THE NEUBERG CUBIC IN LOCUS PROBLEMS

ZVONKO ČERIN

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Abstract. In this paper we continue the exploration of various locus problems whose solutions involve the Neuberg cubic of the scalene triangle in the plane. We use analytical geometry and the complex numbers to show that the Neuberg equation describes the essential part of the locus in many geometric constructions. In this way we discover new characteristics of the Neuberg cubic that has been extensively studied recently.

1. Introduction

Let ABC be a scalene triangle in the plane. The author has considered in [5] numerous locus problems whose solutions involve the circular cubic N which Neuberg [18] calls the 21-point cubic and which is known today as the Neuberg cubic of the triangle ABC. It is evident from the extensive list of references on this curve given below that the Neuberg cubic has attracted a lot of attention lately. The present paper is yet another such contribution. It adds more than two dozens of new instances when the Neuberg cubic appears in various geometric constructions. Most results utilise the notion of the homology for triangles but there are also those that use the concurrence of lines and the concept of the power of a point with respect to a circle.

Our proofs use the analytical geometry of complex numbers. This choice leads to the simplest expressions and appears to be the most natural for our search for the Neuberg cubic. It is suitable for implementation on computers. In fact, our results are all discovered with the help of a computer (PC Pentium 200 MHz, 64 MB RAM) and the software Maple V (version 4). We leave out many details because Maple V (or any other package with symbolic algebra computation capability) performs all factorisations and simplifications easily.

The paper is organised as follows. After the introduction we describe our notation and give basics on the use of complex numbers in geometry. In the remaining sections we present and prove some new results of our search for the Neuberg cubic that all give new characterisations of this remarkable curve by various geometric constructions or locus problems. The section titles are chosen to suggest the method of recognition.

Of course, since our results are characterisations of the same curve, in some cases one can show easily that one method follows from the other(s). Observations of this kind and other comments on possible extensions and special cases are included in remarks.

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2. Complex Numbers in Geometry

In this paper we shall use complex numbers in proofs because they provide simple expressions and arguments. There are several books, for example [16], [9], [23], [12], [26], and [20], that give introductions into the method which we utilise.

A point P in the Gauss plane is represented by a complex number. This number is called the *affix* of P. It is customary to denote the affix of a point P with the corresponding small Latin letter p and to identify a point and its affix. The complex conjugate of p is denoted $\bar{p}$. This rule has an important exception in that the vertices A, B, and C of the reference triangle are represented by numbers u, v, and w on the unit circle. The letters a, b, and c are reserved for the lengths of sides of ABC. Hence, the circumcentre O of ABC is the origin. The affix of O is number 0 (zero) and complex conjugates of u, v, and w are u^{-1}, v^{-1}, and w^{-1}.

Let φ and ψ denote the first and the second cyclic permutation on triples of letters. For example, $\varphi(a) = b$, $\psi(a) = c$, $\varphi(u\,x) = v\,y$, and $\psi(u\,x) = w\,z$. Finally, if f is an expression, $\mathbb{S}f$ and $\mathbb{P}f$ replace $f + \varphi(f) + \psi(f)$ and $f\,\varphi(f)\,\psi(f)$. The expressions $\varphi(f)$ and $\psi(f)$ are called *relatives* of f.

Most interesting points, curves,... associated with the triangle ABC are expressions that involve symmetric functions of u, v, and w that we denote as follows: $\mu = u\,v\,w$, $\sigma = u + v + w$, $\tau = v\,w + w\,u + u\,v$, $\sigma_a = -u + v + w$, $\sigma_b = \varphi(\sigma_a)$, $\sigma_c = \psi(\sigma_a)$, $\mu_a = v\,w$, $\mu_b = w\,u$, $\mu_c = u\,v$, $\tau_a = -v\,w + w\,u + u\,v$, $\tau_b = \varphi(\tau_a)$, $\tau_c = \psi(\tau_a)$, $\delta_a = v - w$, $\delta_b = \varphi(\delta_a)$, $\delta_c = \psi(\delta_a)$, $\zeta_a = v + w$, $\zeta_b = \varphi(\zeta_a)$, $\zeta_c = \psi(\zeta_a)$. For each $k \geq 2$, σ_k, σ_{ka}, σ_{kb}, and σ_{kc} are derived from σ, σ_a, σ_b, and σ_c with the substitution $u = u^k$, $v = v^k$, $w = w^k$. In a similar fashion we can define analogous expressions using letters τ, μ, δ, and ζ.

Let us close this section on preliminaries with a few words on analytic geometry that we shall use and on triangle notation. Any of the books mentioned above contains more than enough information on basic constructions (line through two points, perpendicular and parallel to a line through a point, condition for concurrence of three lines, condition for collinearity of three points) that are needed to follow our arguments. As a convenience for the reader we repeat them here.

In geometry lines are important so that we have the special notation $[m,\, n]$ for the set of all points P that satisfy the equation $\bar{m}\,p - m\,\bar{p} + n = 0$, where n is purely imaginary.

Let X, Y, and Z be three points with affixes x, y, and z and let ℓ be a line $[f,\, h]$ in the plane. Then the line XY is $[x - y,\, x\,\bar{y} - y\,\bar{x}]$, the parallel to ℓ through X is $[f,\, f\,\bar{x} - \bar{f}\,x]$ and the perpendicular to ℓ through X is $[-f,\, -f\,\bar{x} + \bar{f}\,x]$.

The conditions for points X, Y, and Z to be collinear and for lines $[m,\, n]$, $[p,\, q]$, and $[s,\, t]$ to be concurrent are $\mathbb{S}\,\bar{x}\,(y - z) = 0$ and $\mathbb{S}\,m\,(\bar{p}\,t - \bar{s}\,q) = 0$. If X, Y, and Z are not collinear, they determine the unique circle $k(X,\, Y,\, Z)$ which goes through them.

The centroid, the circumcentre, and the centre of the nine-point circle of the triangle XYZ are $(\mathbb{S}\,x)/3$, $(\mathbb{S}\,x\,\bar{x}\,(y - z))/(\mathbb{S}\,\bar{x}\,(y - z))$, and $(\mathbb{S}\,\bar{x}\,(y^2 - z^2))/(2\,\mathbb{S}\,\bar{x}\,(y - z))$.

