Preference measurement and application to choice theory

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Contents

A brief history (which concerns out topic only)

- Borda (1784) vs. Condorcet (1785) (still today)
- Thurston (1927) introduced pairwise comparison.
- Morgenstern and John von Neumann (1944) — utility theory.
- Savage (1954) reconstruction of attributes and objects probabilities from preferences (axiomatic approach).

Background: probability.

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What is time?

Human vs. exact sciences (again and again)

Humanity (nature)	Techniques (mind)
value difference measurement	*extensive measurement*
preference intensity	measurement unit
half, double	archimedean axiom
consistency	precision
concatenation	algebra
structure	equation
probability	statistics
quality	quantity
feedback	max/min
behaviour	*convergence*

Scientists in humanity are using (imitating) the methods from exact sciences. They should develop their own mathematics.

A solution → *Potential Method*. 

Stochastic preference

S — set of states (objects).
p_{ab} — propensity of choosing state a if the pair of states (a, b) is offered ($p_{aa} = \frac{1}{2}, \forall a \in S$). We suppose that $0 < p_{ab} < 1$ and

$$p_{ab} + p_{ba} = 1.$$

Let us define a relation on the set of states S

$$a \succcurlyeq b \iff p_{ab} \geqslant \frac{1}{2}.$$

Q. Is it possible to represent the relation $(S, \succcurlyeq)$ by real function V such that

$$a \succcurlyeq b \iff V(a) \geqslant V(b).$$

Existence →

Theorem (Representation theorem for choice)

If $p_{ab} \neq 0, \forall a, b$ satisfies the consistency condition

$$\frac{p_{ab}}{p_{ba}} \cdot \frac{p_{ca}}{p_{ac}} = \frac{p_{cb}}{p_{bc}}, \quad \text{for all } a, b, c \in S, \tag{1}$$

then, $\succcurlyeq$ is transitive and there exists a real function V such that

$$p_{bc} = \frac{V(b)}{V(b) + V(c)}. \tag{2}$$

Moreover,

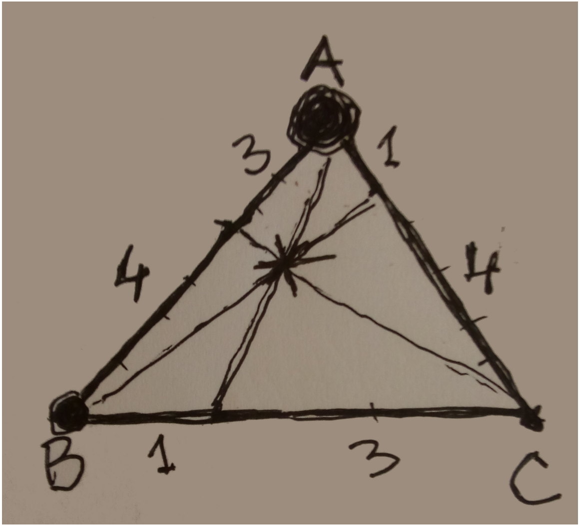
$$a \succcurlyeq b \iff V(a) \geqslant V(b),$$

and function $v(a) = \ln(V(a))$ is measurable value function, i.e.

$$(a \leftarrow b) \succcurlyeq_e (c \leftarrow d) \iff v(a) - v(b) \geqslant v(c) - v(d), \tag{3}$$

where $(a \leftarrow b) \succcurlyeq_e (c \leftarrow d) \iff p_{ab} \geqslant p_{cd}$.

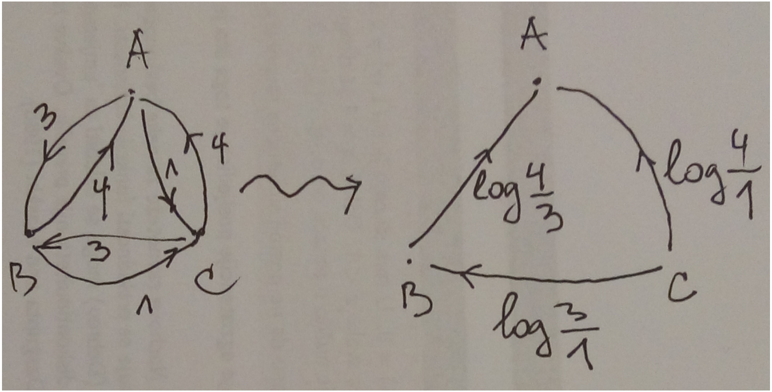
Visual representation: ratio $A : B : C = 4 : 3 : 1$



Central point (star) is the representation of $A : B : C = 4 : 3 : 1$. This is the consistent case which is equivalent to 3 pairwise ratios:
$A : B = 4 : 3$
$A : C = 4 : 1$
$B : C = 3 : 1$.

