UNIVERSITY OF ZAGREB
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Homogenization and Mathematical Analysis of Immiscible Compressible Two-Phase Flow in Heterogeneous Porous Media by the Concept of the Global Pressure

PhD thesis

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Homogenizacija i matematička analiza nemješivog stlačivog dvofaznog toka kroz heterogenu poroznu sredinu pomoću koncepta globalnog tlaka

Doktorska disertacija

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Homogénéisation et Analyse Mathématique d'Écoulement Diphasique Immiscible et Compressible en Milieux Poreux Hétérogènes par le Concept de la Pression Globale

Thèse de Doctorat[1]

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Summary

Mathematical modelling of multiphase flow in porous media is of great practical importance in problems of petroleum and environmental engineering such as petroleum reservoir and groundwater aquifer simulation, radioactive waste management and sequestration of CO2. The variations of the physical properties of the porous medium occur at many distinct space scales and strongly affect the fluid flow and transport through the heterogenous porous media.

In this thesis we study a model of the immiscible compressible two-phase flow in porous media in a new formulation, which has been lately established in [8] and [11]. The usual equations describing such flow are the mass conservation law and the Darcy-Muskat law for each of the phases, which make a system of coupled nonlinear evolutionary partial differential equations. By using the notion of the global pressure, the original system is transformed to a fully equivalent system of nonlinear parabolic equations for the global pressure and the phase saturation, which is more convenient for the mathematical analysis and numerical solving. The main difficulties in the analysis of this problem are the nonlinearity, degeneracy and coupling of the equations.

Direct numerical and analytical methods for problems of flow in a porous medium are impossible or inefficient due to its considerable inhomogeneity. Homogenization theory aims to describe the macroscopic behavior of a highly heterogeneous system by replacing it with a simpler homogenized or effective medium whose global characteristics are a good approximation of the initial ones.

In this thesis we derive three new results on the existence and homogenization for the new fully equivalent global pressure model of immiscible compressible flow in porous media. The previous results for this type of flow were obtained for the phase formulation whereas the notion of the global pressure as introduced in [21, 48] for incompressible immiscible flows is employed to obtain the a priori estimates and compactness results. In comparison to them, the common feature of our contributions is also that the hypotheses on the data are significantly relieved, so that we make only the physically justified assumptions, as the ones that appear in some realistic applications. In particular, our results include the

case of an unbounded capillary pressure function as well as the discontinuous porosity and absolute permeability tensor.

The first result of this thesis is an existence result for the immiscible flow of water and gas. Our result extends the existence result in [12] which is valid for the two strictly compressible phases. Other preceding results on the existence for immiscible compressible flow [68–72, 75–77] set much more restrictive assumptions on the data. Moreover, we cover the case of the non homogenous Dirichlet and Neumann boundary data. The proof uses an appropriate regularization and a time discretization. We use a modified compactness result, as in [12, 76, 95], to pass to the limit in nonlinear terms.

The second contribution in the thesis is the rigorous justification of the homogenization process for the immiscible flow of two compressible fluids in a strongly heterogeneous porous medium of a single-rock type. So far the only homogenization result concerning this type of problem for the compressible flow was [6], where the water-gas flow was studied assuming the boundedness of the capillary pressure. On the microscopic level, the periodic heterogenous porous medium is scaled by ε, $0 < \varepsilon \ll 1$ which represents the typical size of the periodicity blocks with respect to the domain size, and the medium porosity and the permeability are modeled as ε-periodic functions. Passage to the limit as $\varepsilon \to 0$ in the microscopic problem is performed by means of the two-scale convergence technique of [2], with the aid of the compactness result from [6]. We obtain a nonlinear homogenized problem with effective coefficients which are computed via a cell problem.

As a third result of this thesis, we establish the convergence of the homogenization process for a double porosity model of the immiscible gas-water flow in a naturally fractured reservoir. This type of porous media consists of a disconnected periodically spaced system of blocks of usual porous media, matrix, which are separated by a net of thin fractures. The matrix keeps most of the fluid while the flow in the fissures is much readier due to their notably higher permeability. In the double porosity model the ratio of the permeability of the matrix and the fractures is of order ε^2, where ε is the periodicity parameter. Such scaling preserves the flow from the matrix to the fractures. We make use of the classical compactness results of [95] and [5] to pass to the limit as $\varepsilon \to 0$, using the two-scale convergence. This leads to the homogenized problem for the fracture flow where an additional source-like term arises which depicts the matrix-fracture flow. Its non-explicit form is caused by the nonlinearity and the coupling in the system. On the other hand, in the limit as $\varepsilon \to 0$ to each point of the reservoir corresponds a matrix block. The double porosity model comprises the set of the equations for each matrix block, that capture the flow at the medium-size scale. In order to obtain the effective matrix problem and identify the fracture source term we use the notion of the dilation operator, introduced in [25].

Our contribution presents an extension of the result of [14] to the fully equivalent global pressure formulation.

Sažetak

Matematičko modeliranje višefaznog toka kroz poroznu sredinu od velike je praktične važnosti u problemima naftnog inženjerstva i zaštite okoliša, kao što su na primjer simulacije nalazišta ugljikovodika i podzemnih voda, upravljanje odlagalištima radioaktivnog otpada u dubokim geološkim slojevima te sekvestracija CO_2. Varijacije fizičkih svojstava poroznog medija se javljaju na puno različitih prostornih skala i značajno utječu na tok i transport kroz heterogenu poroznu sredinu.

U ovoj disertaciji proučavamo novi model nemješivog dvofaznog toka kompresibilnih fluida u poroznoj sredini koji je nedavno izveden u [8] i [11]. Jednadžbe koje opisuju takav tok su zakon sačuvanja mase te Darcy-Muskatov zakon za svaku od faza, što daje sustav vezanih nelinearnih evolucijskih parcijalnih diferencijalnih jednadžbi. Uvođenjem pojma globalnog tlaka, osnovni sustav se transformira u potpuno ekvivalentni sustav nelinearnih paraboličkih jednadžbi za globalni tlak i zasićenje jedne od faza. Ta je formulacija pogodnija za matematičku analizu i numeričko rješavanje problema. Glavne poteškoće pri analizi ovog problema su nelinearnost, degeneracija i vezanost u jednadžbama.

Zbog nezanemarive heterogenosti porozne sredine, direktne numeričke metode te metode matematičke analize za tok fluida kroz poroznu sredinu su vrlo neefikasne ili čak nemoguće. Cilj teorije homogenizacije je opisati makroskopsko ponašanje snažno heterogenog sustava tako da ga se zamijeni jednostavnijim homogeniziranim ili efektivnim medijem čije globalne karakteristike dobro aproksimiraju karakteristike polaznog medija.

U ovoj disertaciji izvodimo tri nova rezultata o postojanju rješenja i homogenizaciji za spomenuti novi, potpuno ekvivalentni model nemješivog dvofaznog toka kompresibilnih fluida kroz poroznu sredinu. Prethodni rezultati za ovakav tok su dobiveni za model u faznoj formulaciji, a pri tome je koncept globalnog tlaka korišten u obliku u kojem je izveden za nestlačivi nemješivi tok u [21, 48], te je upotrijebljen za dobivanje apriornih ocjena i rezultata kompaktnosti. U usporedbi s tim rezultatima, rezultati ove radnje su dobiveni uz znatno oslabljene ulazne pretpostavke koje su fizikalno opravdane, tako da je dozvoljen slučaj neograničenog kapilarnog tlaka te diskontinuiteta poroznosti i tenzora apsolutne permeabilnosti.

Prvi doprinos ove disertacije je rezultat postojanja slabih rješenja za nemješivi tok vode i plina. Naš rezultat proširuje rezultat egzistencije za dva stlačiva fluida koji je dobiven u [12]. Postoje i drugi rezultati egzistencije za nemješivi tok stlačivih fluida [68–72, 75–77], koji se odnose na aproksimacijske modele te kod kojih su pretpostavke na podatke (naročito na funkciju kapilarnog tlaka) znatno restriktivnije. Rezultat koje je dokazan u ovoj radnji uključuje nehomogene Dirichletove i Neumannove rubne podatke. Dokaz se temelji na odgovarajućoj regularizaciji i vremenskoj diskretizaciji. Za prijelaz na limes u nelinearnim članovima koristi se modificirani rezultat kompaktnosti, kao u [12, 76, 95].

Nadalje, u disertaciji je prikazano strogo opravdanje procesa homogenizacije za nemješivi tok stlačivih fluida u jako heterogenoj poroznoj sredini koja se sastoji od jednog tipa stijene. Jedini raniji rezultat homogenizacije za ovaj tip problema je [6], gdje je razmatran tok vode i plina uz pretpostavku ograničenosti kapilarnog tlaka. Periodička heterogena porozna sredina se na mikroskopskoj razini opisuje malim parametrom $\varepsilon > 0$ koji predstavlja odnos karakteristične veličine periodičkog bloka i veličine cijele domene. Tada su poroznost i permeabilnost sredine predstavljene kao periodičke funkcije perioda ε. Za prijelaz na limes kad $\varepsilon \to 0$ u mikroskopskom problemu koristi se tehnika dvoskalne konvergencije razvijena u [2]. Pri tome se za nelinearne funkcije upotrebljava rezultat kompaktnosti iz [6]. Na taj način uspostavlja se nelinearni homogenizirani problem te se efektivni koeficijenti iskazuju kao rješenja odgovarajućih lokalnih problema.

