Decision Making

Decision Graph

One criterion. The set of vertices equals the set of alternatives $V = \{i, j, k, \ldots\}$. A pair $\alpha = (j, i)$ is in the set of arcs $\mathcal{A}$ iff i and j are compared and i is more preferable than j. An arc α is weighted by nonnegative number F_α. In case of equal preference $F_\alpha = 0$ and arc orientation is arbitrary.

$$(i) \xleftarrow{\alpha,\, F_\alpha \geq 0} (j)$$

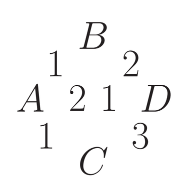
$F : \mathcal{A} \to \mathbb{R}$ we call `preference flow`.

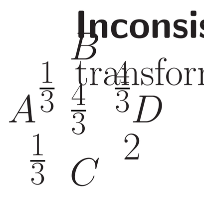
Multiple criteria – group flow. Each criterion (group member) has its own preference graph. For i-th criterion C_i, with weight w_i ($\sum w_i \leq 1$) and preference flow F_i on $(V_i, \mathcal{A}_i)$, $V_i \subseteq V$, `consensus graph` $(V, \mathcal{A})$ and `consensus flow` F are defined as follows: For $\alpha = (u, v)$ calculate

$$F_\alpha := \sum_{\substack{i=1 \\ \pm\alpha \in \mathcal{A}_i}}^{k} w_i F_i(\alpha). \qquad (1)$$

- If $F_\alpha \geq 0$ then: $\alpha \in \mathcal{A}$ and $F(\alpha) := F_\alpha$;
- Otherwise: $-\alpha \in \mathcal{A}$ and $F(-\alpha) := -F_\alpha$.

An example for two criteria.





$w_1 = 1/3 \quad w_2 = 2/3$ Consensus

Demonstration: [use refresh button, F5]
`http://pc205.math.hr/Decision/show.php`

Normal Integral

Consistent flow. A preference flow F we call `consistent` if there exists a potential $X : V \to \mathbb{R}$ such that

$$BX = F$$

where B denotes incidence matrix of the preference graph.

Normal integral of a given flow F defined on a connected graph is potential $X : V \to \mathbb{R}$, a (unique) solution of

$$B^\tau BX = B^\tau F, \quad \sum_{i=1}^{m} X_i = 0. \qquad \text{(Int)}$$

Weight function $w : V \to \mathbb{R}$ is

$$w = \frac{a^X}{\|a^X\|_1} \qquad (a = 2 \quad \text{for the moment}).$$

Inconsistency measure invariant on positive affine transformations in F-space is defined by

$$\mathrm{Inc}(F) = \frac{\|F - BX(F)\|_2}{\|BX(F)\|_2} \qquad \text{(Inc)}$$

which measures the angle between F and column space of B. Evidently,

Theorem 1. *F is consistent iff* $\mathrm{Inc}(F) = 0$.

PM and Stochastic Preference

Stochastic preference. The classical approach to stochastic preference can be find in [French]. Problem is the following:

> To each pair of alternatives a, b decision maker assigns probability p_{ab} of choosing a when offered the choice between a and b. We assume $p_{ab} + p_{ba} = 1$, with convention $p_{aa} = \frac{1}{2}$. Find a condition on numbers p_{ab} to generate a value function U on the set of alternatives, i.e.
>
> $$p_{ab} \geq \frac{1}{2} \Leftrightarrow U(a) \geq U(b).$$

A binary relation P on the set of alternatives we call *stochastic preference* if

$$aPb \Leftrightarrow p_{ab} \geq \frac{1}{2}.$$

Theorem 2. ([French, p. 101]) *If stochastic preference satisfies*

$$\frac{p_{ab}}{p_{ba}} \cdot \frac{p_{bc}}{p_{cb}} = \frac{p_{ac}}{p_{ca}} \qquad (2)$$

for all $a, b, c \in A$ then P is necessarily a weak order.

If we define $F(ab) = \log p_{ba}$ we get a preference flow on the set of alternatives. Theorem 2. gives a necessary condition for consistency of F. In that case normal integral of F represents utility.