Let P and Q be points and let ℓ be a line. Then $pa(P,\, \ell)$, $pe(P,\, \ell)$, $pr(P,\, \ell)$, $re(P,\, \ell)$, and $re(P,\, Q)$ denote the parallel to ℓ through P, the perpendicular to ℓ through P, the projection of P onto ℓ, the reflection of P in ℓ, and the reflection of P at Q, respectively.

For a point P not on the circumcircle of a triangle XYZ, let $ig(P, XYZ)$ be the isogonal conjugate of P with respect to XYZ. This point is the intersection of lines which make equal angles with internal angle bisectors as do the lines XP, YP, and ZP.

Let G, O, H, F, K, I_v, and I_u be the centroid, the circumcentre, the orthocentre, the centre of the nine-point circle, the symmedian or Grebe-Lemoine point, the first isogonic point, and the second isogonic point of the base triangle ABC.

We shall need triangles $A_xB_xC_x$, where the index x is either e, r, t, u, and v. In order to describe $A_xB_xC_x$ it suffices to give description of the vertex A_x because B_x and C_x are its relatives. The point A_x is the excentre on AI, the reflection $re(A, BC)$, the intersection of tangents to the circumcircle at B and C, and the apexes of equilateral triangles constructed inwards and outwards on BC, respectively. $A_eB_eC_e$ is the excentral, $A_tB_tC_t$ the tangential, and $A_rB_rC_r$ the three images triangle of ABC.

Triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are *homologous* if lines X_1X_2, Y_1Y_2, and Z_1Z_2 are concurrent.

3. Homology of Triangles — Circumcentres

Among the oldest known methods of recognising the Neuberg cubic are the following two theorems which use the condition that two families of triangles are families of homologous triangles. In this and the next five sections we consider some other uses of this method for recognition of the Neuberg cubic N. Division into sections reflects different ways of defining families of variable triangles.

Let W_0 be the complement of the union of sidelines of the base triangle ABC in the plane. For a point P in the plane, let O_α, O_β, and O_γ denote the circumcentres of the triangles BCP, CAP, and ABP. Neuberg [18] has first proved the following result. As a convenience to the reader we shall give easy proofs of this and the next theorem using complex numbers.

Theorem 3.1. *The locus of all points P in W_0 such that ABC is homologous to $O_\alpha O_\beta O_\gamma$ is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. The O_α is $\mu_a\, M/n_a$, while O_β and O_γ are its relatives, where n_a and M are equations $p + \mu_a\,\bar{p} - \zeta_a$ and $p\,\bar{p} - 1$ of the sideline BC and of the circumcircle. The line AO_α has the form $[\,f/n_a,\ g/(u\,n_a)\,]$, where $f = u\,p + \mu\,\bar{p} - u\,p\,\bar{p} - \tau_a$ and $g = M\,(u^2 - \mu_a)$, while lines BO_β and CO_γ are its relatives. The triangles ABC and $O_\alpha O_\beta O_\gamma$ are homologous if and only if $M\,N\,\mathbb{P}\,\delta_a\,n_a^{-1} = 0$, where $N = \tau\,p^2\,\bar{p} - \mu\,\sigma\,\bar{p}^2\,p + \mu\,\tau\,\bar{p}^2 - \sigma\,p^2 + \sigma_2\,p - \tau_2\,\bar{p}$ is the equation of the Neuberg cubic [16]. □

The following result is proved on page 199 of [16]. It was well known to readers of Mathesis (see [13]) and was mentioned again recently in [21].

Theorem 3.2. *The Neuberg cubic of ABC is the locus of all points P such that ABC is homologous to the triangle on the reflections of P in the sidelines of ABC.*

Proof. The reflection R_α of the point P in the side BC is $\zeta_a - \mu_a\,\bar{p}$ and the line AR_α is $[\,\mu_a\,\bar{p} - \sigma_a,\ u^{-1}\,\mu_a\,\bar{p} - \mu_a^{-1}\,u\,p + \mu^{-1}\,\zeta_a\,(u^2 - \mu_a)\,]$. Hence, triangles ABC and $R_\alpha R_\beta R_\gamma$ are homologous if and only if $\mu_2^{-1}\,N\,\mathbb{P}\,\delta_a = 0$. □

Our first theorem is similar to the Theorem 3.1. We just replace a point P with its isogonal conjugate Q with respect to ABC. Let W_1 denote the complement of the circumcircle of ABC in the plane.

Theorem 3.3. *The locus of all points P in W_1 such that ABC is homologous to the triangle on the circumcentres of BCQ, CAQ, and ABQ is the intersection with W_1 of the Neuberg cubic of ABC.*

Proof. The affix of the isogonal conjugate Q is $(p + \tau\,\bar{p} - \mu\,\bar{p}^2 - \sigma)/M$ and the circumcentre S_a of BCQ is n_a/M. The circumcentres S_b and S_c of CAQ and ABQ have analogous affixes. The triangles ABC and

$S_a S_b S_c$ are homologous if and only if $M^{-2}\,N\,\mathbb{P}\,\delta_a/u^2 = 0$. ☐

In the next theorems, we shall replace circumcentres of triangles BCQ, CAQ, and ABQ with the circumcentres of $B_r C_r Q$, $C_r A_r Q$, and $A_r B_r Q$, where A_r, B_r, and C_r are vertices of the three images triangle of ABC.

The locus of all points P whose isogonal conjugates lie on the sideline $B_r C_r$ of $A_r B_r C_r$ is a conic Δ_a. Let W_3 denote the complement in the plane of the union of the circumcircle of ABC, of the conic Δ_a, and of two other related conics Δ_b and Δ_c.