Treći prilog ove radnje je dokaz konvergencije postupka homogenizacije za model dvostruke poroznosti za nemješivi tok vode i plina u ležištu s pukotinama. Ovaj tip porozne sredine sastoji se od matrice - nepovezanog sustava periodički raspoređenih blokova koji funkcioniraju kao standarna porozna sredina, i mreže uskih pukotina koje okružuju matricu. Najveći dio fluida se nalazi u matrici, a s druge strane tok se većinom odvija u pukotinama koje su znatno propusnije. U modelu dvostruke poroznosti se propusnost matrice skalira s ε^2 (gdje je ε parametar periodičnosti), što čuva tok iz matrice u pukotine od degeneracije ili neograničenog rasta kad $\varepsilon \to 0$. U dokazu se za prijelaz na limes koriste klasični rezultati kompaktnosti [95], [5] te dvoskalna konvergencija. To daje homogenizirani problem za tok u pukotinama u kojem se javlja član koji predstavlja izvor fluida iz matrice na makroskopskoj razini. Zbog nelinearnosti i vezanosti u sustavu jednadžbi, ovaj član nije u eksplicitnom obliku. S druge strane, kad $\varepsilon \to 0$ za svaku točku domene se dobiva po matrični blok te model dvostruke poroznosti uključuje i sustav jednadžbi za svaki od tih blokova, kojima je sačuvana mikroskala u efektivnom problemu. Pomoću operatora dilatacije koji je uveden u [25] izvode se efektivne matrične jednadžbe te se identificiraju izvori u frakturama. Prikazani rezultat predstavlja proširenje rezultata [14] na potpuno ekvivalentnu formulaciju pomoću globalnog tlaka.

Contents

Chapter 1

Introduction

The subject of this thesis is mathematical modelling of immiscible compressible two-phase flow in heterogenous porous media with the main applications of water-gas migration through engineered and geological barriers of a deep repository for radioactive waste and air-water flow in hydraulic structures. In particular, we investigate three typical problems for a new model of such flow problems, which has been lately derived by the concept of the global pressure in [8] and [11] and which is fully equivalent to the original equations. Namely, this work contributes to the area of the mathematical analysis of multiphase flow in porous media by a new existence result of weak solutions of the system modelling the immiscible flow of one incompressible and one compressible phases (such as water and gas), given in Chapter 4, as well as by new homogenization results for the immiscible compressible two-phase flow which are presented in Chapter 5 for the case of an ordinary heterogenous porous media, and in Chapter 6 for the naturally fractured reservoir.

Multiphase flow in porous media

Many petroleum and environmental engineering problems notably rely on the modelling and prediction of fluid flow and transport through heterogeneous porous media. For instance, some of the techniques in production of hydrocarbons from petroleum reservoirs enforced during the secondary and tertiary oil recovery are water or gas injection with intent to increase reservoir's pressure. The processes occurring therein are mathematically modeled as multiphase and multicomponent flow in porous media. CO_2 sequestration is a relatively new technique aimed to prevent the release of large quantities of carbon dioxide into the atmosphere from fossil fuel use in power plants and other industries. The idea of this method is the sequestration of CO_2 into underground geologic formations (for instance, depleted oil or gas fields), wherein a flow of water and gas comes about. The long-term management of the hazardous radioactive waste produced by nuclear industry has been an issue of an increasing concern [20, 88]. One of the methods investigated and implemented since the 1950s is the disposal of the radioactive waste in engineered facilities located deep underground in stable geologic formations which act as the natural geologic barriers and are selected according to their capability to isolate radioactive waste from the biosphere.The corrosion of metallic components used in the repository design, such as steel lines, waste containers, and water radiolysis by radiation issued from nuclear waste produce gas, mainly hydrogen; certain amounts of methane and carbon dioxide are usually generated by the microbial activity which is also likely to transform some hydrogen into methane. Therefore, one faces a problem of possible two-phase flow of (ground)water and gas, which may result in a pollution of the groundwaters. Also, the creation and transport of a gas phase is an issue of concern with regard to capability of the engineered and natural barriers to evacuate the gas and avoid overpressure, thus preventing mechanical damages. The underground water flow is an object of study in hydrology and soil science for numerous applications to civil and agricultural engineering. In many countries the supply of drinking water for more than half of the population comes from the groundwater [28, 73]. It is accordingly of great importance to maintain the acceptable groundwater quality, which can be threatened by incidental spills of harmful substances generated by industry, by the disposal dumps or by leaking of the storage tanks. The environmental remediation technologies which deal with the removal of the pollution or contaminants from the environmental media such as soil, groundwater, sediment, or surface water are based on the properties of underground fluid flow. In general, mathematical models and numerical simulations of multiphase flows help to gain a better understanding of the process, to predict the flow behavior and ultimately to develop and optimize the remediation or harmful materials storage techniques with respect to cost and efficiency.

This thesis is concerned with the immiscible compressible isothermal two-phase fluid flow in a porous medium, concerning capillary effects, gravity and heterogeneity. A standard way of modelling that problem is to use the mass conservation equations and the Darcy-Muskat law for each of the fluids, which gives the system of two highly coupled nonlinear evolutionary partial differential equations. The disadvantage of this formulation is the degeneracy of the relative permeability functions which does not allow to derive the uniform estimates for the phase pressure gradients. Also, the phase pressures are mathematically not well defined over time globally in the domain since their evolution terms disappear in the regions without the corresponding fluid. With a view to achieve a more tractable form of the governing system, these starting equations are algebraically converted into distinct alternate forms and the primary variables of the system can be selected in several ways. This choice strongly affects the mathematical analysis and numerical methods used in the problem solving and simulations. It has been known for some time that the fractional flow approach, where the flow of the two phases is observed as a total flow of one fictional mixed fluid and the single phases act as fractions of the total flow, is particularly convenient. Additional decoupling of the equations is accomplished by introducing a new variable called the global pressure which is taken as an independent variable in this formulation, along with one of the phase saturations.

The notion of the global pressure originates from [21, 48] and since then many authors investigated this idea from the mathematical analysis point of view as well as for the numerical simulations in hydrology and petroleum reservoir engineering (see for example [28, 51–53]). The fractional flow / global pressure formulation is more suitable for the mathematical treatment due to the clear type of the equations and the coupling between the equations of the system being relieved. It also enables one to establish uniform bound on the global pressure gradient. Moreover, this form of the multiphase immiscible flow has been verified as advantageous over the other formulations with regards to the computational efficiency [28, 53]. In [48] the global pressure was introduced for the incompressible two-phase flow. As far as two compressible phases, in [48] formulation with the global pressure was obtained assuming that the capillary pressure is low and that the phase mass densities and the other pressure dependant coefficients can be evaluated at the global pressure instead of the corresponding phase pressures, with a neglectable error. Although this approximate model is widely used in applications, the results and the numerical simulations of [11] show that for some types of immiscible compressible two-phase flows, such as water-air system in hydrogeological applications, this assumption is not satisfied and in that setting approximation of this kind can result in unacceptably large errors, especially in the loss of the mass balance for the nonwetting phase. Only recently the fractional flow formulation which is fully equivalent to the original system has been established in [8] for the immiscible flow of water and gas, and in [11] for the general case of two immiscible compressible fluids; that model is a topic of our thesis. The said procedure leads to a degenerate coupled system consisting of a nonlinear parabolic equation for the global pressure and a nonlinear convection-diffusion equation for the saturation. In [13] the numerical simulations were performed for this model in the case of porous media with several rock types. A fully equivalent formulation for a three-phase flow is derived in [47].

Two-phase flow in porous media has been extensively mathematically analyzed for a long time, followed by the numerous literature and many developed methods. In the case of two incompressible phases, the questions of the existence and regularity of weak solutions have been studied in [4, 21, 23, 36, 48, 49, 61, 65, 78, 79, 101, 102]. The existence results for the multi-component model have been lately established in [46, 64, 80, 84, 96, 97], and the miscible compressible flow in porous media and the corresponding existence issues have been treated by [17–19, 55, 67]. Still, the first results on the existence for immiscible compressible two-phase flow have been derived just recently. Namely, the authors of [45, 48, 68–71] considered the immiscible two-phase flow with one or more compressible fluids in certain approximate models based on the aforementioned assumption on the mass densities, with the global pressure being introduced in the way it was done in [21, 48] for incompressible immiscible flows. In [68] some terms related to compressibility were disregarded, while in [68–72] more regularity was assumed for the porosity, absolute permeability and the capillary pressure functions which excludes the case of discontinuous medium coefficients and unbounded capillary pressure as it appears in some applications, such as gas migration through engineered and geological barriers for a deep radioactive waste repository. A more general immiscible compressible two-phase flow model was studied by [72, 75–77]. In these works the models in phase formulations were studied while the feature of the global pressure inducted as in [48] for the incompressible flow is employed in order to establish the a priori estimates. These results are established under the assumption that the capillary pressure is bounded and no discontinuity of the porosity and the permeability is permitted. Existence results of weak solutions for the fully equivalent global pressure formulation for the two-phase compressible flows are obtained in [12] under some realistic assumptions on the data which cover the cases of unbounded capillary pressure function, and the discontinuous porosity and absolute permeability tensors.

