GARCH processes

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ARCH(p) MODEL

Assume: \( Z_t \sim \text{IID}(0,1) \)

\( \alpha_0 > 0, \alpha_p > 0, \alpha_1, ..., \alpha_{p-1} \geq 0 \)

\( X_t = \nabla_t \cdot Z_t \)

\( \nabla^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 \quad t \in \mathbb{Z} \)

Define: \( U_t = \nabla_t \cdot (\nabla_t^2 - 1), \quad \phi(t) = 1 - \sum_{i=1}^{p} \alpha_i \nabla_i^t \)

\( \Rightarrow \phi(B) X_t^2 = \alpha_0 + U_t \)
So, $\text{ARCH}(p)$ process squared $(X_t^2)$ can be viewed as $\text{AR}(p)$ process with noise which is not iid.

**EXE 3** If $E\nu_t^4 < \infty, E\zeta_t^4 < \infty$, show that $(U_t)$ is white noise.

It turned out that $\text{ARCH}(p)$ do not fit log-returns very well unless $p$ is large → idea: add MA part to recursion.
**GARCH\((p,q)\) Model**

Assume: \(Z_t \sim \text{IID}(0, 1)\)

\[X_t = \sqrt{\tau_t} Z_t\]

\[\alpha_0, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \geq 0 \quad \& \quad \alpha_0 \alpha_p \beta_2 > 0\]

\[\tau_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \tau_{t-j}^2 \quad t \in \mathbb{Z}\]

**Example 2** (GARCH\((1,1)\))

Note

\[\tau_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \tau_{t-1}^2\]

\[= \alpha_0 + (\alpha_1 \tau_{t-1}^2 + \beta_1) \tau_{t-1}^2\]

↑ SRE again
From this \( \Rightarrow \) GARCH(1,1) has
unique, causal, stationary solution
if \( \mathbb{E} \log(x_t^2 + \beta_t) < 0 \)

\[
\text{IGARCH (integrated GARCH) is} \\
\text{GARCH(p,q) process with} \\
\sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{q} \beta_j = 1 \\
\Rightarrow \text{infinite variance} \ ?! ?
\]

estimators of \( \alpha \)'s/\( \beta \)'s in practice often
approximately satisfy this.
REMARK

- GARCH(p,q) models fit real-like financial data reasonably well over not too long periods.
- they allow simple forecast for condit. distribution of X_{t+1}
- they are related to classical ARMA models
- statistical estimation of parameters is not too difficult
GAUSSIAN QUASI-MAXIMUM LIKELIHOOD

Assume $Z_t \sim N(0, 1)$, then

$$X_t \mid X_{t-1}, X_{t-2}, \ldots \sim N(0, \sigma^2)$$

$\Rightarrow$ one can write conditional densities of $X_t$'s given $X_1, \ldots, X_p$ easily

$$f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = f_{X_1, \ldots, X_n \mid X_1, \ldots, X_p} \cdot f_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

$\Rightarrow$ we optimize this wrt $\alpha$'s, $\beta$'s

$\uparrow$ we ignore this!

to get Gaussian quasi-maximum-likelihood estimators
Turns out

- asymptotic normality with rate \( T_n \)
  holds for these estimators for a large class of noise distributions
- sometimes "more realistic" assumptions on \( \varepsilon_t \)'s can produce non-consistent estimators
- in practice initial values of \( X_0, X_{-1}, \ldots, \gamma_0, \gamma_{-1}, \ldots \) are not known & have to be initialized somehow → this can be justified theoretically.
TAILS OF S.R.E.

Assume \((Y_t)\) is a stationary solution of s.r.e \(Y_t = A_t Y_{t-1} + B_t + e_t \in \mathbb{Z}\) for some iid sequence \((A_t, B_t)_t \in \mathbb{R}^2_+\).

**Theorem 2 (Goldie)**

Suppose for some \(K > 0, \varepsilon > 0\)
\[
EA_t^K = 1, \quad EB_t^{K+\varepsilon} < \infty
\]

then
\[
P(Y_t > u) \sim c \cdot u^{-K} \quad u \to \infty \quad (\star)
\]

for some constant \(c > 0\).
The tail of $Y_t$ in thin 2 is called power-law tail (very popular subject in contemporary statistics & probability).