PM and AHP

Eigenvalue method. A pairwise comparison matrix

$$W = (w_{ij}), \quad i, j = 1, \ldots, n$$

is given. We suppose that $w_{ij} > 0$ and $w_{ij} = w_{ji}^{-1}$. The second requirement defines `reciprocal matrix`. In the case of exact measurements the matrix is of the form

$$w_{ij} = \frac{w_i}{w_j}, \quad i, j = 1, \ldots, n$$

for some positive vector $w = (w_1, \ldots, w_n)$. In this case

$$W = \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ w_n \end{bmatrix} \begin{bmatrix} \frac{1}{w_1} & \cdots & \frac{1}{w_n} \end{bmatrix},$$

the spectrum $\sigma(W) = \{0, n\}$ where 0 has multiplicity $n - 1$ and n has multiplicity one with w as eigenvector. We call such matrix a `consistent matrix`. The following characterization of positive consistent matrix can be found in [Sa96].

Theorem 3. *For a positive reciprocal matrix $A = (a_{ij})$ $i, j = 1, \ldots, n$ the following statements are equivalent:*

i) *A is consistent;*
ii) *The maximum positive eigenvalue λ_{max} equals the order of the matrix;*
iii) $a_{ij} a_{jk} = a_{ik}$, $i, j, k = 1, \ldots, n$.

If we define a preference flow $F_{ji} = \log a_{ij}$ on the set of alternatives theorem 3. *iii)* gives a sufficient and necessary condition for consistency of F.

PM can be applied even if graph is not connected which is not true for EM.

via Potential Method

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PM and Kemeny's median

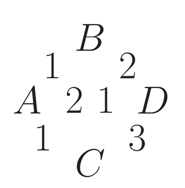
An experiment. Students were asked to rank their lecturers with respect to *Teaching qualities*, *Professional competence* and *Attitude towards students*:

- at Dept. of Psychology, Univ. of Zagreb
- 48 students
- forced to use all criteria and alternatives.

PM ranking for criteria ()

CRITERIA	CLUST. 1	CLUST. 2	GROUP
TeachQual	0.407	0.363	0.389
ProfComp	0.384	0.322	0.356
AttStud	0.209	0.315	0.255

Kemeny's social preference flow is



Low weights of the Kemeny's social preference may lead to conclusion that those qualities are 'almost equaly prefered'. In PM they are strongly separated because *the weight of the preference* has no value for Kemeny.

Another experiment was made with students, 29 of them, at Math. Dept. They were allowed to select criteria and alternatives of their own choice.

Conclusions

- Criteria profile can be formed if each group member use all criteria (Psycho-group example). Criteria clustering can be done as well.
- Inconsistency measure is not a valuable information in group decision. Each cluster can have small inconsistency and group inconsistency can be big, and vice versa. This is a reason for doing group clustering.
- Dissimilarity matrix is highly sensitive on input data. This means that 'small' change of flow, from the point of view of MCDM, generates new clusters.

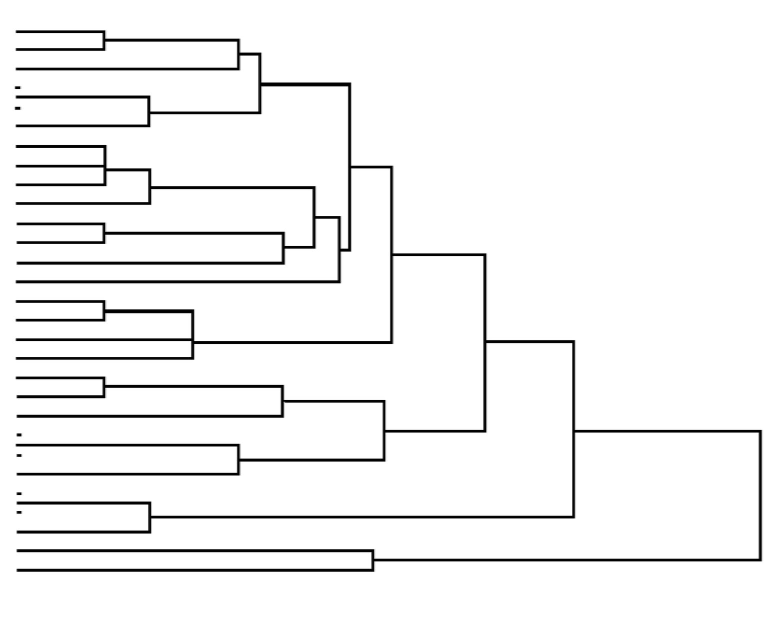
Clustering

Dissimilarity measure. Let $\pi = (F_1, \ldots, F_n)$ be a *group profile* of individual preference flows for a group of decision makers $\mathcal{G} = \{1, 2, \ldots, n\}$. Denote by X_i the normal integral of F_i. We define

$$\delta(F_1, F_2) := \|X_1 - X_2\|_2 \qquad \text{(FDist)}$$

Dissimilarity measure (FDist) allows to calculate the distance between preference flows of two decision makers, even if they have different criteria.

