8 = f^5 - 2f^4 - f^3 - 2f^2 - f - 3$, $A_8 = 3f^5 - 2f^4 + 2f^3 - 3f^2 - f - 4$, $G_8 = 4f^5 - 2f^4 + 4f^3 - 3f^2 + 2f - 5$, $F_8 = 2f^5 + f^4 + 2f^3 + f - 2$, $E_8 = f^3 + 2f^2$, $D_9 = F_{10}$, $C_9 = F_{10}$, $B_9 = G_{10}$, $A_9 = A_{10}$, $G_9 = B_{10}$, $F_9 = C_{10}$ and $E_9 = U_{10}$. Since $f^{2k}(U_0 - V_0) + V_0$ and $f^{2k}(D_0 - V_0) + V_0$ for $k$ from 1 to 6 are $E_8$, $F_8$, $G_8$, $A_8$, $B_8$, $C_8$ and $E_9$, $F_9$, $G_9$, $A_9$, $B_9$, $C_9$ we conclude that $U_0C_8B_8A_8G_8F_8E_8$ and $D_9C_9B_9A_9G_9F_9E_9$ are regular heptagons inscribed in $n$.\hfill \Box
This time we get

$$A_{10} = (1 + 3 \eta) \left( f^5 + \frac{1 - 14 \eta}{11} f^3(f - 1) - f^2 - 2 \right),$$

$$B_{10} = f^5 - (1 + 2 \eta)f^4 + (1 - \eta)f^3 - (1 + 5 \eta)f^2 - \eta f - 2(1 + \eta).$$

The condition for the lines $AB$ and $A_{10}B_{10}$ to be parallel again holds because it contains $p_-$ as a factor. Since $A_{10}B_{10}C_{10}D_{10}E_{10}F_{10}G_{10}$ is obviously a regular heptagon it follows that its sides are parallel with the corresponding sides of $ABCDEFG$.  

**Theorem 11.** The points $A_7$, $D_{10}$, $G_7$, $C_{10}$, $F_7$, $B_{10}$, $E_7$, $A_{10}$, $D_7$, $G_{10}$, $C_7$, $F_{10}$, $B_7$, $E_{10}$ are the vertices of the regular 14-gon (see Fig. 10).

**Proof.** Since $A_{10} = f(D_7 - V_0) + V_0$, it follows that by rotating $D_7$ for the angle of $\frac{\pi}{11}$ radians we get $A_{10}$. This implies the claim of the theorem. Notice that the regular heptagons $A_7B_7C_7D_7E_7F_7G_7$ and $A_{10}B_{10}C_{10}D_{10}E_{10}F_{10}G_{10}$ are symmetric with respect to the perpendicular at $O$ to the line $OA$.  

Figure 9: Another two easily constructible regular heptagons inscribed in the circle $n$. (Theorem 10.)
Figure 10: Two regular heptagons on midpoints of shorter arcs inscribed in the circle $n$ and the regular 14-gon from their vertices. (Theorems 8–11).

REFERENCES

This note is an example of a new approach to geometry offered by computers. In this appendix we will reveal how one can check our results on a computer.

The figures are made in the software The Geometer’s Sketchpad that could also be used for approximate verification of statements and in the discovery of new theorems about geometric objects like regular heptagons.

Our mathematically correct proofs were realized on a computer in the software Maple V (version 8). We will describe how to prove Theorems 1–3 in Maple V.

First we give points as ordered pairs \([p, q]\) of the complex number \(p\) and its conjugate \(q\). The complex number \(f\) is the 14\(^{th}\) root of unity.

\[
\begin{align*}
hA &:=[1, 1]; hB := [f^2, f^{12}]; hC := [f^4, f^{10}]; hD := [f^6, f^8]; \\
hE := [f^8, f^6]; hF := [f^{10}, f^4]; hG := [f^{12}, f^2]; hO := [0, 0]; \\
hP := [f^{13}, f]; & \quad hQ := [f, f^{13}];
\end{align*}
\]

Here we use \(hA\) instead of \(A\) as a name of the first vertex because with plain letters we run into problems as some letters are reserved in Maple V (for example \(D\)).