Ratio $A : B : C : D$ may be represented as a point in tetrahedron.

Flow representation: ratio $A : B : C = 4 : 3 : 1$



Left side: multigraph with parallel edges which represent the ratio.
Right side: aggregated graph ready for analysis with Potential Method.

Axiom of choice

Axiom (Luce, 1959)

Suppose R is a subset of S; then the choice probabilities for the choice set R are assumed to be identical to the choice probabilities for the choice set S conditional on R having been chosen, i.e., for $a \in R$

$$P_R(a) = P_S(a|R)$$

Consequences (equivalence):

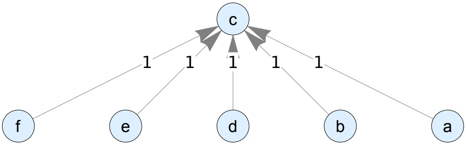
– $p_{ab}p_{bc}p_{ca} = p_{ac}p_{cb}p_{ba}$ (product rule)

– $p_R(a) = \frac{V(a)}{\sum_{x \in R} V(x)}$ (logit, strict utility model)

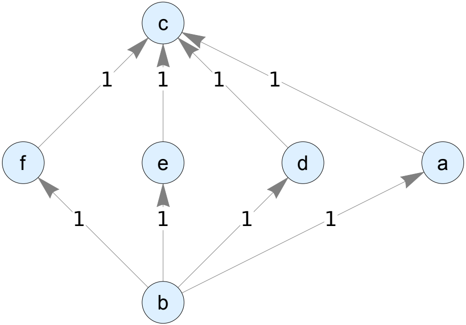
– consistency

What happens if the axiom of choice is not satisfied and (or) $p_{ab} = 0$ for some pair (a, b)? In that case data are not consistent, we have now value function V, but we may calculate *potential X*.

Ballot. Example.



most preferred



least preferred

Generators of choice data

– In a survey:

(a) Please choose one possibility from the given four: A, B, C, D.

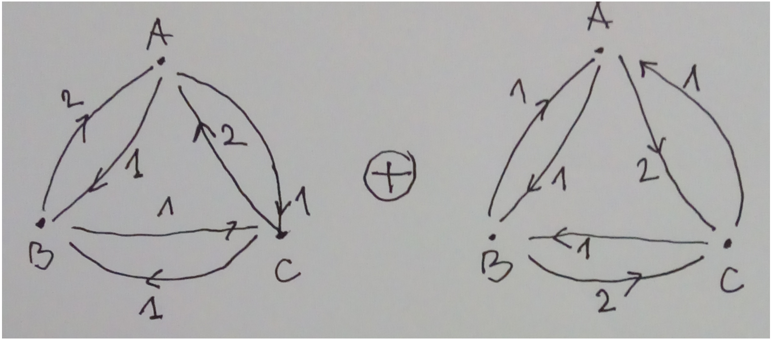
(b) How sure you are in yoou choice? (0–100)

– Promotion in marketing.
– Individual choice by triad interface.
– Recommendation (of a restaurant, option, . . .)
– Product development
– Public transport (organization)

. . .