Homogenization

The fluid flow and transport in the subsurface is considerably affected by the inhomogeneity of the porous medium. Namely, the porous medium is characterized by several distinct spacial scales and its permeability and the porosity vary on many different length scales. Mathematically, this feature is expressed by the rapidly oscillating coefficients of the equations which describe multiphase flow in heterogenous media. In order to provide realistic analysis and predictions of the flow and transport behavior, one needs the models which accurately account for the strong heterogeneity effects of the medium. However, these variations make direct analytical and numerical methods for solving boundary value problems for the equations of this type at the field-scale extremely difficult and often practically infeasible. It is therefore desirable to use methods that represent the effects of subgrid scale variations on larger scale flow results in a way which allows the use of a coarse computational grid. A standard approach is to average or upscale physical parameters such as porosity and absolute permeability to obtain the macroscopic laws capturing their integral effects on multiphase flow. The homogenization theory has been developed with a view to rigorously mathematically describe the various physical processes in highly inhomogeneous materials, in particular, in the porous media, such as in oil reservoir simulation or hydrogeology. The objective of homogenization is to replace the governing equations by a simpler set of equations (homogenized or effective equations) for which the solution can be resolved on a reasonable coarse-scale mesh and which provide a good approximation of the average behavior of the solution of the governing equations. In its simplest form, the coefficients of the equations describing the original processes are replaced by effective or macroscopic coefficients. As a result, the characteristics of the original, highly heterogeneous material are well-approximated by those of the effective locally homogeneous material.

The mathematical theory of homogenization or upscaling has been comprehensively developed since the early 1960s and the methods as well as literature in this area have been numerous. We may refer to to the classical books [31], [59] and [93] for an advanced general presentation of mathematical homogenization, and in [74] one can find an extensive collection of applications of homogenization to porous media. For recent reviews on other upscaling methods, see for instance [62, 66] and the bibliography therein.

The homogenization of one phase flow in the framework of the geological radioactive waste disposal was considered in [33, 34, 40, 41, 44]. Many authors have studied the homogenization and upscaling of incompressible immiscible two-phase flow in porous media, for instance [30,35,37,38,42,43,73,74,83,89,90,94,99]. Homogenization results for compressible miscible two-phase flow in porous media were obtained in [17, 56]. On the other hand, for immiscible compressible two-phase flow, the first homogenization result was only recently established in [6]. In that work the model of water-gas flow in original phase form is used with the simplified global pressure formulation used for obtaining the a priori estimates and compactness results. Moreover, the bounded capillary pressure function is assumed which is too restrictive for some realistic problems.

A distinct kind of porous medium is noticed in a naturally fractured reservoir which is often met in hydrology and petroleum applications, in particular the sedimentary rocks that compose a hydrocarbon reservoir. This type of porous medium consist of a discontinuous system of periodically repeating matrix blocks of ordinary porous media surrounded by a connected system of thin fissures. It has been first observed in the engineering literature in 1960s that flow in such fractured reservoirs is quite unlike that in an ordinary, unfractured porous media. Instead, the flow behaves as if the reservoir possessed two porous structures, one associated to the fractures, and the other to the porous rock, with disparate features: the permeability of the fractures is much higher than those of the matrix and hence the majority of fluid flow takes place through the fracture system while the matrix occupies most of the volume of the reservoir and most of the fluid is situated there. A double porosity model was proposed on physical grounds by [27, 85, 91, 100] to describe the flow of one or more fluids in a naturally fractured reservoir. Since then it has been widely used for engineering applications in geohydrology, petroleum reservoir engineering, civil engineering and soil science to model the effect of natural fractures on subsurface fluid flow and subsequent reservoir performance. Namely, in this model the fractured reservoir is replaced by an equivalent imaginary coarse grained porous medium for which the fractures act as pores, while the matrix blocks could be seen as fictive grains.

Regarding the mathematical homogenization of flow in a naturally fractured porous medium, the first result was the description of a general form of the double porosity model for a single phase flow in [22]. Then in [25] this general model was rigorously justified from the point of view of homogenization theory and also the dilation operator was introduced. The global behavior of single phase flow in fractured media is studied as well in [15, 16], where the variable ratio of the block permeability to the fractures permeability as well as the fractures to the blocks dimensions are considered. The first contribution on the derivation of the double porosity model for two-phase flow in a fractured medium is [26], where the effective equations of the double porosity model are established formally by asymptotic expansion for completely miscible incompressible flow, and immiscible incompressible two-phase flow. In [39] the double porosity model for immiscible incompressible two-phase flow in a reduced pressure formulation is rigorously justified by periodic homogenization; the same problem was considered by [103] who studied the cases of the ratio of the permeabilities in the matrix blocks and in the fractures being of order ε^2, smaller than ε^2 and greater than ε^2, respectively, where ε is a small parameter depicting the size of a matrix block with respect to the domain size. This study revealed that in the first case, the limit model is of a dual porosity type, the second one leads to a single-porosity model for the fracture flow, while the last one yields another type of single-porosity model for the fractures, with an additional source term arising from matrix blocks. For the displacement of one compressible miscible fluid by another in a naturally fractured reservoir, the double porosity model was rigorously derived in [54] and for compositional three-phase flow it was established by the formal asymptotic expansion in [24, 50]. Furthermore, [101] investigates the existence of weak solutions for the model of the immiscible two-phase flow in fractured porous media. Finally, the first result on the immiscible compressible two-phase flow in this context is [14] where only lately the double porosity model has been established for the water-gas flow in a global pressure formulation, whose existence is given in [75].

Overview of the thesis

In Chapter 2 we firstly exhibit the standard notions and equations for the two-phase immiscible flow in porous media, as in [28]. Afterwards we present the global pressure formulation of the problem for the compressible flow, both the simplified and the fully equivalent formulation, as well as the global pressure formulation of the incompressible flow [48]. At the end of the Chapter a review of existence results concerning the two-phase flow in porous media is provided.

In Chapter 3 we explain the concepts of homogenization and upscaling. In particular, the notion of the two-scale convergence is defined and the standard results regarding it are quoted. We also introduce a dilation operator and list its properties. The Chapter is concluded by a review of homogenization results for the two-phase flow.

The existence of weak solutions for the two-phase immiscible compressible flow in porous media in a fully equivalent global pressure - saturation formulation is established in Chapter 4. The results of this Chapter have been published in [9]. We study the case of an incompressible wetting phase and a compressible non-wetting phase. The model takes into account gravity, capillary effects and heterogeneity, and an isothermal condition is assumed. On the boundary of the domain we impose non-homogeneous Dirichlet and Neumann conditions. We also present some physically reasonable assumptions on the data, as in [12], as well as on the boundary data. The proof of the existence of weak solutions under such assumptions is based on using an appropriate regularization and a time discretization. The main difficulty is the degeneracy of the equations caused by annulation of the diffusivity coefficient. Hence we add a small constant $\eta > 0$ to it, and in the same time we regularize the singular capillary pressure function to overcome the integrability problems. This brings us to a regularized problem with a parameter $\eta > 0$. To establish the existence of weak solutions for the evolution equations in the regularized system, we discretize in time with a small parameter $h > 0$, which leads to a sequence of elliptic approximations. Schauder's fixed point theorem is used to establish the existence for a discretized problem. A priori estimates with respect to the space and time variable, uniform in h and η are obtained by using convenient test functions, proposed in [72]. In order to pass to the limit in nonlinear terms as $h \to 0$ we also employ a generalization of compactness lemma from [48] and from [95] (see also [11, 71, 75]) which allows the discontinuities of the porosity, and some auxiliary results. Thus the existence for a regularized system is shown. Analogous arguments are used to pass to the limit as $\eta \to 0$ in the regularized problem which proves the main result.

The last two Chapters contain the new homogenization results for the new, fully equivalent global pressure formulation for the two-phase immiscible compressible flow.

Therein we rigorously justify the homogenization process for a single and a double porosity model by using the two-scale convergence technique.

More precisely, Chapter 5 is concerned with a flow of two compressible phases through a highly heterogeneous porous medium consisting of a rock of a single type and possessing a periodic microstructure. This Chapter comprises the results from [10]. Our purpose is to describe the effective problem for this model and to prove the convergence of the weak solutions of the microscopic system to the weak solutions of this effective problem, under certain physically justified assumptions on the data. At the beginning we expose the equations describing the problem at the microscopic level depending on a small parameter ε, whose at least one weak solution has been proved to exist in [12]. The appropriate test functions are employed to establish the a priori estimates independent of ε for these weak solutions. To obtain the compactness for the weak solutions of the microscopic problem we use a new compactness result of [6]. This enables to pass to the limit as $\varepsilon \to 0$ in the microscopic equations by employing the two-scale convergence arguments. Thereby the homogenized equations are obtained.

Finally, in Chapter 6 we consider an immiscible flow of an incompressible wetting phase and a compressible non-wetting phase in a fractured porous medium. Namely, the objective is to write up the corresponding homogenized problem and to show the convergence of the weak solutions of the microscopic problem to weak solutions of the homogenized problem. We firstly present the microscopic problem which depends on ε and recall the corresponding existence result from Chapter 4 which is valid under some realistic assumptions on the data. Similarly as in the two previous Chapters, the a priori estimates uniform with respect to ε are established. By using the classical compactness results of [95] and [5] we then establish the compactness for the weak solutions of the microscopic problem. The two-scale convergence is employed to obtain the effective equations for the fracture flow. However, the nonlinearities and the coupling of the system give rise to a non-identified term which represents an upscaled matrix-to-fractures flow source term. Moreover, the deduced two-scale convergence results for the matrix solutions are insufficient to establish the homogenized matrix system. These inconveniences are resolved by transforming the weak solutions with help of the dilation operator. More precisely, first we obtain the equations for the dilated solutions for ε as well as their limit functions as $\varepsilon \to 0$, which correspond to the two-scale limits of the non-transformed matrix solutions. We establish also a certain compactness result which enables to pass to the limit in the equations satisfied by the dilated functions as well as in the corresponding boundary conditions. Eventually it allows us to finish the proof of the main result.