Theorem 3.4. *The locus of all points P in W_3 such that ABC is homologous to the triangle on the circumcentres of the triangles $B_r C_r Q$, $C_r A_r Q$, and $A_r B_r Q$ is the intersection with W_3 of the union of the sidelines and the Neuberg cubic of ABC and a quartic which goes through the vertices of ABC.*

Proof. From the proof of previous theorem we know the affix of Q and since the affixes of B_r, and C_r are τ_b/v and τ_c/w, we can find the affix of the circumcentre S_a of $B_r C_r Q$. The circumcentres S_b and S_c of $C_r A_r Q$ and $A_r B_r Q$ have analogous affixes. The triangles ABC and

$S_a S_b S_c$ are homologous if and only if $K\,M^{-2}\,N\,\mathbb{P}\,\delta_a\,n_a/(u\,L_a) = 0$, where K and L_a denote expressions $2\,(\tau_2\,p^3\,\bar{p} + \mu^2\,\sigma_2\,p\,\bar{p}^3) + (4\,\tau^3 - \sigma^2\,\tau^2 + 4\,\mu\,\sigma^3 - 15\,\mu\,\sigma\,\tau + 12\,\mu^2)\,p^2\,\bar{p}^2 + (4\,\mu - \sigma\,\tau)(p^3 + \mu^2\,\bar{p}^3) + (8$ $-\tau^2 - 2\,\mu\,\sigma)(2\,p^2 - \sigma\,p) + (\sigma\,\tau^2 - \mu\,\sigma^2 - 2\,\mu\,\tau)(2\,\mu\,\bar{p}^2 - \tau\,\bar{p}) + 3\,(\sigma^2\,\tau^2 - 2\,\mu\,\sigma^3 - 2\,\tau^3 + 7\,\mu\,\sigma$ $\tau - 12\,\mu^2)\,p\,\bar{p} + 2\,(\mu\,\sigma^3 - 6\,\mu\,\sigma\,\tau + 8\,\mu^2)$ and $\tau_a\,p^2 + (\zeta_{2a}\,u^2 - \zeta_a\,(\zeta_{2a} - \mu_a)\,u + \mu_a\,\zeta_{2a})\,p\,\bar{p} - u$ $(\zeta_{2a} + u\,\zeta_a)\,p + \mu\,\mu_a\,\sigma_a\,\bar{p}^2 - \mu_a\,(\mu_a\,\zeta_a + u\,\zeta_{2a})\,\bar{p} + (v^2 + \mu_b)(w^2 + \mu_c)$. Notice that $K = 0$ is the equation of a quartic which goes through the vertices of ABC while $L_a = 0$ is the equation of the conic Δ_a. ☐

Instead of the homology with ABC, the following result looks at the homology of $O_\alpha O_\beta O_\gamma$ with the triangle on circumcentres of $B_r C_r P$, $C_r A_r P$, and $A_r B_r P$. Let W_{r0} denote the complement of the union of sidelines of triangles ABC and $A_r B_r C_r$ in the plane.

Theorem 3.5. *The locus of all points P in W_{r0} such that the triangle $O_\alpha O_\beta O_\gamma$ on circumcentres of BCP, CAP, and ABP is homologous to the triangle on the circumcentres of the triangles $B_r C_r P$, $C_r A_r P$, and $A_r B_r P$ is the intersection with W_{r0} of the union of the circumcircle of $A_r B_r C_r$, the circumcircle of ABC, and the Neuberg cubic of ABC.*

Proof. Since the affixes of A_r, B_r, and C_r are τ_a/u, τ_b/v, and τ_c/w, the circumcentre O_a^r of the triangle $B_r C_r P$ is $(\,\mu\,\tau_a\,p\,\bar{p} + 2\,\mu_a\,(\mu_a - u^2)\,p - \sigma_a\,\tau_b\,\tau_c) B_r C_r(P)^{-1}$, where $B_r C_r(P)$ denotes the value at P of the equation of the line B_rC_r. Now we can determine the line $O_\alpha O_a^r$, find the lines $O_\beta O_b^r$ and $O_\gamma O_c^r$ using the usual substitutions, and discover that these lines are concurrent if and only if $k(A_r, B_r, C_r)(P)\, M\, N\, \mathbb{P}\, u^{-1}\, \delta_a^3\, B_rC_r(P)^{-1}\, BC(P)^{-1} = 0$, where $k(A_r, B_r, C_r)(P)$ is the value at P of the equation of the circumcircle of $A_rB_rC_r$. □

Remark 1. We can get analogous results to the above two theorems by replacing $A_rB_rC_r$ with either $A_uB_uC_u$ or $A_vB_vC_v$.

4. Homology of Triangles — Antipedal Triangles

The common feature of results in this section is that the homology of the antipedal triangle of a variable point P with triangles on circumcentres is used to recognise the Neuberg cubic.

Theorem 4.1. *The locus of all points P in W_0 such that the antipedal triangle $P^\alpha P^\beta P^\gamma$ of P with respect to ABC is homologous to the triangle on the circumcentres of BCH_α, CAH_β, and ABH_γ, where H_α, H_β, and H_γ are orthocentres of BCP, CAP, and ABP, is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. The point H_α has affix $(p^2 - \mu_a\, p\, \bar{p} + \mu_a\, \zeta_a\, \bar{p} - \zeta_{2a})/n_a$ so that the affix of the circumcentre S_a of BCH_α is $(\zeta_a\, p + \mu_a\, \zeta_a\, \bar{p} - \mu_a\, p\, \bar{p} - \zeta_{2a} - \mu_a)/n_a$. The lines $P^\beta S_b$ and $P^\gamma S_c$ are relatives of $P^\alpha S_a$. These three lines are concurrent if and only if $2\, M\, N\, \mathbb{P}\, \delta_a/(u\, n_a) = 0$. □

Remark 2. Let R_α denote the reflection of a point P at the sideline BC. The triangles BCH_α and BCR_α have the same circumcentre so that in the above theorem the orthocentres H_α, H_β, and H_γ could be replaces with the reflections R_α, R_β, and R_γ of P at the sidelines of ABC.

Let W_2 be the complement in the plane of the union of the three sidelines of ABC and the three circles with sides of ABC as diameters.