**Remark**

- $(*) \Rightarrow \mathbb{E}Y_t^{*+\varepsilon} = +\infty$  \quad \forall \varepsilon > 0$
- more general are regularly varying tails
KOR. 3

If \( (X_t) \) is a stat. AR(4)(1) process with stand. Gaussian noise, let

\[
E(x, x^2)^k = 1 \quad \text{for } k > 0
\]

\[
\Rightarrow \quad P(X_t > x) \sim \frac{c}{2} \cdot x^{-2k} \quad x \to \infty
\]
REMARK

- for $\alpha_1 \in (0,1)$ $X_t$ is stationary with finite variance
- for $1 \leq \alpha_1 < 2e^{\delta} \approx 3.56$ ($\delta = \text{Euler const.}$) $(X_t)$ has infinite variance
- for $\alpha_1 \geq 2e^{\delta}$ one cannot find station. causal solution (see thm 1).
Spectral analysis of time series

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SPECTRAL ANALYSIS

OF TIME SERIES

Assume: \((X_t)\) is weakly stationary time series with autocov. function \(\gamma_x\) s.t.
\[
\sum_{h \in \mathbb{Z}} |\gamma_x(h)| < \infty
\]

Then, the series
\[
f_x(\lambda) = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \gamma_x(h) e^{-i\lambda h} \tag{1}
\]
is abs. convergent, uniformly in \(\lambda \in \mathbb{R}\)
Function \( f_X \) in (1) is called the spectral density of \((X_t)\). It is clearly periodic \(2\pi\), so we shall only consider it on interval \([-\pi, \pi]\).

Moreover, uniform convergence in (1) allows us to interchange sum & integral to obtain

\[
y_X(n) = \int_{-\pi}^{\pi} e^{i n \lambda} f_X(\lambda) \, d\lambda
\]

**Inversion Formula**
In analysis (1) represents Fourier series of function $f_x$; and by (2) $x(h)$ are corresponding Fourier coefficients.

Functions $(e^{ih})_{h \in \mathbb{Z}}$ form a basis of $L_2((-\pi, \pi], \text{Leb})$.

Condition $\sum |x(h)| < \infty$ is more restrictive than necessary, ($\sum x^2(h) < \infty$ is enough) but sufficient for us.
Lemma 1

Function \( f_x \) is also even and nonnegative.

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Four series in (1) will not converge unless \( \mu(h) \rightarrow \infty \) sufficiently fast, so in such cases spectral density will not exist. Still we will be able to find the spectral measure.
**Theorem 2** (Herglotz)

For every weakly stationary sequence \( (X_t) \) there exists a unique finite measure \( F_x \) on \( (-\pi, \pi) \) s.t.

\[
y_x(h) = \int e^{ihx} \, dF_x(x), \quad h \in \mathbb{Z}
\]

**Definition** The measure \( F_x \) above is called the **spectral measure** of \( (X_t) \).

If it has density w.r.t. the Lebesgue measure \( f_x \), then \( f_x \) is called the **spectral density**.
EXAMPLE 1  \((\text{White noise})\)

\[ y_x(h) = 0 \quad h \neq 0 \quad \text{for } X \sim \mathcal{N} \]

\[ \Rightarrow f_X(x) = \frac{1}{2\pi} y_x(0) + \mathbb{1} \]

We'll say \(w\) noise contains all possible frequencies in the same amount.

EXAMPLE 2  \((\text{Deterministic periodic sequence})\)

Let

\[ x_t = A \cos \omega t + B \sin \omega t, \quad \omega \in (0,\pi) \]

\[ \mathbb{E}A = \mathbb{E}B = 0, \quad \text{Var}A = \text{Var}B = \sigma^2, \quad \text{Corr}(A,B) = 0 \]
We showed
\[ f_x(h) - \xi^2 \cos h \lambda = \frac{\xi^2}{2} (e^{ih\lambda} + e^{-ih\lambda}) \]

\[ \Rightarrow \text{F has mass } \frac{\xi^2}{2} \text{ at points } \lambda, 2\lambda. \]

REMARK: For real valued time series spectral measure is symmetric (show it)
So we will ignore point \(-\lambda, \lambda\)

call only \(\lambda\) the frequency
of this time series
EXE 1

a) If \((X_t),(Y_t)\) are two uncorr. stationary seq., show that the spec. measure of \((X_t+Y_t)\) is the sum of corresponding spectral measures.

b) Find a seq. with symmetric spectral measure concentrated at points \( \pm \lambda_i \), \( i = 1, \ldots, k \) for arbitrary \( \lambda_i \in (0, \pi) \).

c) Find a seq. s.t. \( F_x \) has all its mass at 0.

d) Find a seq. s.t. \( F_x \) has all its mass at \( \pi \).

e) Show that every finite measure on \((-\pi, \pi]\) is the spectral meas. of some t. series.
SPECTRAL ANALYSIS

- Inference about time series using spectrum, as opposed to the usual analysis using acf’s which is called “analysis in time domain”

- Also called “analysis in frequency domain”
FILTER & SPECTRUM

Recall a concept of filter \((\psi_j)_{j \in \mathbb{Z}}\)
acting on a stat. time series to obtain a linear process.