Chapter 2

Modelling immiscible two-phase flow in porous media by the concept of global pressure

2.1 Introduction

This Chapter contains the presentation of the model describing immiscible compressible two-phase fluid flow in porous media in the global pressure formulation. We begin by introducing in Section 2.2 the basic notions and presenting the standard system of macroscopic equations for the two-phase immiscible compressible flow; herein we follow the references [28, 29, 48, 53, 73]. In Section 2.3 we briefly explain the transformation of the standard system to the fractional flow form, or the global pressure saturation form, in the case of two incompressible phases, as in [48]. The idea behind the fractional flow reformulation is to consider the two-phase flow as a total fluid flow of a single mixed fluid, in which the individual phases can be seen as fractions of the total flow. Introducing a new variable (global pressure P) leads to a system where the coupling between the two equations is weakened and the new formulation is more suitable for mathematical and numerical analysis of the problem. An overview of the existence results for the immiscible incompressible two-phase flow is included. Next, following [11, 104], [48], the concept of the global pressure applied to two-phase immiscible compressible fluid flow is presented in Section 2.4. Firstly in Subsection 2.4.1 we derive a fractional flow formulation of the original equations by considering the total flow. By this procedure, the coupling between the two coupled equations is weakened, and the problem gains a well defined mathematical structure. In order to further decouple the system, a new variable called the global pressure was introduced in [21, 48] and a formulation with the global pressure and one saturation as primary variables was developed for the incompressible flow. Until recently, a global pressure formulation for two and three phase compressible flow models had been used only in a certain approximate form proposed by [48]. More precisely, it is assumed that one can disregard the error created by evaluating the phase densities at the global pressure instead of the corresponding phase pressures. It was shown [11] that for some types of immiscible compressible two-phase flows this assumption is not satisfied. In Subsection 2.4.2 we present the fully equivalent fractional flow formulation which was recently established without any simplifying assumptions in [8] for the immiscible flow of water and gas, and in [11] for the general case of two immiscible compressible fluids. Next, in Subsection 2.4.3 we consider a simplified fractional flow formulation based on approximate calculation of mass densities. Lastly in Section 2.5 we overview the references concerning the existence for the immiscible compressible two-phase flow, and some related results.

2.2 Two-phase immiscible flow in porous media

2.2.1 Definitions

The *porous medium* $\Omega \subseteq \mathbb{R}^n$ is any body that consists of a solid part called *solid matrix* and the connected void space (or *pore space*) that can be filled with one or more fluids. For example, soil, sand, wood, cork, ceramics, sponge, bread, lungs, kidneys, bones can be considered as porous media.

The porous medium can be observed at several different space scales. At the *molecular scale* (about 10^{-9} m) the individual fluid molecules can be detected. These are replaced by a hypothetical continuum on the *microscopic scale* (about 10^{-3} m) which is determined by a characteristic size of a pore. Here the individual solid grains and pore channels are visible. Finally, the scale of order of 10 m is referred to as the *macroscopic scale*. The porous medium is at this scale modeled as a continuum in which one does not distinguish the solid phase from the fluid phases present in the pore space. At this length size the different types of rock with different average grain sizes can be identified.

The flow of one or more fluids in the pore space of a porous medium is described at the microscopic level by the Navier-Stokes equations with appropriate boundary conditions. This model is unapplicable in practice due to the unknown geometry of the pore space and the discrepancy between the dimensions of pores (of order of micrometer) and the space dimensions of the domain (field) (up to few kilometers) which disables the numerical simulations on the microscopic level. Indeed, one is typically not interested in the flow variations at the pore space scale, although multiphase flow problems at the microscale have been studied and different approaches have been used for solving them numerically, see e.g. [92] and the references therein. Therefore, the model is commonly considered at the larger, macroscopic level where the exact description of the microscopic configuration is not needed. Namely, the porous medium is taken as a continuum where each point represents *elementary volume* of the porous medium that is taken large enough to ensure that both solid phase and the fluid in the pore space are contained in it. For more information on the identification of the size of such representative elementary volume [29], [73] or [28] can be consulted. Then each point of the macroscopic continuum is assigned average values over elementary volumes of quantities given at the microscopic level, such as the fluid pressure and velocity.

The basic macroscopic properties of a porous medium are *porosity* and *absolute permeability*. The porosity Φ of a porous medium is defined as the ratio of a pore volume to the total volume of the porous medium. We assume that the rock is not deformable, so that the porosity does not depend on the pressure of the fluid. The absolute permeability tensor $\mathbb{K}$ $[m^2]$ (a symmetric tensor) is a measure of the ability of a porous material to permit the fluids flow through it. We consider the *heterogeneous* porous medium whose macroscopic properties vary in space through the domain, but not in time.

We will consider the flow of two fluids in porous media assuming that at the microscopic level there exists a surface tension at the boundary of the two fluids. As a result, the fluids can not mix and a sharp interface between the fluids is formed. The two fluids separated by the well defined surface are called *immiscible* and are referred to as the *phases*. This type of flow is indicated as an immiscible flow. In multiphase immiscible flow, a wetting and a non-wetting phase are discerned: if a contact angle between the solid surface and the fluid-fluid interface for one of the fluids is less than 90^0 then it is called the wetting phase fluid, the other fluid is then the non-wetting phase fluid. In other words, one phase wets the porous medium more than the other. Generally, water is the wetting fluid relative to oil and gas, while oil is the wetting fluid relative to gas. Throughout this work we will use indices $j \in \{w, g\}$ or $j \in \{w, n\}$ to denote the wetting and the non-wetting phase, respectively.

The individual fluid phases are characterized by the following macroscopic properties: their *densities* ρ_j $[kg/m^3]$ which are assumed to depend only on the *phase pressure* P_j $[Pa] = [N/m^2]$ (the temperature represents just a parameter, i.e., the isothermal flow is considered); the *compressibility* ν_j of the fluid phase which is defined as

$$\nu_j = \frac{1}{\rho_j(P_j)} \frac{\partial \rho_j(P_j)}{\partial P_j} [Pa^{-1}]$$

and the *dynamic viscosity* μ_j $[Pa\ s]$ that is assumed to be constant in this work. The *volumetric phase velocity* (*macroscopic apparent velocity*, *Darcy velocity*) of the phase $j \in \{w, n\}$ is denoted by $\mathbf{q}_j$ $[m/s]$.

For the multiphase flow, *phase saturations* S_j, $j \in \{w, n\}$ are the macroscopic variables measuring the quantities of the volume of individual phases at the point in the macroscopic model. Obviously it is $S_j(x, t) \geq 0$, $j \in \{w, n\}$, and $\sum_j S_j(x, t) = 1$.

At the microscopic level, the two immiscible fluids are separated by a clearly defined curved interface whose form is determined by the surface tension. This phenomenon gives rise to a discontinuity in microscopic pressure throughout the contact surface. The magnitude of that jump of the pressure is called the *capillary pressure* P_c. It is equal to

$$P_c = P_g - P_w \geq 0$$

and it is described by the Young-Laplace law. The macroscopic capillary pressure can in general also depend on temperature and fluid composition due to changes in surface tension but in this work it is assumed to depend on the saturation solely.

In practice, one uses some of the functional correlations between the capillary pressure and the saturation that contain parameters which try to account for the different pore space geometry and are used in order to fit the models to the experimental data. Most commonly used models are the functions of Van Genuchten and Brooks-Corey (see [29, 73]).

The phase *relative permeability* kr_j, $j \in \{w, n\}$, is the macroscopic adimensional quantity indicating to what extent the flow of the phase is prevented by the presence of the other phases in the pore domain. It holds $kr_j(S_j = 0) = 0$ and $S_j \mapsto kr_j(S_j)$ is an increasing function, $j \in \{w, n\}$. The *phase mobility* λ_j is defined by $\lambda_j = \frac{kr_j}{\mu_j}$, $j \in \{w, n\}$.

2.2.2 Governing equations

The standard system of equations describing the immiscible, compressible isothermal two-phase fluid flow in a porous medium at the macroscopic level consists of the mass conservation equations for the individual fluid phases, $j \in \{w, n\}$ ([29, 48, 73]):

$$\Phi \frac{\partial}{\partial t}(\rho_j(P_j) S_j) + \mathrm{div}(\rho_j(P_j)\mathbf{q}_j) = F_j, \tag{2.1}$$

combined with the Darcy-Muskat law for each phase, $j \in \{w, n\}$:

$$\mathbf{q}_j = -\lambda_j(S_j)\mathbb{K}(\nabla P_j - \rho_j(P_j)\mathbf{g}). \tag{2.2}$$

Here Φ and $\mathbb{K}$ are the porosity and the absolute permeability of the porous medium, and for $j \in \{w, n\}$, ρ_j, P_j, S_j, $\mathbf{q}_j$ and λ_j are the mass density, pressure, saturation, volumetric velocity and the mobility of the phase j, respectively; F_j is the source/sink term and $\mathbf{g}$ is the gravity acceleration, a downward-pointing, constant vector. The Darcy-Muskat law is an experimentally obtained relation of the volumetric phase velocity to the corresponding phase pressure gradient. It can also be derived by homogenization or local averaging techniques (see, e.g., [74]) from momentum conservation of the Navier-Stokes equations at the microscopic scale. The system is closed by adding the capillary pressure law as well as the condition that the two phases fill the whole pore space:

$$P_c(S_n) = P_n - P_w, \tag{2.3}$$

$$S_w + S_n = 1. \tag{2.4}$$

In this work the porosity and the absolute permeability are assumed to depend only on the space variable, and the capillary pressure and relative permeabilities are considered as functions of the saturation only.