Theorem 4.2. *The locus of all points P in W_2 such that the antipedal triangle $P^\alpha P^\beta P^\gamma$ of P with respect to ABC is homologous to the triangle on the circumcentres of BCO_α, CAO_β, and ABO_γ, where O_α, O_β, and O_γ are circumcentres of BCP, CAP, and ABP, is the intersection with W_2 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. The point O_α has affix $\mu_a\, M/n_a$ so that the affix of the circumcentre S_a of BCO_α is $\mu_a\, (\mu_a\, (p^2\, \bar{p}^2 - \mu_a\, \bar{p}^2 - 1) - p^2 + \zeta_{2a} - 2\, U)/(n_a\, U)$, where $U = 2\, \mu_a\, p\, \bar{p} - \zeta_a\, p - \mu_a\, \zeta_a\, \bar{p} + \zeta_{2a}$ is the equation of the circle with BC as diameter. The lines $P^\beta S_b$ and $P^\gamma S_c$ are relatives of $P^\alpha S_a$. These three lines concur if and only if $2\, M\, N\, \mathbb{P}\, \delta_a\, (p - u)(u\, \bar{p} - 1)/(U\, n_a) = 0$. □

5. Homology of Triangles — Orthocentres

Here we obtain the Neuberg cubic in homologies with triangles on the orthocentres of variable triangles. The last result also uses the centres of the nine-point circles.

Theorem 5.1. *The locus of all points P in W_0 such that ABC is homologous to the triangle on the orthocentres of the triangles $OO_\beta O_\gamma$, $OO_\gamma O_\alpha$, and $OO_\alpha O_\beta$ is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. The orthocentre H_a of $O O_\beta O_\gamma$ has affix $u\,\zeta_a\,M\,(p-u)/(n_b\,n_c)$. Of course, the other two orthocentres H_b and H_c have analogous affixes. Hence, the triangles ABC and $H_aH_bH_c$ are homologous if and only if $M\,N\,\mathbb{P}\,\delta_a/(u\,n_a)=0$. □

Remark 3. We get an analogous result to the above theorem by replacing ABC with the triangle $G_\alpha G_\beta G_\gamma$ on centroids of BCP, CAP, and ABP.

Theorem 5.2. *The locus of all points P in W_0 such that ABC is homologous to the triangle on the orthocentres of the triangles $GG_\beta G_\gamma$, $GG_\gamma G_\alpha$, and $GG_\alpha G_\beta$ is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. The orthocentre H_a of $GG_\beta G_\gamma$ has affix $(2\,\mu_a\,p\,\bar p+u\,p+\mu\,\bar p-\tau-\mu_a)/(3\,n_a)$. The other two orthocentres H_b and H_c are relatives of H_a. It follows that the triangles ABC and $H_aH_bH_c$ are homologous if and only if $\frac{8}{27}\,M\,N\,\mathbb{P}\,\delta_a/(u\,n_a)=0$. □

Remark 4. We get a similar result to the above theorem by replacing ABC with the triangle $O_\alpha O_\beta O_\gamma$ on circumcentres of BCP, CAP, and ABP.

Theorem 5.3. *The Neuberg cubic of the triangle ABC is the locus of all points P such that ABC is homologous to the triangle on either the centres of the nine-point circles or the orthocentres of $R_\beta R_\gamma P$, $R_\gamma R_\alpha P$, and $R_\alpha R_\beta P$, where R_α, R_β, and R_γ are reflections of P at the sidelines BC, CA, and AB.*

Proof. The orthocentre H_a of $R_\beta R_\gamma P$ is $p+(\tau-\mu_a)\,\bar p+\zeta_a$. The orthocentres H_b and H_c of $R_\gamma R_\alpha P$, and $R_\alpha R_\beta P$ are relatives of H_a. The triangles ABC and $H_aH_bH_c$ are homologous if and only if $\mu_2^{-1}\,N\,\mathbb{P}\,\delta_a=0$.

For the second part observe that the centre of nine-point circle of $R_\beta R_\gamma P$ is collinear with the points A and H_a. □

6. Homology of Triangles — Symmedian and Isogonic Points

The second theorem in this section is analogous to the following theorem which is an exercise on page 200 of [16]. It was restated as the Superior Locus Problem by J. Tabov in [24] and it was resolved by the author in [3] (see also [4]).

Theorem 6.1. *The locus of all points P in W_0 such that the Euler lines of the triangles ABP, CAP, and BCP are concurrent (at the point on the Euler line of ABC) is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. We know the circumcentre O_α of the triangle BCP and since its centroid G_α is $(p+\zeta_a)/3$ it follows that the Euler line $G_\alpha O_\alpha$ of this triangle is

$$[\,n_a^{-1}\,(p^2-2\,\mu_a\,p\,\bar p+\mu_a\,\zeta_a\,\bar p+\mu_a-\zeta_{2a}),\;n_a^{-1}\,M\,(p-\mu_a\,\bar p)\,].$$

Hence, the Euler lines $G_\alpha O_\alpha$, $G_\beta O_\beta$, and $G_\gamma O_\gamma$ concur if and only if $M\,N\,\mu^{-1}\,\mathbb{P}\,\delta_a\,n_a^{-1}=0$. Notice that these lines intersect on the Euler line GO of ABC. □

Recall that the *Brocard diameter* or the *Brocard axis* are the names for the central line joining the circumcentre with the symmedian point (or the Grebe-Lemoine point) of a triangle.

Theorem 6.2. *The locus of all points P in W_0 such that the Brocard diameters of the triangles ABP, CAP, and BCP are concurrent (at the point on the Brocard axis of ABC) is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. The affix of K_α is $(\mu_a\,\zeta_a\,p\,\bar{p} + 2\,(\mu_a - \zeta_{2a})\,p - 2\,\mu_{2a}\,\bar{p} + \mu_a\,\zeta_a)/U_a$, where the complex number U_a is $((2\,p - \zeta_a)(2\,\mu_a\,\bar{p} - \zeta_a) - 3\,\delta_a^2)/2$ and thus is never zero. The affix of O_α is $\mu_a\,M/n_a$, so that the triangles $K_\alpha K_\beta K_\gamma$ and $O_\alpha O_\beta O_\gamma$ are homologous if and only if $8\,M\,N\,\mathbb{P}\,\delta_a\,(p-u)(u\,\bar{p}-1)/(n_a\,U_a) = 0$. Notice that the lines $O_\alpha K_\alpha$, $O_\beta K_\beta$, and $O_\gamma K_\gamma$ intersect on the Brocard axis KO of ABC. □

The second theorem in this section is similar to the Theorem 3.1. In it we replace the circumcentres with isogonic points. Let W_5 be the complement in the plane of the apexes A_u, B_u, and C_u of equilateral triangles built towards inside on the sides of ABC.