A puzzle

Question: $\boxed{A : B : C = 2 : 1 : 1} \oplus \boxed{A : B : C = 1 : 1 : 2}$

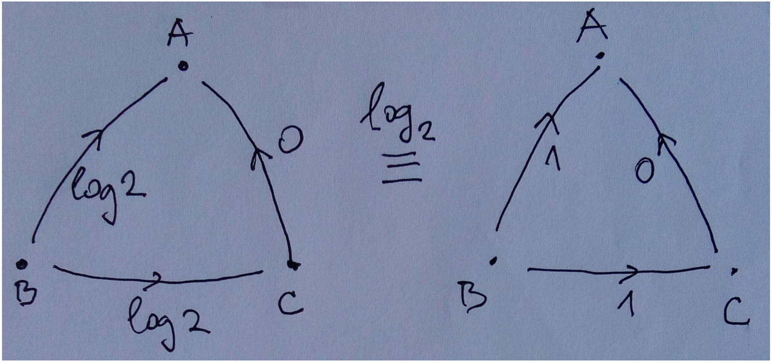


Both multigraphs we shall aggregate by:

1. …taking *log* (like choice) and add parallel edges.
2. …adding parallel edges and *summing* after that.

Puzzle - continued

Answer: $\boxed{A : B : C = 2 : 1 : 1} \oplus \boxed{A : B : C = 1 : 1 : 2}$



Measurable value function

$(S, \succcurlyeq)$ — weak preference, $(S_e, \succcurlyeq)$ weak preference on the set of *exchanges*.

Definition (Measurable value function)

Function $V : S \to \mathbb{R}$ is **measurable value function** if:

$$a \succcurlyeq b \Leftrightarrow V(a) \geqslant V(b) \tag{4}$$

$$(a \leftarrow b) \succcurlyeq_e (c \leftarrow d) \Leftrightarrow V(a) - V(b) \geqslant V(c) - V(d). \tag{5}$$

(4) means that V is ordinal value function on S.
(5) means that $V(a) - V(b)$ is ordinal value function on S_e.

Theorem (Necessary and sufficient condistions for MVF)

Axioms A1–A6 (bellow) are sufficient for existence of measurable value function.
Moreover, A1–A4 and A6 are necessary for existence of measurable value function.

A1. (Weak preference) $\succcurlyeq$ is weak preference, and $\succcurlyeq_e$ is weak preference on the set of exchanges.

A2. (Compatibility $\succcurlyeq$ and $\succcurlyeq_e$) $\forall a, b \in S$

$$a \succcurlyeq b \Leftrightarrow (a \leftarrow b) \succcurlyeq_e (c \leftarrow c), \quad \forall c \in S.$$

A3. (Inversion) $\forall a, b, c, d \in S$

$$(a \leftarrow b) \succcurlyeq_e (c \leftarrow d) \Leftrightarrow (d \leftarrow c) \succcurlyeq_e (b \leftarrow a).$$

A4. (Concatenation) $\forall a, b, c, d, e, f$

$$\left.\begin{array}{l}(a \leftarrow b) \succcurlyeq_e (d \leftarrow e) \\ (b \leftarrow c) \succcurlyeq_e (e \leftarrow f)\end{array}\right\} \Longrightarrow (a \leftarrow c) \succcurlyeq_e (d \leftarrow f).$$

A5. (Solvability) $(\forall b, c, d \in S)$ $(\exists x \in S)$ tako da je

$$(x \leftarrow b) \sim_e (c \leftarrow d). \tag{a}$$

$(\forall b, c \in S)$ $(\exists x \in S)$ such that

$$(b \leftarrow x) \sim_e (x \leftarrow c). \tag{b}$$

A6. (archimedean) Each strictly bounded standard sequence is finite.

de Finetti. Qualitative probability.

S — the set and $(\mathcal{P}(S), \succcurlyeq)$ the relation on the set of subsets. We are looking for representation $P : A \mapsto P(A) \in \mathbb{R}$ such that

$$A \succcurlyeq B \iff P(A) \geqslant P(B). \tag{6}$$

[1] Complete and and transitive.

[2]

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Axioms for **qualitative probability**:

Z1 (Completeness) $\succcurlyeq$ is weak preference[1]. Let use denote ani-symmetric and symmetric part by $\succ$ & $\sim$.

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Z2 (Independence[2] from common part) For subsets A, B, C such that $A \cap C = B \cap C = \varnothing$

$$A \succcurlyeq B \iff A \cup C \succcurlyeq B \cup C.$$

[1] Complete and and transitive.

[2] Known in the literature as the *Axiom of independent alternative*.