In signal processing & physics by filter or transfer function we refer to
\[
\psi(x) = \sum \psi_n e^{i n x}
\]

THEOREM 3 (effect of filtering on spectrum)

For a stationary seq \((X_t)\) with spectral measure \(F_X\), & filter \((\psi_j)\) s.t. \(\sum |\psi_j| < \infty\)

Define \(Y_t = \sum \psi_j X_{t-j}\)

then \(dF_Y(x) = |\psi(x)|^2 dF_X(x)\) is sp. meas. of \((Y_t)\)
EXAMPLE 3 (MA(1))

Since for $Z_t \sim WN(0, \sigma^2)$, $f_{Z}(\lambda) = \frac{\sigma^2}{2\pi}$

THM3 $\Rightarrow$ for $X_t = Z_t + \psi Z_{t-1}$

$$f_{X}(\lambda) = (1 + \psi e^{-i\lambda}) \frac{\sigma^2}{2\pi} = (1 + 2\psi \cos \lambda + \psi^2) \frac{\sigma^2}{2\pi}$$

Note: for $\psi > 0$ small frequencies dominate
for $\psi < 0$ larger frequencies dominate
EXAMPLE 4 (complex valued periodic seq.)

Assume \( EA = 0 \), \( \text{Var} A = \sigma^2 < \infty \), let \( \lambda \in (0, \pi) \) be

\[ X_t = A e^{i \lambda t} \Rightarrow \mu_X(h) = e^{ih \lambda} \sigma^2 \]

\( \Rightarrow \) \( F_X \) has mass \( \sigma^2 \) at point \( \lambda \).

\( \Rightarrow \) \( Y = \sum y_j X_{t-j} \) has mass \( |\psi(\lambda)|^2 \sigma^2 \) at point \( \lambda \).

EXE 2 \( \rangle \) Find the spec. measure for

\( X_t = A e^{i \lambda t} \) for \( \lambda \notin (-\pi, \pi] \).
EXAMPLE 5 (band pass filter)

Consider
\[ \psi(x) = \begin{cases} 0 & 1 - 2\lambda > \delta \\ 1 & 1 - 2\lambda \leq \delta \end{cases} \]

for fixed frequency \( 2\omega \) and bandwidth \( \delta \).

This filter by Ex. 4 kills all frequencies outside \([10-\delta,\lambda+\delta]\). Spectral density of so filtered signal

\[ Y_t = \sum \psi_j X_{t-j} \]

(\( \text{if it exists} \) is)

\[ f_Y(x) = |\psi(x)|^2 f_X(x) = \begin{cases} 0 & 1 - 2\lambda > \delta \\ f_X(x) & 1 - 2\lambda \leq \delta \end{cases} \]
For small $\delta$, 

$$\text{Var} Y_t = \mu_y(0) = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} f_x(\lambda) d\lambda \approx 2\delta f_x(\lambda_0)$$

$\Rightarrow f_x(\lambda_0)$ is "proportional to variance of subsegment in $X_t$ of frequency $\lambda_0$."

Note: band pass filter is theoretical filter only! In practice only smooth transfer functions can be implemented.
REMARK

- Instead of frequencies \( \lambda \) we can use periods (e.g. \( e^{i \omega t} \) is periodic with period \( \frac{2\pi}{\omega} \)). E.g. monthly series with period 12 months will have visible peak in the spectrum at frequency \( \frac{2\pi}{12} \).

- Frequency \( \Pi \) is highest possible (so called Nyquist frequency), this is because we only observe time series at integer times.
Consequence of Thm 3 is

**THEOREM (Spectral density of ARMA process)**

A causal ARMA process \((X_t)\)

has spectral density

\[
S_X(\lambda) = \frac{\pi^2}{2\pi^2} \left| \frac{\gamma(e^{i\lambda})}{\gamma(e^{-i\lambda})} \right|^2 \lambda e^{-\pi^2/\lambda^2}
\]
SPECTRAL DECOMPOSITION

It turns out that:

any stationary time series can be written as a randomly weighted sum of single frequency signals $e^{i\lambda t}$.