Theorem 6.3. *The locus of all points P in W_5 such that ABC is homologous to the triangle $I_{u\alpha}I_{u\beta}I_{u\gamma}$ on the second isogonic points of BCP, CAP, and ABP is the intersection with W_5 of the union of the equilateral hyperbola through A_u, B_u, and C_u with the centre at the first isogonic point I_v of ABC and the Neuberg cubic of ABC.*

Proof. The point $I_{u\alpha}$ is $(U\,\eta + V)/(X\,\eta + Y)$, where η denotes $-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$ (the cube root of unity), the letter U is an abbreviation for $\mu_a\,\zeta_a\,p\,\bar{p} + 2\,(\mu_a - \zeta_{2a})\,p - 2\,\mu_a^2\,\bar{p} + \mu_a\,\zeta_a$, the letter V for $v\,\mu_a\,p\,\bar{p} - \delta_a\,p^2 + (\mu_a - \zeta_{2a})\,p - \mu_a\,(\mu_a - \zeta_{2a})\,\bar{p} + w\,\mu_a + \delta_{3a}$, and the letters X and Y for $2\,\mu_a\,p\,\bar{p} - \zeta_a\,p - 2\,\mu_a\,\zeta_a\,\bar{p} + 4\,\mu_a - \zeta_{2a}$ and $\mu_a\,p\,\bar{p} - (\delta_a + v)\,p - \mu_a\,(\delta_a + v)\,\bar{p} + v^2 + 2\,w\,\delta_a$. The other two second isogonic points $I_{u\beta}$ and $I_{u\gamma}$ are relatives of $I_{u\alpha}$. It follows that the triangles ABC and $I_{u\alpha}I_{u\beta}I_{u\gamma}$ are homologous if and only if $H_u\,N\,\mathbb{P}\,\delta_a/(u^2\,(X\,\eta + Y)) = 0$, where $H_u = 0$ is the equation (in p) of the hyperbola from the statement of the theorem. In order to see that $X\,\eta + Y = 0$ only when $p = m$, where m is the affix of A_u, it suffices to note that the value of $X\,\eta + Y$ at $m + n$ is equal $(1 + 2\,\eta)\,n\,\bar{n}\,v\,w$. □

Remark 5. Of course, there is a dual result to the above theorem with first isogonic points of BCP, CAP, and ABP. The hyperbola of the locus has its centre at the second isogonic point of ABC.

7. Homology of Triangles — Reflections

In this section we use homology with triangles whose vertices are reflections in appropriate lines. Let W_4 be the complement in the plane of the vertices A, B, and C of the triangle ABC.

Theorem 7.1. *The locus of all points P in W_4 such that ABC is homologous to the triangle on reflections in sidelines of ABC of inversions of A, B, and C with respect to the circles $k(B, C, P)$, $k(C, A, P)$, and $k(A, B, P)$ is the intersection with W_4 of the union of the sidelines, the circumcircle, and the Neuberg cubic of ABC.*

Proof. The inversion of A with respect to the circle $k(B, C, P)$ is $(\tau_a\,M - u\,n_a)/(u\,M - n_a)$ and its reflection T_a in BC is $(\mu\,M - \tau_a\,n_a)/(\mu_a\,M - u\,n_a)$. The other two reflections T_b and T_c are relatives of T_a. The triangles ABC and $T_aT_bT_c$ are homologous if and only if $M\,N\,\mathbb{P}\,\delta_a^3\,n_a/(u^2\,(u\,M - n_a)(\mu_a\,M - u\,n_a)) = 0$. From this our theorem follows immediately provided one observes that up to a constant $\mu_a\,M - u\,n_a$ is a complex conjugate of $u\,M - n_a$ and both are zero only at the affixes of B and C. □

Theorem 7.2. *The locus of all points P in W_4 such that ABC is homologous to the triangle on reflections in sidelines of the extriangle $A_eB_eC_e$ of inversions of A, B, and C with respect to the circles $k(B, C, P)$, $k(C, A, P)$, and $k(A, B, P)$ is the intersection with W_4 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. In this proof, in order to avoid the appearance of square roots, we shall assume that the vertices A, B, and C have affixes u^2, v^2, and w^2 for some unimodular numbers u, v, and w. The reflection T_a in B_eC_e of the inversion of A with respect to the circle $k(B, C, P)$ is $(U\,M - u^4\,c_2(n_a))/c_2(\mu_a\,M - u\,n_a)$, where $U = \mu_a\,(u^4 + (\mu_a - \zeta_{2a})\,u^2 + \mu_{2a})$ and c_2 performs the substitution $u \to u^2$, $v \to v^2$, $w \to w^2$. The other two reflections T_b and T_c are relatives of T_a. The triangles ABC and $T_aT_bT_c$ are homologous if and only if $M^4\,c_2(N)\,\mathbb{P}\,c_2(\delta_a)^3/(u^2\,c_2(u\,M - n_a)\,c_2(\mu_a\,M - u\,n_a)) = 0$. □

Theorem 7.3. *If ABC has no right angle, then the locus of all points P in W_4 such that the tangential triangle $A_tB_tC_t$ is homologous to the triangle on reflections in BC, CA, and AB of the second intersections of lines AP, BP, and CP with the circumcircle of ABC is the intersection with W_4 of the union of the sidelines and the Neuberg cubic of ABC.*

Proof. Since the affix of A_t is $2\,\mu_a/\zeta_a$, the affix of the second intersection S_a of AP with the circumcircle of ABC is $(u - p)/(u\,\bar{p} - 1)$, and the affix of the reflection T_a of S_a in BC is $(\zeta_a\,p + \mu\,\bar{p} - \tau)/(p - u)$, the triangles $A_tB_tC_t$ and $T_aT_bT_c$ are homologous if and only if $2\,N\,\mathbb{P}\,\delta_a\,n_a/(u\,\zeta_a\,(p - u)(u\,\bar{p} - 1)) = 0$. □

Theorem 7.4. *The locus of all points P in W_4 such that the pedal triangle $P_\alpha P_\beta P_\gamma$ of P with respect to ABC is*

homologous to the triangle on reflections in $P_\beta P_\gamma$, $P_\gamma P_\alpha$, and $P_\alpha P_\beta$ of P is the intersection with W_4 of the union of the sidelines, the circumcircle, and the Neuberg cubic of ABC.