The governing system (2.1)-(2.4) with the primary unknowns P_j, S_j and $\mathbf{q}_j$ consists of the two nonlinear partial differential equations (2.1), (2.2) which are highly coupled through the two algebraic relations (2.3), (2.4). These basic equations can be algebraically manipulated and combined into different modified forms. Also, there are few possible ways to choose primary (independent) variables in this system and concomitantly to eliminate the remaining unknowns (dependent variables). As a result, different mathematical formulations of the same model can be obtained. Mathematical analysis and especially numerical methods used in the simulations for the given model considerably depend on the choice of the form of the model (see e.g. [11, 48, 51–53]).

The formulation of the flow equations that is derived using the individual phase mass balance laws and choosing one phase pressure and one phase saturation as primary unknowns has two major deficiencies that make it not tractable for a mathematical study: the types of the equations are not evident; the equations are degenerate due to the vanishing of the relative permeabilities in the zones where the corresponding phases disappear. Therefore, another approach to modelling multiphase flow in porous media has been employed in order to obtain a different formulation of the model, with advantageous mathematical properties. Namely, by introducing appropriate new functions, the original equations are transformed into a formulation that we will refer to as the *fractional flow formulation* or the *global pressure - saturation formulation*, which employs a new, "pressure-like" variable called the global pressure and one saturation as the primary variables. The transformed system is less coupled and has a well defined mathematical structure. Another advantage of using the global pressure is that enables one to obtain uniform bound on the global pressure gradient, while the uniform estimates for the phase pressure gradients are not available due to the degeneracy of the relative permeabilities. The global pressure formulation for immiscible two-phase flow in porous media is presented in the next Section. Since it was introduced, the global pressure formulation has been utilized in a wide range of numerical simulations, particularly in hydrology and petroleum reservoir engineering (see for instance [48,53] and the references therein). It has been proven that this fractional flow approach is far more efficient than the original phase formulation from the computational point of view [53].

2.3 Immiscible incompressible two-phase flow in porous media by the concept of global pressure

Consider now the case of two-phase flow with incompressible fluids. Following [48], the aim of this Section is to present the fractional flow formulation of the system (2.1)-(2.4) and introduce the global pressure for the incompressible case, as in [21, 48].

2.3.1 Fractional flow formulation for the incompressible case

In order to distinguish the coefficients of the fractional flow formulation in the incompressible case from the corresponding coefficients for compressible phases, considered in the next Section, a superscript i is used.

With the choice of S_n and P_w as the main unknowns in (2.1)-(2.4), the *total mobility* is defined by

$$\lambda^i(S_n) = \lambda_w(S_n) + \lambda_n(S_n),$$

and the *fractional flow* functions are

$$f_j^i(S_n) = \frac{\lambda_j(S_n)}{\lambda^i(S_n)}, \ j \in \{w, n\}.$$

Next, the *total velocity*

$$\mathbf{q}_t = \mathbf{q}_w + \mathbf{q}_n$$

is introduced and from (2.2) it can be expressed as

$$\mathbf{q}_t = -\mathbb{K}\left(\lambda_w(S_n)\nabla P_w + \lambda_n(S_n)\nabla P_n\right) + \mathbb{K}\mathbf{g}\left(\lambda_w(S_n)\rho_w + \lambda_n(S_n)\rho_n\right).$$

From the initial system (2.1)-(2.4), after summing up the two mass conservation equations one obtains the following system:

$$\mathrm{div}(\mathbf{q}_t) = F_w/\rho_w + F_n/\rho_n, \tag{2.5}$$
$$\mathbf{q}_t = -\lambda^i(S_n)\mathbb{K}\left(\nabla P_w + f_n^i(S_n)\nabla P_c(S_n) - \rho^i(S_n)\mathbf{g}\right), \tag{2.6}$$
$$\Phi\frac{\partial}{\partial t}S_n + \mathrm{div}(f_n^i(S_n)\mathbf{q}_t - \mathbb{K}\mathbf{g}b^i(S_n)) + \mathrm{div}(\mathbb{K}a^i(S_n)\nabla S_n) = \frac{F_n}{\rho_n}, \tag{2.7}$$

where the coefficients are defined by

$$\begin{aligned}
\rho^i(S_n) &= (\rho_w\lambda_w(S_n) + \rho_n\lambda_n(S_n))/\lambda^i(S_n),\\
\alpha^i(S_n) &= \lambda_w(S_n)\lambda_n(S_n)/\lambda^i(S_n),\\
a^i(S_n) &= -\alpha^i(S_n)P_c'(S_n),\\
b^i(S_n) &= \alpha^i(S_n)(\rho_w - \rho_n).
\end{aligned}$$

The pressure equation (2.5), (2.6) is coupled to the equation for the saturation (2.7) via the gradient of the capillary pressure. In the following Subsection it is shown citing [21,48] that by introducing an appropriate new variable, the coupling between the two equations can be additionally relieved.

2.3.2 Global pressure formulation for the incompressible case

The idea introduced in [21,48] is to induct a new pressure-like variable P, called the ***global pressure***, in such a way that (2.6) takes the form of a Darcy law for the pressure P, with a non-degenerate coefficient. That is, a function $\gamma(S_n)$ and a quantity P that represents some mean pressure are needed such that

$$\lambda_w(S_n)\nabla P_w + \lambda_n(S_n)\nabla P_n = \gamma(S_n)\nabla P. \tag{2.8}$$

Using (2.3) it is easy to see that (2.8) is fulfilled is we take $\gamma(S_n) = \lambda^i(S_n)$. Then we have

$$\nabla P_w + f_n^i(S_n)P_c'(S_n)\nabla S_n = \nabla P,$$

which is satisfied if

$$P = P_w + \int_0^{S_n} f_n^i(s)P_c'(s)ds. \tag{2.9}$$

Using (2.3), the relation (2.9) implies

$$P = P_n - P_c(0) - \int_0^{S_n} f_w^i(s) P_c'(s) ds. \tag{2.10}$$

We note that by choosing the non-wetting saturation as a primary unknown, the capillary pressure $S \mapsto P_c(S)$ is increasing function. Therefore, (2.9) and (2.10) yield

$$P_w \le P \le P_n.$$

Finally, taking into account (2.10), the system (2.5)-(2.7), and therefore the original governing equations (2.1)-(2.2), is transformed into the following system with primary variables P and S_n:

$$\operatorname{div}(\mathbf{q}_t) = F_w/\rho_w + F_n/\rho_n, \tag{2.11}$$

$$\mathbf{q}_t = -\lambda^i(S_n)\mathbb{K}\left(\nabla P - \rho^i(S_n)\mathbf{g}\right), \tag{2.12}$$

$$\Phi\frac{\partial S_n}{\partial t} + \operatorname{div}(f_n^i(S_n)\mathbf{q}_t - \mathbb{K}\mathbf{g}b^i(S_n)) + \operatorname{div}(\mathbb{K}a^i(S_n)\nabla S_n) = \frac{F_n}{\rho_n}, \tag{2.13}$$

The new system (2.11)-(2.13) is referred to as the fractional flow or global pressure-saturation formulation of the system (2.1)-(2.4) for the two incompressible fluids. It consists of the global pressure equation (2.11), which is an elliptic equation with the parameter $t \in]0, T[$, and the nonlinear convection-diffusion equation for the non-wetting saturation (2.13), whose diffusion term a^i degenerates as it satisfies $a^i(S_n = 0) = a^i(S_n = 1) = 0$. These two equations are coupled through the total velocity $\mathbf{q}_t$ and the coefficients which depend on S_n. By introducing the global pressure, the transformed system is established that is less strongly coupled and the derived equations are well mathematically structured. Introduction of the global pressure for the two-phase immiscible compressible flow is considered in the next Section.

2.4 Immiscible compressible two-phase flow in porous media by the concept of global pressure

For two and three-phase compressible flow, the concept of the global pressure was introduced in [48]. The authors derived the formulation with the global pressure and one saturation as primary variables under the assumption that the phase densities vary slowly with the pressure and that the capillary pressure, a difference between P_n and

P_w, is small. In this case, it was considered that the error caused by evaluating phase density ρ_j at the global pressure P instead of the phase pressure P_j can be ignored. This assumption has been since employed in petroleum engineering applications [51, 53]. A numerical analysis performed in [11] revealed that such simplified models based on a mass density approximation can be used safely in applications where the mean field pressure is high, capillary pressure is low and the wetting phase is not highly compressible, such as the oil reservoir simulations, but are inadequate in many underground gas and water flows where the difference between the phase pressures and the global pressure can be significant. Only recently, the global pressure formulation for two-phase flow in porous media has been derived without any simplifying assumptions. Namely, the formulation that is fully equivalent to the original phase equations, where the phase pressures and the phase saturations are primary unknowns, was established for the water-gas flow in [8], and for the general case of two compressible fluids in [11]. A fully equivalent global pressure formulation for three-phase flow was derived in [47]. The global pressure formulation is more suitable for mathematical and numerical analysis, for more details see [12, 13, 104]. The current Section begins by establishing the fractional flow formulation of two-phase compressible flow in Subsection 2.4.1. Then the fully equivalent global pressure formulation for two-phase compressible immiscible flow is presented in Subsection 2.4.2, and a brief display of the simplified model is given in Subsection 2.4.3.