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Z3 (Nontriviality) $S \succ \varnothing$ (strong preference) and $A \succcurlyeq \varnothing, \forall A \subseteq S$.

[1]Complete and and transitive.

[2]Known in the literature as the *Axiom of independent alternative*.

Theorem (de Finetti[3])

Let us suppose Z1-Z6, then there exist P such that (6).

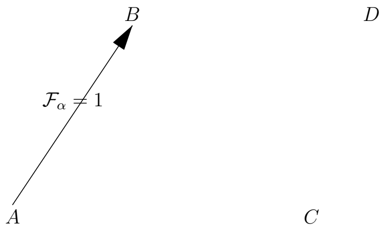
Z4 (Referent test) Decision maker is capable to identify the event on the probability wheel (PW).

Z5 (Continuity) $\forall A \subset S$ decision maker is capable to identify sector $\tilde{A}$ on the PW such that $A \sim \tilde{A}$. Let us denote by $\alpha(A)$ the central angle of $\tilde{A}$.

Z6 (Sure thing principle) $\alpha(S) = 360^o$.

[3] In fact he almost had a theorem.

Potential Method[4]



Preference flow $\mathcal{F}$

Incidence matrix $A \in \mathbb{R}^{m \times n}$

	nodes$_n$				flow
arcs$_m$	A	B	C	D	$\mathcal{F}$
α	-1	1	0	0	1

[4] Čaklović (2012); Čaklović and Kurdija (2017)

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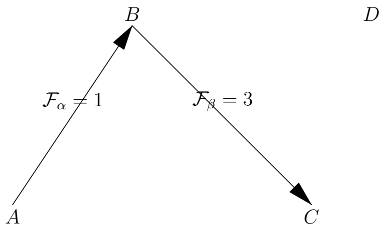

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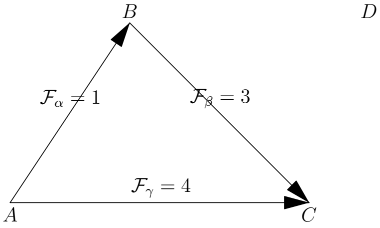

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γ	-1	0	1	0	4

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$$\mathcal{F}_\alpha + \mathcal{F}_\beta - \mathcal{F}_\gamma = 0$$

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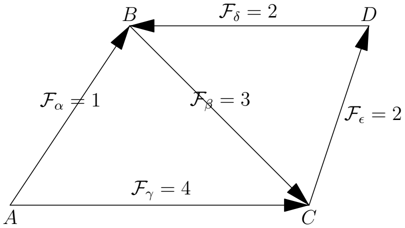

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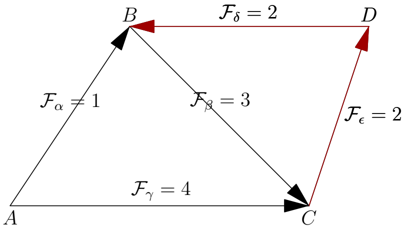

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$$\mathcal{F}_\alpha + \mathcal{F}_\beta - \mathcal{F}_\gamma = 0$$
$$\mathcal{F}_\epsilon + \mathcal{F}_\delta + \mathcal{F}_\beta = 7$$

$\mathcal{F}$ cycle DBCD is not consistent!

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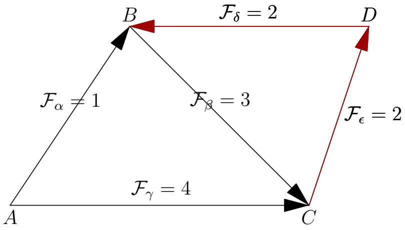

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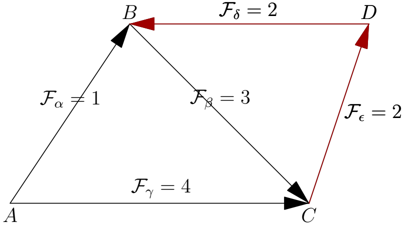

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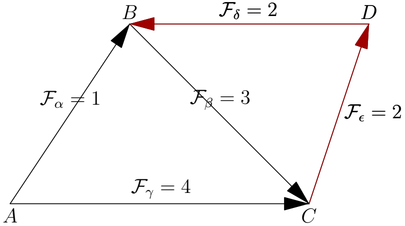