Proof. Since the affix of P_α is $(p - \mu_a\,\bar{p} + \zeta_a)/2$ and the affix of the reflection T_a of P in $P_\beta P_\gamma$ is $(p^2 - u\,\zeta_a\,p\,\bar{p} + \zeta_a\,p - \mu\,u\,\bar{p}^2 + u\,(\tau + \mu_a)\,\bar{p} - \zeta_b\,\zeta_c)/(2\,(p - u))$, the triangles $P_\alpha P_\beta P_\gamma$ and $T_aT_bT_c$ are homologous if and only if $\frac{1}{16}\,M^2\,N\,\mathbb{P}\,\delta_a\,n_a/(u^2\,(p - u)(u\,\bar{p} - 1)) = 0$. □

Remark 6. Since the triangle $R_\alpha R_\beta R_\gamma$ on reflections of a point P in sides of ABC is homothetic to the pedal triangle $P_\alpha P_\beta P_\gamma$ from P, the above theorem holds also for $R_\alpha R_\beta R_\gamma$ in place of $P_\alpha P_\beta P_\gamma$.

Theorem 7.5. *The locus of all points P in W_0 such that the antipedal triangle $P^\alpha P^\beta P^\gamma$ of P with respect to ABC is*

homologous to the triangle on reflections in $P^\beta P^\gamma$, $P^\gamma P^\alpha$, and $P^\alpha P^\beta$ of P is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.

Proof. The affix of P^α is $(\mu_a\,p\,\bar{p} - p^2 + \zeta_a\,p - 2\,\mu_a)/n_a$ and the affix of the reflection T_a of P in $P^\beta P^\gamma$ is $2\,u - p$, so that the triangles $P^\alpha P^\beta P^\gamma$ and $T_aT_bT_c$ are homologous if and only if $16\,M\,N\,\mathbb{P}\,\delta_a/(u\,n_a) = 0$. □

8. Homology of Triangles — Isogonal Conjugacy

Here we encounter the Neuberg cubic in homologies with triangles whose vertices are isogonal conjugates of various points with respect to appropriate variable triangles.

Theorem 8.1. *The union of the circumcircle and the Neuberg cubic of the triangle ABC is the locus of all points P such that the pedal triangle $P_\alpha P_\beta P_\gamma$ of the point P with respect to ABC is homologous to the triangle on isogonal conjugates of P_α, P_β, and P_γ with respect to triangles $PP_\beta P_\gamma$, $PP_\gamma P_\alpha$, and $PP_\alpha P_\beta$.*

Proof. The vertex P_α has affix $(p - \mu_a\,\bar p + \zeta_a)/2$ while the isogonal conjugate T_a of P_α with respect to the triangle $PP_\beta P_\gamma$ has affix $(2\,p - u\,M)/2$. Hence, the triangles $P_\alpha P_\beta P_\gamma$ and $T_a T_b T_c$ are homologous if and only if $\frac{1}{16}\,M^2\,N\,\mathbb{P}\,\delta_a/u^2 = 0$. □

Remark 7. The above theorem holds also for the triangle on reflections of a point P in sides of ABC instead of the pedal triangle $P_\alpha P_\beta P_\gamma$.

Theorem 8.2. *The locus of all points P in W_0 such that the triangle $O_\alpha O_\beta O_\gamma$ on the circumcentres of triangles BCP, CAP, and ABP is homologous to the triangle on the isogonal conjugates of O_α, O_β, and O_γ with respect to triangles $PO_\beta O_\gamma$, $PO_\gamma O_\alpha$, and $PO_\alpha O_\beta$ is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC.*

Proof. The vertex O_α has affix $\mu_a\,M/n_a$ while the isogonal conjugate T_a of O_α with respect to the triangle $PO_\beta O_\gamma$ has affix $u\,M/(u\,\bar p - 1)$. It follows that the triangles $O_\alpha O_\beta O_\gamma$ and $T_a T_b T_c$ are homologous if and only if $M^4\,N\,\mathbb{P}\,\delta_a/(n_a\,(p-u)(u\,\bar p - 1)) = 0$. □

Theorem 8.3. *The locus of all points P in W_4 such that the triangle $S_\alpha S_\beta S_\gamma$ on the second intersections of lines AP, BP, and CP with the circumcircle of ABC is homologous to the triangle on the isogonal conjugates of S_α, S_β, and S_γ with respect to triangles $PS_\beta S_\gamma$, $PS_\gamma S_\alpha$, and $PS_\alpha S_\beta$ is the intersection with W_4 of the union of the sidelines, the circumcircle, and the Neuberg cubic of ABC.*

Proof. The complex number $(u-p)/(u\,\bar p - 1)$ is the affix of the vertex S_α. On the other hand, $(\zeta_a\,p^2\,\bar p + \mu\,p\,\bar p^2 - p^2 - (\tau + \mu_a)\,p\,\bar p + u\,p + \mu_a)/((u-p)(v\,\bar p - 1)(w\,\bar p - 1))$ is the affix of the isogonal conjugate T_a of S_α with respect to the triangle $PS_\beta S_\gamma$. Hence, the triangles $S_\alpha S_\beta S_\gamma$ and $T_a T_b T_c$ are homologous if and only if $M^6\,N\,\mathbb{P}\,\delta_a\,n_a/((p-u)^3\,(u\,\bar p - 1)^3) = 0$. □

Theorem 8.4. *The locus of all points P in W_1 such that the triangle $H_\alpha H_\beta H_\gamma$ on the orthocentres of triangles BCP, CAP, and ABP is homologous to the triangle on isogonal conjugates of A, B, and C with respect to those triangles is the intersection with W_1 of the union of the sidelines and the Neuberg cubic of ABC.*