2.4.1 Fractional flow formulation for the compressible case

We consider the system (2.1)-(2.4) in the case of two compressible fluids and select the non-wetting saturation S_n and the wetting phase pressure P_w for primary variables. Following the path used in the incompressible case, we introduce the *total flux*

$$\mathbf{Q}_t = \rho_w(P_w)\mathbf{q}_w + \rho_n(P_n)\mathbf{q}_n,$$

and the following nonlinear coefficients: the total mobility function

$$\lambda(S_n, P_w) = \rho_w(P_w)\lambda_w(S_n) + \rho_n(P_n)\lambda_n(S_n), \tag{2.14}$$

the fractional flow functions

$$f_j(S_n, P_w) = \frac{\rho_j(P_j)\lambda_j(S_n)}{\lambda(S_n, P_w)},\ j \in \{w, n\}, \tag{2.15}$$

and the nonlinear functions

$$\begin{aligned}
\rho(S_n, P_w) &= (\rho_w(P_w)^2\lambda_w(S_n) + \rho_n(P_n)^2\lambda_n(S_n))/\lambda(S_n, P_w), \\
\alpha(S_n, P_w) &= \rho_w(P_w)\rho_n(P_n)\lambda_w(S_n)\lambda_n(S_n)/\lambda(S_n, P_w), \\
a(S_n, P_w) &= -\alpha(S_n, P_w)P_c'(S_n), \\
b(S_n, P_w) &= \alpha(S_n, P_w)(\rho_w - \rho_n).
\end{aligned}$$

After summing the equations (2.1)-(2.4), one obtains the transformed system

$$\Phi\frac{\partial}{\partial t}(\rho_w(P_w)(1-S_n) + \rho_n(P_n)S_n) + \mathrm{div}\mathbf{Q}_t = F_w + F_n, \quad (2.16)$$

$$\mathbf{Q}_t = -\lambda(S_n, P_w)\mathbb{K}\left(\nabla P_w + f_n(S_n, P_w)\nabla P_c(S_n) - \rho(S_n, P_w)\mathbf{g}\right), \quad (2.17)$$

$$\Phi\frac{\partial}{\partial t}(\rho_n(P_n)S_n) + \mathrm{div}(f_n(S_n, P_w)\mathbf{Q}_t - b(S_n, P_w)\mathbb{K}\mathbf{g}) + \mathrm{div}(a(S_n, P_w)\mathbb{K}\nabla S_n) = F_n. \quad (2.18)$$

In regards to the primary system (2.1)-(2.4), the new system for the unknowns P_w and S_n is less coupled and its structure is more evident. Namely, (2.18) is a nonlinear convection-diffusion equation for the saturation, while the pressure equation (2.16) is a nonlinear parabolic equation that is still strongly coupled to the saturation equation through the gradient of capillary pressure and the time derivative term.

As mentioned before, the idea of [21,48] is to introduce the global pressure P in order to further decouple the equations (2.16)-(2.18). This accounts for posing

$$\nabla P_w + f_n(S_n, P_w)\nabla P_c(S_n) = \omega(S_n, P)\nabla P, \quad (2.19)$$

where a function $\omega(S_n, P)$ and the variable P are to be determined.

Let us point out that the fractional flow formulation in the compressible case could have alternatively be derived by using the total velocity $\mathbf{q}_t = \mathbf{q}_w + \mathbf{q}_n$ instead of the total flux, which would induce equations in a non-conservative form. This formulation has been studied in [6, 14, 68–72, 75–77]. However, the total flux behaves more smoothly [52].

2.4.2 Fully equivalent model

In this part we present the fully equivalent global pressure formulation for two compressible fluids, following [11, 104] which we refer to for details.

In order to resolve (2.19), we assume that the wetting-phase pressure is an unknown function P_w related to a new variable P (the global pressure) by

$$P_w = P_w(S_n, P). \tag{2.20}$$

Combining (2.19) and (2.20) leads to

$$\frac{\partial P_w}{\partial S_n}(S_n, P)\nabla S_n + \frac{\partial P_w}{\partial P}(S_n, P)\nabla P = \omega(S_n, P)\nabla P - f_n(S_n, P_w(S_n, P))P_c'(S_n)\nabla S_n.$$

Since P and S_n are independent variables this yields the equations

$$\frac{\partial P_w}{\partial S_n}(S_n, P) = -f_n(S_n, P_w(S_n, P))P_c'(S_n), \tag{2.21}$$

$$\frac{\partial P_w}{\partial P}(S_n, P) = \omega(S_n, P). \tag{2.22}$$

By integrating (2.21) one obtains the equation for the wetting pressure function $P_w(S_n, P)$:

$$P_w(S_n, P) = P - \int_0^{S_n} f_n(s, P_w(s, P))P_c'(s)ds, \tag{2.23}$$

where it is set $P_w(0, P) = P$. Then the formula for the non-wetting phase pressure follows,

$$P_n(S_n, P) = P + P_c(0) + \int_0^{S_n} f_w(s, P_w(s, P))P_c'(s)ds. \tag{2.24}$$

After introducing the capillary pressure $u = P_c(S_n)$ as an independent variable, which simplifies the form of the equation, the integral equation (2.23) can be rewritten as a Cauchy problem for an ordinary differential equation with the parameter P as follows [11, 104]:

$$\begin{cases} \dfrac{d\widehat{P}_w(u, P)}{dS} = -\dfrac{\rho_n(\widehat{P}_w(u, P) + u)\hat{\lambda}_n(S)}{\lambda(\rho_w(\widehat{P}_w(u, P))\hat{\lambda}_w(S) + \rho_n(\widehat{P}_w(u, P) + u)\hat{\lambda}_n(S))}, \quad u > 0 \\ \widehat{P}_w(0, P) = P - P_c(0), \end{cases} \tag{2.25}$$

where we denote $\widehat{f}(u) = f(S_n(u))$ utilizing the monotonicity of the capillary pressure. The problem (2.25) has a global solution $\widehat{P}_w(u, P)$ [11] and we put

$$P_w(S_n, P) = \widehat{P}_w(P_c(S_n), P).$$

For known $P_w(S_n, P)$, the function ω is determined from (2.22) and it holds [11]

$$\omega(S_n, P) = \frac{\partial P_w(S_n, P)}{\partial P} = \frac{\partial P_n(S_n, P)}{\partial P}. \tag{2.26}$$

Next, the coefficients in (2.16)-(2.18) are now functions of the global pressure P instead of the phase pressures P_g and P_w, and of the non-wetting saturation. Thus we denote (using the same letters for the new functions)

$$\begin{aligned}
\rho_w(S_n, P) = \rho_w(P_w(S_n, P)),\ \rho_n(S_n, P) = \rho_n(P_w(S_n, P) + P_c(S_n)),& \\
\lambda(S_n, P) = \rho_w(S_n, P)\lambda_w(S_n) + \rho_n(S_n, P)\lambda_n(S_n),& \qquad (2.27)\\
f_j(S_n, P) = \frac{\rho_j(S_n, P)\lambda_j(S_n)}{\lambda(S_n, P)},\ j \in \{w, n\},& \qquad (2.28)\\
\rho(S_n, P) = \rho(S_n, P_w(S_n, P)),\ \alpha(S_n, P) = \alpha(S_n, P_w),& \\
a(S_n, P) = a(S_n, P_w(S_n, P)),\ b(S_n, P) = b(S_n, P_w(S_n, P)).&
\end{aligned}$$

Then one can calculate [11]

$$\omega(S_n, P) = \exp\left(- \int_0^{S_n} (\nu_n(s, P) - \nu_w(s, P)) \frac{\rho_w(s, P)\rho_n(s, P)\lambda_w(s)\lambda_n(s)P_c'(s)}{(\rho_w(s, P)\lambda_w(s) + \rho_n(s, P)\lambda_n(s))^2}\, ds \right),$$

where the phase compressibilities are given by

$$\nu_w(S_n, P) = \frac{\rho_w'(P_w(S_n, P))}{\rho_w(P_w(S_n, P))}, \quad \nu_n(S_g, P) = \frac{\rho_n'(P_n(S_n, P))}{\rho_n(P_n(S_n, P))}.$$

Finally, we replace P_w by $P_w(S_n, P)$ in the equations (2.16)-(2.18) and employ (2.19) to establish the following system of equations for S_n and P:

$$\Phi \frac{\partial}{\partial t}(\rho_w(S_n, P)(1 - S_n) + \rho_n(S_n, P)S_n) + \mathrm{div}\mathbf{Q}_t = F_w + F_n, \tag{2.29}$$

$$\mathbf{Q}_t = -\lambda(S_n, P)\mathbb{K}\left(\omega(S_n, P)\nabla P - \rho(S_n, P)\mathbf{g}\right), \tag{2.30}$$

$$\Phi \frac{\partial}{\partial t}(\rho_n(S_n, P)S_n) + \mathrm{div}(f_n(S_n, P)\mathbf{Q}_t - b(S_n, P)\mathbb{K}\mathbf{g}) + \mathrm{div}(a(S_n, P)\mathbb{K}\nabla S_n) = F_n. \tag{2.31}$$

Let us emphasize that the equations (2.29)-(2.31) are fully equivalent to the system (2.16)-(2.18), and therefore to the initial standard two-phase flow equations (2.1)-(2.4), as argued in [11, 104].

Remark 1 *Having the global pressure at the disposal, the total flow* $\boldsymbol{Q}_t$ *is rewritten in the form of the Darcy-Muskat law in* (2.30)*, as intended. Hence the global pressure can be* *seen as a mixture pressure where the two phases are considered as mixture constituents, and* (2.29) *may be interpreted as the mass conservation law of the "idealized" compressible fluid replacing the mixture of the two fluids. Furthermore, from* (2.23) *and* (2.24) *one can obtain the "energy equality" (see [11])*

$$\begin{aligned}\rho_w(S_n,P)\lambda_w(S_n)\mathbb{K}\nabla P_w\cdot\nabla P_w+\rho_n(S_n,P)\lambda_n(S_n)\mathbb{K}\nabla P_n\cdot\nabla P_n\\=\lambda(S_n,P)\omega(S_n,P)^2\mathbb{K}\nabla P\cdot\nabla P+\alpha(S_n,P)\mathbb{K}\nabla P_c(S_n)\cdot\nabla P_c(S_n),\end{aligned}\tag{2.32}$$

which indicates physical relevance of the global pressure and will be used to obtain the a priori estimates on the solutions. Moreover, it is $P_w \le P \le P_n$.