For uncorrelated $Z_1, \ldots, Z_k$ with mean 0 and arbitrary $\lambda_1, \ldots, \lambda_k \in (-\pi, \pi)$,

$$X_k = \sum_{j=1}^{k} Z_j e^{i\lambda_j t}$$

has sp. mean

$$F_x = \sum_{j=1}^{k} E|Z_j|^2 \delta_{\lambda_j}$$
Thus

\[ X_t = \sum_{j=1}^{k} Z_j e^{i \omega_j t} = \int e^{i \omega t} \sum_{j=1}^{k} Z_j \delta \omega_j (dt) \]

\[ \downarrow \]

\[ X_t \] is the sum of uncorrelated single-frequency signals of stochastic amplitudes

Interestingly: any zero mean stationary \((X_t)\) with discrete sp. measure has such a decomposition
More interestingly: any mean zero stationary time series has such a decomposition only the sum becomes the integral

\[ X_t = \int e^{\lambda t} dZ(\lambda) \]

w.r.t. some random measure \( Z \).

**Def.** A random measure with orthogonal increments \( Z \) is a collection of r.v.'s \( \{ Z(B) : B \in \mathcal{B} \} \) with mean zero, on some \((\Omega, \mathcal{F}, \mathbb{P})\) s.t. for some finite Borel measure \( \mu \) on \((-\pi, \pi]\)

\[ \text{Cov}(Z(B_1), Z(B_2)) = \mu(B_1 \cap B_2) + \mu(B_1^c B_2^c) \]
Remark: Definition of $Z \Rightarrow Z$ is $\tau$-additive.

Also, $Z_x = Z(-\tau, x) : x \in (-\tau, \tau)$ is a stochastic process with uncorrelated increments.

Problem: How to define integral w.r.t. $Z$. 
Idea:

Step 1. Set $\int 1_B \, d\pi = \pi(B)$

$$\int \sum_{j} \alpha_j 1_{B_j} \, d\pi = \sum_{j} \alpha_j \pi(B_j)$$

for any $\alpha_j, B_j \subset \mathbb{B}$

Step 2.

For $f \in L_2(\mu)$ take sequence of step functions $f_n \uparrow f$ in $L_2(\mu)$

Note: $\Phi f \mapsto \int f \, d\pi$ is linear isometry on step functions

Set

$$\int f \, d\pi = \lim_{n} \int f_n \, d\pi \quad \text{in} \quad L_2(\mathbb{R}^d \mathbb{P})$$
REMARK \[ \Phi : L_2(-T,T,\mu) \rightarrow L_2(S^2,F,P) \] is a linear isometry between two Hilbert spaces.

**THEOREM 5**

For any mean zero stationary t.s. \((X_t)\) with spec. meas. \(F_x\) there exists a random measure \(Z\) with orth. inc. s.t.

\[
X_t = \int e^{i\omega t} dZ(\omega) \quad \text{as} \quad t \in \mathbb{Z}
\]

\[ \underbrace{\text{Spectral Decomposition}} \]
ESTIMATION OF THE SPECTRAL DENSITY

Recall, if $\Sigma |y(k)| < \infty$, $(X_t)$ stationary

$$f_X(x) := \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-ihx} j_X(x)$$

It is natural to replace $j_X$ with $\tilde{j}_X$ to get an estimator. If we assume

$X_t$'s are centered, we could use

$$\tilde{j}_X(x) = \begin{cases} \frac{1}{n} \sum_{t=1}^{n-1} X_t X_{t+1} & |h| < \delta \\ 0 & \text{otherwise} \end{cases}$$
The natural estimator is thus

\[ I_{n,n} (x) = \frac{1}{2\pi} \sum_{|h|\leq n} e^{-i2\pi x h} \hat{f}_x (h) \]

\[ = \frac{1}{2\pi} \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} X_t X_s e^{-i2\pi (t-s)} \]

\[ = \frac{1}{2\pi} \left| \frac{1}{n} \sum_{t=1}^{n} e^{-i2\pi X_t} \right|^2 \]

We usually evaluate \( I_{n,n} \) only at Fourier frequencies:

\[ \lambda_j = \frac{2\pi j}{n}, \quad 0 < j < \lfloor n/2 \rfloor \]

i.e. \( j \in (0, \pi] \).
Since $\sum_{t=1}^{n} e^{i\lambda t} = 0$ at Fourier frequencies