Proof. Since $ig(A,\,BCP)$ is $(p^2 + u\,\zeta_a\,p\,\bar p - \sigma\,p - \mu\,\bar p + \mu_a)/(u\,M)$ and the vertex H_α has the affix $(p^2 - \mu_a\,p\,\bar p + \mu_a\,\zeta_a\,\bar p - \zeta_{2a})/n_a$ it is easy to check that the triangles $H_\alpha H_\beta H_\gamma$ and $ig(A,\,BCP)ig(B,\,CAP)ig(C,\,ABP)$ are homologous iff $M^{-3}\,N\,\mathbb{P}\,\delta_a\,n_a\,u^{-3} = 0$. □

9. Concurrent Parallels

Results in this section use the condition that three lines are concurrent. However, these lines are not lines joining corresponding vertices of two triangles as in previous sections but are parallels to lines.

Theorem 9.1. *The Neuberg cubic of the triangle ABC is the locus of all points P such that the parallels through A, B, and C to the Euler lines of triangles $PP_\beta P_\gamma$, $PP_\gamma P_\alpha$, and $PP_\alpha P_\beta$ formed by P and the vertices of its pedal triangle are concurrent.*

Proof. The parallel $pa(A,\, G_aO_a)$ with the Euler line G_aO_a of $PP_\beta P_\gamma$ through the vertex A is the line $[\,(u\,\zeta_a\,\bar{p} - p - \sigma_a)/2,\, (\mu\,\sigma\,\bar{p} - \tau\,p + \zeta_a\,(u^2 - \mu_a))/(2\,\mu)]$. The other two parallels $pa(B,\, G_bO_b)$ and $pa(C,\, G_cO_c)$ are relatives of $pa(A,\, G_aO_a)$. The condition for these lines to concur is $\frac{1}{8}\,N\,\mathbb{P}\,\delta_a\,u^{-2} = 0$. □

Remark 8. The above theorem is true also when parallels to the Euler lines of $PP_\beta P_\gamma$, $PP_\gamma P_\alpha$, and $PP_\alpha P_\beta$ are drawn through vertices of either the pedal triangle of P with respect to ABC or the triangle on reflections of P in sidelines of ABC.

Theorem 9.2. *The locus of all points P in W_0 such that the parallels through the vertices P^α, P^β, and P^γ of its antipedal triangle to the Euler lines of triangles $PP^\beta P^\gamma$, $PP^\gamma P^\alpha$, and $PP^\alpha P^\beta$ are concurrent is the intersection with W_0 of the union of*

the circumcircle and the Neuberg cubic of ABC.

Proof. As in the proof of the previous theorem, we first find the parallel $pa(P^\alpha,\, G_aO_a)$ with the Euler line G_aO_a of $PP^\beta P^\gamma$ through the vertex P^α. This line has a rather complicated polynomial of order five in p and $\bar{p}$ as the second term. Of course, the other two parallels $pa(B,\, G_bO_b)$ and $pa(C,\, G_cO_c)$ are relatives of $pa(A,\, G_aO_a)$. These lines concur if and only if $48\,M^2\,N\,\mathbb{P}\,\delta_a\,(p-u)(u\,\bar{p}-1)/n_a = 0$. □

Remark 9. The above theorem remains true when parallels to the Euler lines of $PP^\beta P^\gamma$, $PP^\gamma P^\alpha$, and $PP^\alpha P^\beta$ are drawn through vertices of the triangle on the second intersections of lines AP, BP, and CP with the circumcircle of ABC.

10. Characterisations with Power

Neuberg [18] noticed the following theorem which requires the notion of the power of a point with respect to a circle that we recall now.

Let P be a point and k be a circle in the plane with the centre S and the radius r. Then the power $\omega(P,\, k)$ of the point P with respect to the circle k is the number $|PS|^2 - r^2$. For points X and Y in the plane, let $k(X,\, Y)$ denote the circle with the centre at X which passes through Y.

Theorem 10.1. *The Neuberg cubic of ABC is the locus of all points P in the plane such that the product of powers of the point P with respect to the circles $k(A,\, B)$, $k(B,\, C)$, and $k(C,\, A)$ is equal to the product of powers of the point P with respect to the circles $k(A,\, C)$, $k(B,\, A)$, and $k(C,\, B)$.*

Proof. Let $W = p\,\bar{p} - u^{-1}\,p - u\,\bar{p}$. Since $W + \mu_c^{-1}(\zeta_{2\,c} - \mu_c)$ and $W + \mu_b^{-1}(\zeta_{2\,b} - \mu_b)$ are the powers $\omega(P,\, k(A,\, B))$ and $\omega(P,\, k(A,\, C))$, the difference $\mathbb{P}\,w(P,\, k(A,\, B)) - \mathbb{P}\,w(P,\, k(A,\, C))$ is equal to $\mu_2^{-1}\,N\,\mathbb{P}\,\delta_a$. □

The above result uses circles determined by two points (the centre and a point on it). Much more interesting is to consider powers with respect to circles which are given by three points.

For a point P and triangles UVW and XYZ, let $\mathbb{P}\,\omega(U,\, k(P,\, Z,\, X)) - \mathbb{P}\,\omega(U,\, k(P,\, X,\, Y))$ be $\nu(P,\, UVW,\, XYZ)$.