Lastly, by introducing the functions

$$\Lambda_j(S_n,P)=\rho_j(S_n,P)\lambda_j(S_n)\omega(S_n,P),\ j\in\{w,n\},\tag{2.33}$$

the system (2.29)-(2.31) is rewritten as ([8,11]):

$$\begin{aligned}\Phi\frac{\partial}{\partial t}(\rho_w(S_n,P)(1-S_n))-\operatorname{div}(\Lambda_w(S_n,P)\mathbb{K}\nabla P)+\operatorname{div}(a(S_n,P)\mathbb{K}\nabla S_n)\\+\operatorname{div}(\lambda_w(S_n)\rho_w(S_n,P)^2\mathbb{K}\mathbf{g})=F_w,\end{aligned}\tag{2.34}$$

$$\begin{aligned}\Phi\frac{\partial}{\partial t}(\rho_n(S_n,P)S_n)-\operatorname{div}(\Lambda_n(S_n,P)\mathbb{K}\nabla P)-\operatorname{div}(a(S_n,P)\mathbb{K}\nabla S_n)\\+\operatorname{div}(\lambda_n(S_n)\rho_n(S_n,P)^2\mathbb{K}\mathbf{g})=F_n.\end{aligned}\tag{2.35}$$

The system (2.34)-(2.35) for the unknowns P and S_n contains two nonlinear degenerate parabolic equations which are highly coupled. Chapter 4 of this work is devoted to proving that there are weak solutions for this model of two-phase immiscible flow in porous media in the case of one compressible and one incompressible fluid.

2.4.3 A simplified model

As in [48], now we reconsider the equation (2.19) adopting a hypothesis that the phase densities (and the other nonlinear functions defined via phase pressures) can be computed at the global pressure P instead of the corresponding phase pressures, in other words, that the phase pressure in the non-wetting fractional flow function can be replaced by the global pressure P. In this situation, (2.19) is reduced to

$$\nabla P_w+f_n(S_n,P)\nabla P_c(S_n)=\omega(S_n,P)\nabla P\tag{2.36}$$

and it is easy to see that (2.36) will be fulfilled if one puts

$$P_w = P + \gamma(S_n, P), \tag{2.37}$$

where

$$\gamma(S_n, P) = -\int_0^{S_n} f_n(s, P_w(s, P)) P_c'(s) ds. \tag{2.38}$$

From (2.36) and (2.37) it follows that

$$\omega(S_n, P) = 1 + \frac{\partial}{\partial P}\gamma(S_n, P). \tag{2.39}$$

Taking into account the initial approximation assumption, $\rho_w(P_w)$ is replaced by $\rho_w(P)$, $\rho_n(P_n)$ by $\rho_n(P)$ and consequently the coefficients for a simplified model are defined as follows:

$$\begin{aligned}
\lambda^{sim}(S_n, P) &= \rho_w(P)\lambda_w(S_n) + \rho_n(P)\lambda_n(S_n),\\
f_j^{sim}(S_n, P) &= \frac{\rho_j(P)\lambda_j(S_n)}{\lambda(S_n, P)},\ j \in \{w, n\},\\
\rho^{sim}(S_n, P) &= (\rho_w(P)^2\lambda_w(S_n) + \rho_n(P)^2\lambda_n(S_n))/\lambda(S_n, P),\\
\alpha^{sim}(S_n, P) &= \rho_w(P)\rho_n(P)\lambda_w(S_n)\lambda_n(S_n)/\lambda(S_n, P),\\
a^{sim}(S_n, P) &= -\alpha(S_n, P)P_c'(S_n),\\
b^{sim}(S_n, P) &= \alpha(S_n, P)(\rho_w(P) - \rho_n(P)).
\end{aligned}$$

Finally one obtains the simplified global pressure formulation of the system (2.16)-(2.18):

$$\Phi\frac{\partial}{\partial t}(\rho_w(P)(1 - S_n) + \rho_n(P)S_n) + \mathrm{div}\mathbf{Q}_t = F_w + F_n, \tag{2.40}$$

$$\mathbf{Q}_t = -\lambda^{sim}(S_n, P)\mathbb{K}\left(\omega(S_n, P)\nabla P - \rho^{sim}(S_n, P)\mathbf{g}\right), \tag{2.41}$$

$$\Phi\frac{\partial}{\partial t}(\rho_n(P)S_n) + \mathrm{div}(f_n^{sim}(S_n, P)\mathbf{Q}_t - b^{sim}(S_n, P)\mathbb{K}\mathbf{g}) + \mathrm{div}(a^{sim}(S_n, P)\mathbb{K}\nabla S_n) = F_n. \tag{2.42}$$

Again, the total flux is expressed in a form of Darcy's law in (2.41).

Let us remark that the simplifying assumption on the phase densities considered in this Subsection leads to a fractional flow model in which the coefficients are calculated from the mass densities, the relative permeabilities and the capillary pressure, in contrast to the formulation (2.29)-(2.31) which demands solving a large number of the Cauchy problems for ordinary differential equation. For this reason the approximate model (2.40)-(2.42) is attractive in the applications. In [11, 104] the error introduced by replacing the phase pressures with the global pressure in the calculations of the mass densities is investigated by comparison of the coefficients of the two models and by performing the numerical simulations. As mentioned previously, this study detected that the simplified model can be used securely in applications with high mean field pressure and relatively small capillary pressure, if the wetting phase is not highly compressible, such as oil-gas systems. However, in hydrogeological applications, where capillary pressures may be increased with respect to mean field pressure, this approximation can cause unacceptably large errors, especially in the prediction of total mass of the non-wetting phase [11].

2.5 A review of the existence results for the two-phase flow

The partial differential equations describing the flow of multiple phase flow in porous media have been studied by many authors in the past few decades. In particular, the existence and regularity of weak solutions for the incompressible immiscible two-phase flow has been shown under various assumptions on physical data in [4, 21, 23, 36, 48, 49, 61, 65, 78, 79, 101, 102]. For immiscible two-phase flows of one or more compressible fluids with exchange between the phases, i.e. for partially miscible flow or multi-component model (like hydrogen dissolved in water), existence of weak solutions to these equations under some assumptions on the compressibility of the fluids and the finite transfer velocity among the phases has been recently established in [46, 64, 80, 84, 96, 97]. The miscible compressible flow in porous media and the corresponding existence issues have been investigated in [17–19, 55, 67]. In [63] the existence for three-phase immiscible incompressible flow in porous medium is proved under Chavent's "total differentiability" condition. When a porous medium is exposed to a mechanical deformation, that is, in the case when the porosity depends on the pressure, the existence for a two phase incompressible flow was shown in [61].

On the other hand, for immiscible compressible two-phase flow in porous media only recently several results have been obtained. Regarding immiscible two-phase flows with one or more compressible fluids without any exchange between the phases, some approximate models were studied in [45, 48, 68–71]. Namely, in these works the phase mass densities are assumed to depend not on the physical pressure, but on the global pressure. This assumption was introduced in [48] taking advantage of the fact that the densities vary slowly with the pressure and that the difference between the phase pressures, i.e. the capillary pressure is small, so the density can be evaluated at intermediate global pressure. As shown in [11], the models based on the mass density approximation can be suitable in oil reservoir simulations but are inadequate in many underground gas and water flows where the difference between the phase pressures and the global pressure can be significant. Moreover, [45] is concerned with the existence for compressible immiscible flow of two fluids if the porosity depends on the global pressure and on the space variable. In [68] certain terms related to the compressibility are neglected. The contributions [68–72] pose stronger assumptions on the regularity of the porosity, absolute permeability and the capillary pressure function which excludes the case of discontinuous medium coefficients and unbounded capillary pressure.

The authors of [72, 75–77] consider a more general immiscible compressible two-phase flow model in porous media. In these contributions, the models are based on phase formulations, i.e. the main unknowns are the phase pressures and the saturation of one phase, and the feature of the global pressure as introduced in [21, 48] for incompressible immiscible flows is used to establish a priori estimates. The results are established under the restrictive assumption that the capillary pressure is bounded, and no discontinuity of the porosity and the permeability is permitted.

Existence results of weak solutions for the fully equivalent global pressure formulation for the two-phase compressible flows are obtained in [12] under some realistic assumptions on the data which cover the cases of unbounded capillary pressure function, and the discontinuous porosity and absolute permeability tensors.

In Chapter 4 of this thesis a new existence result is established for a model of water-gas flow in porous media in the fully equivalent formulation using the concept of the global pressure. This work extends the results of [12] to the case of an incompressible phase (water) and a compressible phase (gas). Due to the incompressibility of one phase, establishing a priori estimates and passage to the limit is more involved in this case. In comparison to earlier existence results for this type of flow, the required hypotheses on data are significantly weakened, so that only physically relevant assumptions are made. In particular, our result includes the cases of unbounded capillary pressure function, and the discontinuous porosity and absolute permeability tensors. Also, the non homogenous Dirichlet and Neumann conditions on the boundary are allowed.