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$$N(A^T) \oplus R(A) = \mathbb{R}^m$$

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$\mathcal{F}$ is consistent iff $\mathcal{F} \in R(A)$

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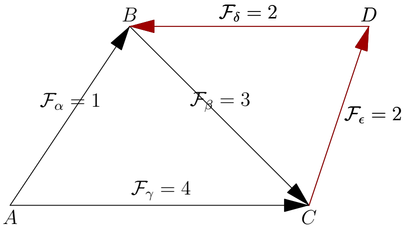

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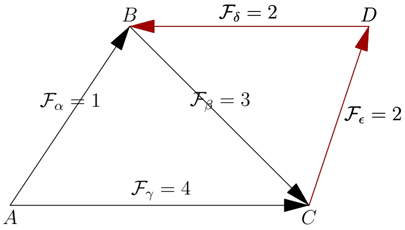

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$\mathcal{F}$ is consistent iff $\mathcal{F} \in R(A)$

$\mathcal{F}$ je consistent iff $AX = \mathcal{F}$

$\mathcal{F}$ je consistent iff $c \perp \mathcal{F}, \forall c$

$c \in N(A^\tau)$ cycle

[4] Čaklović (2012); Čaklović and Kurdija (2017)

Potential of preference graph

A — incidence matrix, $n = \#$Vertices, $m = \#$Arcs.
$\mathcal{F}$ — preference flow.

Ranking of the vertices is given by *potential* X:

$$A^{\tau} A X = A^{\tau} \mathcal{F}.$$

$A^{\tau}\mathcal{F}$ — flow gain in vertices
$L = A^{\tau} A$ — *Laplace matrix* of the graph.

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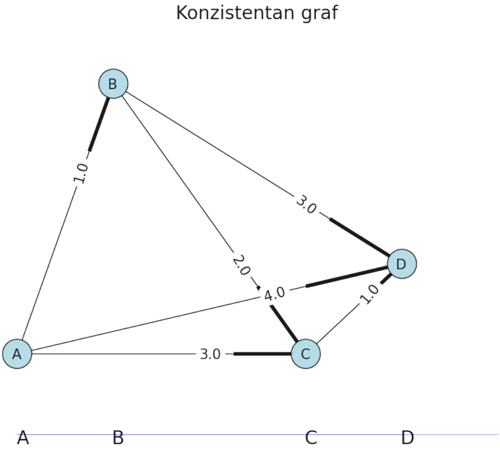
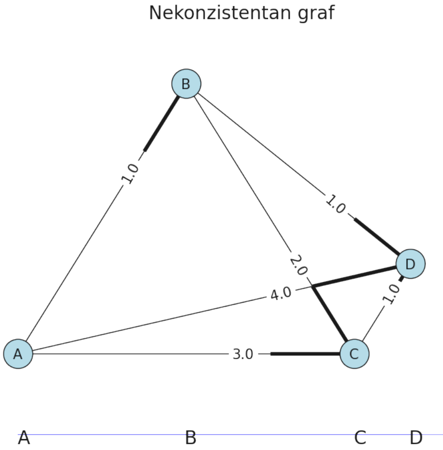
For connected graph, the matrix A has range $n-1$, the kernel is generated by the vector of ones $\mathbb{1} = [1, 1, \ldots, 1]^\tau$. For uniqueness of X we put the condition

$$\sum_{i=1}^{n} x_i = 0.$$

Konsistency (bis)





Bibliografija

Čaklović, L. (2012). Measure of Inconsistency for the Potential Method. In Torra, V., Narukawa, Y., López, B., and Villaret, M., editors, *MDAI*, volume 7647 of *Lecture Notes in Computer Science*, pages 102–114. Springer.

Čaklović, L. and Kurdija, A. S. (2017). A universal voting system based on the Potential Method. *European Journal of Operational Research*, 259:677–688.

Roberts, F. S. and Luce, R. D. (1968). Axiomatic Thermodynamics and Extensive Measurement. *Synthese*, 18(4):311–326.