Theorem 10.2. *The locus of all points P in W_0 such that $\nu(P,\, ABC,\, O_\alpha O_\beta O_\gamma) = 0$ is the intersection with W_0 of the union of the circumcircle and the Neuberg cubic of ABC, where O_α, O_β, and O_γ are circumcentres of triangles BCP, CAP, and ABP.*

Proof. Since the quotient $((\tau-\mu_b)\,p\,\bar p-\mu_a\,\zeta_c\,\bar p-\zeta_c\,p+\mu_b+v^2)(p-u)(u\,\bar p-1)/(u\,n_a\,n_c)$ is the power $\omega(A,\,k(P,\,O_\gamma,\,O_\alpha))$ and the power $\omega(A,\,k(P,\,O_\alpha,\,O_\beta))$ is the analogous quotient $((\tau-\mu_c)\,p\,\bar p-\mu_a\,\zeta_b\,\bar p-\zeta_b\,p+\mu_c+w^2)(p-u)(u\,\bar p-1)/(u\,n_a\,n_b)$ and all the remaining four powers which appear in $\nu(P,\,ABC,\,O_\alpha O_\beta O_\gamma)$ are their relatives, $\nu(P,\,ABC,\,O_\alpha O_\beta O_\gamma)=0$ is true if and only if $M\,N\,\mathbb{P}\,\delta_a\,(p-u)(u\,\bar p-1)/(u\,n_a^2)=0$. □

Remark 10. The above theorem remains true when $\nu(P,\,ABC,\,O_\alpha O_\beta O_\gamma)=0$ is replaced by the equation $\nu(O,\,ABC,\,O_\alpha O_\beta O_\gamma)=0$, where O is the circumcentre of ABC.

Theorem 10.3. *The Neuberg cubic of ABC is the locus of all points P in the plane such that $\nu(P,\,ABC,\,R_\alpha R_\beta R_\gamma)=0$, where R_α, R_β, and R_γ are reflections of P in the sidelines BC, CA, and AB, respectively.*

Proof. Since $(u\,p+v\,\mu_c\,\bar p-\mu_c\,p\,\bar p+\mu_c-\zeta_{2c})/\mu_c$ and $(u\,p+w\,\mu_b\,\bar p-\mu_b\,p\,\bar p+\mu_b-\zeta_{2b})/\mu_b$ are powers $\omega(A,\,k(P,\,R_\gamma,\,R_\alpha))$ and $\omega(A,\,k(P,\,R_\alpha,\,R_\beta))$ and the other four powers which appear in $\nu(P,\,ABC,\,R_\alpha R_\beta R_\gamma)$ are their relatives, $\nu(P,\,ABC,\,R_\alpha R_\beta R_\gamma)=0$ is true if and only if $N\,\mathbb{P}\,\delta_a\,u^{-2}=0$. □

Let W_6 be the complement of the union of the circumcircles of triangles BCO, CAO, and ABO in the plane. If P is a point different from the circumcentre O of ABC, let R denote the inversion of P with respect to the circumcircle of ABC.

Theorem 10.4. *The intersection of the Neuberg cubic of ABC with W_6 is the locus of all points P in W_6 such that $\nu(R,\,ABC,\,R_\alpha R_\beta R_\gamma)=0$.*

Proof. Since $(u\,p+v\,\mu_a\,\bar p-\mu_b-v^2)(p-u)(u\,\bar p-1)/(\mu_c\,(\zeta_b\,M-n_b))$ is $\omega(A,\,k(R,\,R_\gamma,\,R_\alpha))$ and $\omega(A,\,k(R,\,R_\alpha,\,R_\beta))$ is $(u\,p+w\,\mu_a\,\bar p-\mu_c-w^2)(p-u)(u\,\bar p-1)/(\mu_b\,(\zeta_c\,M-n_c))$ and all the other four powers which appear in $\nu(R,\,ABC,\,R_\alpha R_\beta R_\gamma)$ are their relatives, it follows that $\nu(R,\,ABC,\,R_\alpha R_\beta R_\gamma)=0$ is true if and only if $N\,\mathbb{P}\,\delta_a\,(p-u)(u\,\bar p-1)/(u^2\,(\zeta_a\,M-n_a))=0$. From this our theorem follows immediately if we observe that $\zeta_a\,M-n_a=0$ is the equation of the circle $k(B,\,C,\,O)$ (or the sideline BC when the angle A is right). □

When the angle A is right, let k_a denote the sideline BC of ABC. Otherwise, we use k_a for a circle which passes through the points B and C and which has the lines joining these points with the circumcentre of ABC as tangents. The (lines) circles k_b and k_c are defined analogously. Let W_7 be the complement in W_1 of the union of k_a, k_b, and k_c. For a point P outside the circumcircle of ABC, let Q denote its isogonal conjugate with respect to ABC.

Theorem 10.5. *The intersection of the union of the sidelines and the Neuberg cubic of ABC with W_7 is the locus of all points P in W_7 such that $\nu(Q,\,ABC,\,R_\alpha R_\beta R_\gamma)=0$.*

Proof. Since $((\tau-\mu_c)\,p\,\bar p-\zeta_a\,p-\mu_b\,\zeta_a\,\bar p+\mu_c+w^2)(p-u)(u\,\bar p-1)\,n_a/(\mu\,M\,(\zeta_b\,M-2\,n_b))$ is the power $\omega(A,\,k(R,\,R_\gamma,\,R_\alpha))$ and the power $\omega(A,\,k(R,\,R_\alpha,\,R_\beta))$ is also the quotient $((\tau-\mu_b)\,p\,\bar p-\zeta_a\,p-\mu_c\,\zeta_a\,\bar p+\mu_b+v^2)(p-u)(u\,\bar p-1)\,n_a/(\mu\,M\,(\zeta_c\,M-2\,n_c))$ and the other powers which appear in $\nu(Q,\,ABC,\,R_\alpha R_\beta R_\gamma)$ are their relatives, $\nu(Q,\,ABC,\,R_\alpha R_\beta R_\gamma)=0$ if and only if $M^{-2}\,N\,\mathbb{P}\,\delta_a\,n_a\,(p-u)(u\,\bar p-1)/(u^3\,(\zeta_a\,M-2\,n_a))=0$. From this our theorem follows provided one observes that $\zeta_a\,M-2\,n_a=0$ is the equation of the circle (line) k_a. □

Remark 11. Let $\nu_0(P,\,UVW,\,XYZ)=\mathbb{P}\,\omega(P,\,k(V,\,W,\,Y))-\mathbb{P}\,\omega(P,\,k(V,\,W,\,Z))$ for a point P and triangles UVW and XYZ. It is interesting that in all results in this section replacing the function ν with the above function ν_0 the Neuberg cubic of ABC will again appear. However, the exception sets are more complicated and the locus might include curves of higher order.

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Kopernikova 7, 10010 Zagreb, Croatia

E-mail address: cerin@math.hr