Outlayer discovery. In a group of students that were asked to compare their lecturers the last two group members were significantly distant from the others.



Finally, their flows were not taken into account in group consensus flow.

PM and Expected utility

Decision table. A standard decision table is given bellow. We have n states of nature or circumstances θ_j, which can be seen as criterions, and m actions a_i, which can be seen as alternatives. Each state has probability P_i. Numbers v_{ij} represent payoff for action a_i taken under the circumstance θ_j.

		States of nature			
		θ_1	θ_2	$\cdots$	θ_n
Actions	a_1	v_{11}	v_{12}	$\cdots$	v_{1n}
	a_2	v_{21}	v_{21}	$\cdots$	v_{21}
	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
	a_m	v_{m1}	v_{m2}	$\cdots$	v_{mn}

Expected utility theory defines utility of action

$$U(a_i) := \textstyle\sum_j P(\theta_j) v_{ij}$$

and maximizes it over the set of all actions to choose the 'best action'.

Consensus flow for decision table is defined on the graph with actions as vertices and according to formula (1)

$$F_{jk} = \sum_{i=1}^{n} P(\theta_i)(v_{ki} - v_{ji}), \quad k, j = 1, \ldots, m. \qquad (3)$$

Note that such flow is complete.

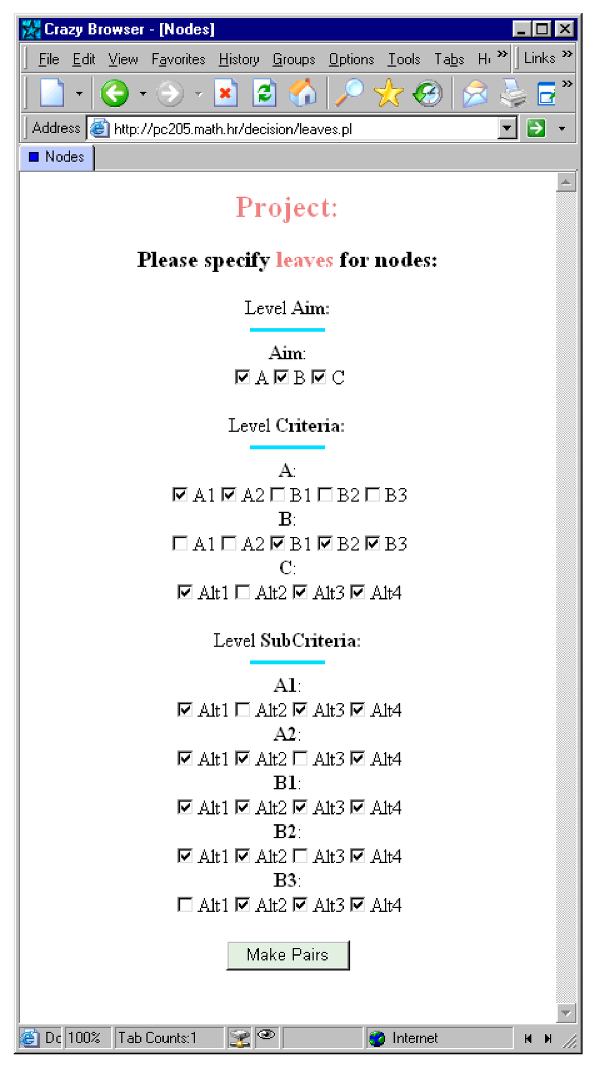
Theorem 4. *Expected utility and PM ranking are equivalent. More precisely*

$$X_k \geq X_l \iff U_k \geq U_l,$$

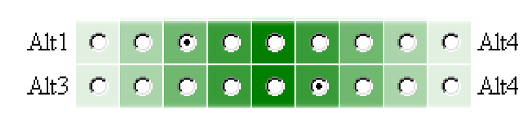
where X is normal integral of F on the set of actions.

Web interface

Setting leaves



Pairwise comparison step



Result