$\downarrow$

Periodogram is the same whether we center $X_t$ or not

$\downarrow$

$I_{n,x}(\lambda^*) = \frac{1}{2\pi} \left| \frac{1}{n} \sum_{t=1}^{n} e^{-i\lambda^* t} (X_t - \bar{X}_n) \right|^2$

$= \frac{1}{2\pi} \sum_{|h| < n} e^{-ih\lambda^*} \hat{g}(h)$
**EXAMPLE 6** (periodogram of Gaussian w. noise)

Assume

\[ Z_t \sim WN(0, \sigma^2) \text{ Gaussian,} \]

Since

\[ I_{n,2} (\lambda) = \frac{1}{2\pi} \left| \frac{1}{n} \sum_{t=1}^{n} e^{i\lambda t} Z_t \right|^2 \]

observe inner product of two complex Gaussian r.v.'s

\[ E \left( \frac{1}{n} \sum_{t=1}^{n} e^{-i\lambda t} Z_t \right) \left( \frac{1}{n} \sum_{s=1}^{n} e^{i\lambda s} Z_s \right) = \langle X_{\lambda}, X_{\lambda} \rangle \]

\[ = \frac{1}{n} \sum_{t=1}^{n} E Z_t^2 e^{-i(\lambda_j - \lambda_k)t} \]

\[ = \cdots = \begin{cases} \sigma^2 & j = k \\ 0 & \text{otherwise} \end{cases} \]
\[ X_j, X_k \text{ are mean zero, uncorrelated Gaussian r.v.'s} \Rightarrow X_i \sim \text{iid } N(0, \tau^2) \]

Observe

\[ I_{n, 2}(\lambda_j) = \frac{1}{2\pi} \quad |X_j|^2 \]

\[ = \frac{1}{2\pi} \left( \frac{1}{n} \sum_{t=1}^{n} z_t \cos \lambda_j t \right)^2 + \frac{1}{2\pi} \left( \frac{1}{n} \sum_{t=1}^{n} z_t \sin \lambda_j t \right)^2 \]

\[ \uparrow \quad \text{independent} \]

\[ \downarrow \quad \text{& both } N(0, \tau^2/2) \text{ distributed} \]

\[ 2\pi I_{n, 2} \text{ has } \frac{\tau^2}{2} \chi^2 \text{ distribution} = \tau^2 \cdot \text{Exp}(1) \]
Thus the periodogram of iid Gaussian seq. at the Fourier frequencies has values which are iid exponential with mean $\frac{\sigma^2}{2\pi}$.

Recall the spectral density was $\frac{\sigma^2}{2\pi}$.

$\Rightarrow$ Periodogram is not consistent estimator

$\Rightarrow$ Fisher derived g-test for Gaussian white noise from this.
Still periodogram is not far from consistency

**Proposition 6**

$(X_t)$ mean zero, stationary, $\Sigma|y(t)| < \infty$

$\Rightarrow \mathbb{E}I_{n,X}(\omega) \to f_X(\omega) \neq 0 \in (0,1]$

**Exercise 3** Prove Proposition 6.

Example 6 really points the asymptotic properties of periodogram
THEOREM 7

Assume \((X_t)\) is a linear process with abs. summable coefficients \((\psi_j)\) driven by white noise with variance \(\sigma^2\).

For fixed frequencies \(0 < \omega_1 < \ldots < \omega_m < \pi\)

\[
(I_{\eta, x(\omega_j)})_j \overset{d}{\to} \left( \frac{\sigma^2}{2\pi} \right)^{1/2} \psi_j(e^{-i\omega_j}) \cdot E_j,
\]

\[
= (I_{x(\omega_j)} \cdot E_j),
\]

for \(E_j \sim \text{Exp}(1)\), \(j = 1, \ldots, n\).
Although, it is inconsistent, smoothed periodogram can be used to get consistent estimator.

For some weights: \((W_n(k))_{1 \leq k \leq m}\)

\[ W_n(k) = W_n(-k), \quad \sum_{1 \leq k \leq m} W_n(k) = 1, \quad \sum_{1 \leq k \leq m} W_n^2(k) \to 0 \]

Use

\[ \hat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{j} W_n(j) \hat{F}_n \left( g_n(\lambda) + 2\pi \frac{j}{n} \right) \]

where \( g_n(\lambda) \) is closest Fourier freq \( \frac{2\pi j}{n} \) to the point \( \lambda \).
In theory we need to let
\[ m = m_n \to \infty \quad \text{and} \quad m_n / n \to 0 \quad \text{as} \quad n \to \infty \]

The simplest weights
\[ W_n (k) = \frac{1}{2m+1} \]

are called Daniell weights.