We introduce the shortening \(FS\) for the simultaneous use of commands \texttt{factor} and \texttt{simplify} to reduce typing.

\[FS := x \rightarrow \text{factor}(\text{simplify}(x));\]

The following function computes the square of the distance between two points \(a\) and \(x\).

\[
di := (a, x) \rightarrow FS((a[1] - x[1]) \cdot (a[2] - x[2]));
\]

Lines are represented as ordered triples \([u, v, w]\) of coefficients of their equations \(u z + v \bar{z} + w = 0\). The function \(li\) gives the line through two different points.

\[
\]

The function \(ins\) gives the intersection of two lines. (The names \texttt{in} and \texttt{int} are reserved!). When its usage results in the error message \texttt{Error, numeric exception: division by zero} then the lines are parallel (when they do not have an intersection).

\[
\]

This short introduction into the analytic plane geometry via complex
numbers concludes with the simple functions for the midpoint of two given points and the parallel and the perpendicular through a given point to a given line.

\[
\text{mid} := (a, b) \rightarrow FS([[a[1]+b[1]]/2,(a[2]+b[2])/2]):
\]

\[
\text{par} := (t, p) \rightarrow FS([p[1], p[2], -t[1]*p[1]-t[2]*p[2]]):
\]

\[
\text{per} := (t, p) \rightarrow FS([p[1], -p[2], t[2]*p[2]-t[1]*p[1]]):
\]

The points \(U\) and \(V\) are now obtained as follows:

\[
\text{hU} := \text{ins}(\text{li}(hC, hE), \text{li}(hD, hG)):
\]

\[
\text{hV} := \text{ins}(\text{li}(hB, hP), \text{li}(hG, hQ)):
\]

The circle \(m\) is the locus of all points whose square of distance to the point \(V\) is equal to 2. The following function \(hm\) associates to a point the difference of the square of its distance from \(V\) and 2. A point \(T\) will lie on the circle \(m\) if and only if the value \(hm(T)\) is zero.

\[
\text{hm} := x \rightarrow FS(\text{di}(x, hV) - 2):
\]

We check now the values of \(hm\) in the points \(C\), \(F\), and \(U\).

\[
\text{hm}(hC); \text{hm}(hF); \text{hm}(hU);
\]

The output for the first two inputs is \(p_{-}K\) where \(K\) is

\[
f^{18} - f^{17} - f^{15} + f^{14} - f^{13} + f^{12} - 2 f^{11} +
\]

\[
f^{10} - f^{9} + 3 f^{8} + f^{7} - f^{5} + f^{4} - 2 f - 2
\]

while for the third is \(\frac{p_{-}M}{N^2}\) where \(N = (f^2 + f + 1)(f^2 - f + 1)\) and

\[
M = -2 - 2 f - 9 f^{10} + 3 f^{11} - 4 f^3 - 7 f^5 - 10 f^{15} + 11 f^{14} -
\]

\[
11 f^{17} + 9 f^{16} - 3 f^{23} - 6 f^{21} - 4 f^2 - f^{25} + f^{26} + 9 f^{10} - 4 f^9 +
\]

\[
4 f^8 - 5 f^7 - f^6 + 7 f^{18} + 5 f^{20} + 4 f^{22} + 3 f^{24} - 5 f^4 + 13 f^{12} - 7 f^{13}.
\]

Since all of these expressions contain \(p_{-}\) as a factor we infer that they are equal to zero.

The points \(I\), \(N\), \(J\), \(M\), and \(K\) are defined as follows.

\[
\text{hI} := \text{ins}(\text{li}(hB, hC), \text{per}(hG, \text{li}(hG, hO))):
\]

\[
\text{hN} := \text{ins}(\text{li}(hE, hF), \text{per}(hG, \text{li}(hG, hO))):
\]

\[
\text{hJ} := \text{ins}(\text{li}(hF, hG), \text{per}(hB, \text{li}(hB, hO))):
\]

\[
\text{hM} := \text{ins}(\text{li}(hC, hD), \text{per}(hB, \text{li}(hB, hO))):
\]

\[
\text{hK} := \text{ins}(\text{li}(hI, hN), \text{li}(hJ, hM)):
\]

We compute the values of \(hm\) in the points \(I\), \(N\), \(J\), and \(M\) to verify that they lie on the circle \(m\). Next we find the midpoint of the segment \(KO\) and show that it is at the distance zero from the point \(V\).

\[
\text{hm}(hI); \text{hm}(hN); \text{hm}(hJ); \text{hm}(hM); \text{di}(hV, \text{mid}(hK, kO));
\]
For the last claim we will use the following functions that test if two triangles are directly or reversely similar (see [5]).

\[
\text{sid} := ((a, b, c), (p, q, r)) \rightarrow \text{FS}(a[1]*q[1]-a[1]*r[1]-b[1]*p[1]+b[1]*r[1]+c[1]*p[1]-c[1]*q[1]):
\]

\[
\text{sir} := ((a, b, c), (p, q, r)) \rightarrow \text{FS}(q[2]*a[1]-r[2]*a[1]-p[2]*b[1]+r[2]*b[1]+c[1]*p[2]-q[2]*c[1]):
\]

\[
\text{sid}((hD,hE,hG),(hJ,hK,hG)); \text{sid}((hD,hE,hG),(hF,hN,hG));
\]

\[
\text{sir}((hD,hE,hG),(hI,hK,hB)); \text{sir}((hD,hE,hG),(hC,hM,hB));
\]