Chapter 3

A review of homogenization of two-phase flow in porous media

This Chapter is intended to provide a preparatory material for the homogenization results for two-phase immiscible compressible flow in the case of ordinary porous media and fractured porous media, which will be presented in Chapters 5 and 6, respectively. The current chapter is organized as follows. In Section 3.1 we indicate the basic ideas of mathematical homogenization theory and display the definitions and results concerning the notion of two-scale convergence which is going to be employed in the homogenization process. Section 3.3 provides a detailed exposition of the concept of double porosity for the flow in fractured porous media and in Subsection 3.3.1 we introduce the dilation operator, which is going to be used in Chapter 6. Finally, Section 3.4 contains a review of contributions in mathematical homogenization of flow in porous media, with the results on two-phase immiscible flow pointed out.

3.1 Introduction

Many relevant scientific and engineering problems in physics, chemistry or geology describe phenomena that occur at various length and time scales, for example heat, sound, current and stress distribution in composite materials, macroscopic properties of crystalline or polymer structures, atmospheric turbulence, and in particular our primary interest, flow and transport in porous media. Composite materials are characterized by the fact that they contain two or more finely mixed constituents. Therefore, a common feature of composite materials is presence of two length scales which are well separated: the macroscopic scale, describing the overall behavior of the composite, and the microscopic one, depicting the heterogeneities which are small with respect to the global dimension of the material.

In general, one is interested in the overall properties of the composite materials. These are usually "better" than the average behavior of their individual constituents and due to that trait, the composites are widely used in industry nowadays. Likewise, since the characteristic dimension of a porous medium domain is much larger than the characteristic dimension of a pore, one observes two or more various spacial scales characterizing a porous medium.

Physical properties of a heterogenous porous medium, such as porosity or absolute permeability, may notably vary from one point of the domain to another. This counts to strong variations of the characteristic functions of the medium which figure in the governing equations. Numerical simulation models of flow and transport in porous medium should be capable to determine the value of these functions at every point of the domain. If a highly heterogenous medium is placed in a standard framework, one faces difficulties in computations performed on the basis of the original equations because heterogeneity occurs at many different length scales. More precisely, the mesh fineness should fit a size of a heterogeneity block, which is very small and therefore, in view of the size of the model domain, a full numerical simulation of the flow and transport in porous medium directly on a microscopic or pore scale over many time steps becomes infeasible, even with the modern computers and parallel computing technology at disposal. Hence, it is desired to develop methods for representing the effects of finer scale variations on larger scale flow results. The standard approach for numerical simulations of flow in heterogeneous reservoirs is to average or upscale physical parameters such as porosity and absolute permeability, which allows the use of a coarse computational grid.

It is often reasonable to assume that the microstructure of the porous medium is periodic. Taking this into account, the problems of flow and transport in heterogenous porous medium, for instance in oil reservoir simulation or hydrogeology, are modeled by using partial differential equations with periodically and rapidly oscillating coefficients. The periodic homogenization is a fundamental tool for treating this type of problems.

In general, the aim of homogenization theory is to establish the macroscopic behavior of a system which is microscopically heterogeneous by taking into account its microscopic properties, in order to describe some characteristics of the considered heterogeneous medium (for instance, thermal or electrical conductivity of a composite material, or the porous medium permeability and porosity). In other words, a strongly heterogeneous medium is replaced by a fictitious homogeneous one (the 'homogenized' or effective medium), whose global characteristics are a good approximation of the initial ones. From a mathematical point of view, the idea of the homogenization is to model the problem using techniques of asymptotic analysis to account for the fine scale variations. In this sense, the microscopic heterogeneities are represented by a small positive length parameter ε, the typical size of a pore. Rather than considering a single heterogeneous medium with a fixed length scale ε_0, the microscopic model is embedded in a series of similar problems with periodic coefficients parametrized by a length scale ε, from which, as ε tends to zero, the *homogenized macroscopic or effective* problem is obtained. Mathematically, the work that needs to be done in the homogenization procedure is to establish rigorously the convergence in some sense of the corresponding solutions of a sequence of boundary value problems, depending on a small parameter, to some limit, and to explicitly describe a limit boundary value problem that the limit function solves. In other words, the goal is to replace the governing equations with highly oscillatory coefficients by a simpler set of equations with uniform macroscopic or effective coefficients whose solution can be numerically resolved on a reasonable coarse-scale mesh and this solution approximates the average behavior of the solution of the governing equations.

Let us remark that in this context the Darcy-Muskat law can be understood as a "first level of homogenization" for a passage from the pore scale to the macroscale, where a porous medium is homogenized or averaged in a sense that the pore and the matrix are no longer distinguishable.

Since the initial contributions in the mathematical theory of homogenization (Spagnolo) in the late 1960s, the literature in this area has been vast. For an advanced general presentation of mathematical homogenization one can consult the classical books [31], [59] and [93]. An extensive collection of applications to porous media can be found in [74].

In subsequent Section we overview the standard methods of mathematical homogenization that are going to be employed in the following.

3.2 Methods of homogenization

Historically premier and anyhow basic technique of periodic homogenization is the *two-scale asymptotic expansion* or *method of multiple scales* (see [31, 59, 74, 93]).

Let the periodic structure of the porous medium Ω be described by a small parameter $\varepsilon > 0$ representing the ratio of a cell size to the size of the domain. Denote the reference cell by $Y =]0,1[^d$. The solutions of the microscopic problem with rapidly oscillating coefficients in Ω depend on ε and on the position $x \in \Omega$. Let u^ε be such a solution for $\varepsilon > 0$. Two scales describe the problem: the variable x is the macroscopic or the "slow" variable, while $y = \frac{x}{\varepsilon}$ depicts the microscopic scale ("fast" variable). Indeed, if $x \in \Omega$, by the definition of the reference unit cell Y, there exists a unique $k \in \mathbb{Z}^d$ such that $\frac{x}{\varepsilon} = y + k$ for some $y \in Y$. Accordingly, x gives the position of a point in the physical domain Ω of the microscopic model while y denotes its position in the standard cell Y. This suggests that one may assume u^ε having the (formal) asymptotic expansion, with respect to ε, of the form

$$u^\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \cdots, \tag{3.1}$$

where the coefficients $u_i(x,y)$ are Y-periodic in second variable and $y = \frac{x}{\varepsilon}$ (the dependence of the solution on time is left out since it is not subject to scaling by ε). The derivatives follow the rule

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y, \ \mathrm{div} = \mathrm{div}_x + \frac{1}{\varepsilon}\mathrm{div}_y, \ \triangle = \triangle_x + 2\frac{1}{\varepsilon}\triangle_{xy} + \frac{1}{\varepsilon^2}\triangle_y.$$

After inserting (3.1) into the equations and comparing the coefficients of the different powers of ε, a series of equations for the unknowns u_i is obtained. Finally, the homogenized limit of the starting equation is the equation for u_0 which is obtained by averaging with respect to y of the equation for ε^0.

Mathematically, the method of two-scale asymptotic expansions is only formal because, a priori, there are no arguments to ensure that (3.1) holds true. Also, the process is carried out arguing as if the coefficients were smooth, while in applications they are not even continuous. Still, the model obtained by this type of inference can be utilized to anticipate the limit macroscopic model which afterwards needs to be rigorously justified. Moreover, the process provides a better apprehension of the structure of the effective model.

Generally speaking, one needs to prove the convergence of the sequence (u^ε) as $\varepsilon \to 0$ in some sense to a solution of the effective problem that was formally obtained by the asymptotic expansion. In order to prove the needed convergence results, several different approaches and methods can be employed. We mention here two of the most commonly used: the energy method and the two-scale convergence method.

The *energy method* or *variational method of oscillating test functions* is a homogenization method due to Tartar [98]. In its general form, this method does not require any geometrical assumptions on the behavior of the coefficients of the partial differential differential equations describing the microstructure, like periodicity or statistical properties. The key idea is to use an appropriate set of oscillating test functions, instead of a fixed test function, in the weak formulation of the problem. The special form of the chosen test functions permits to replace products of weakly convergent sequences with products of a weakly and a strongly convergent sequence. Thus, one can pass to the limit in the variational formulation using a compensated compactness argument ([98]; see also [74]). In particular, for the case of periodic coefficients the test functions are periodic. The energy method is more general and successful in homogenizing many different types of equations, but it is also not entirely satisfactory because it consists of two steps, the formal derivation of the homogenized problem and then the rigorous justification of it, which have little in common and are partly redundant.

In this work we will use the mathematical homogenization method as described in [74] and [90] for flow in porous media. Namely, we employ the asymptotic expansion and the *two-scale convergence method*. Contrary to the energy method, the two-scale convergence method is restricted to the periodic homogenization problems but at the same time, it is simpler and more efficient in the periodic setting. The method of the two-scale convergence is based on the new type of convergence which is presented in the next Subsection.

3.2.1 Two-scale convergence

The concept of two-scale convergence has been introduced by G. Nguetseng in [86,87] and the theory was further developed by Allaire in [2,3].

The new notion of convergence is motivated by the following. In problems of periodic homogenization, one works with the sequences of functions $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$, where g is some periodic function. The weak limit as $\varepsilon \to 0$ of g_ε depends only on the average of g over the basic period. In particular, it does not keep any information on the shape of the oscillations present in the sequence g_ε. Furthermore, during the process of rigorous justification of the effective model, there is typically a need to pass to the limit in products of only weakly convergent sequences which is impossible since in this case the limit of the product is not the product of the limits.

This feature of weak limits can be interpreted as an inability of the class of test functions used in the definition of weak convergence. Therefore, the class of test functions should be modified in order to pick up the oscillations via some other type of weak limits. As mentioned earlier, a resolution was proposed by Tartar's "compensated compactness" arguments. Nevertheless, a simpler solution appeared in the concept of the two-scale convergence. Loosely speaking, the test functions that are used in the definition of the two scale convergence are able to describe the oscillations through the two-scale limit. Namely, a two-scale limit is function of variable x and an additional variable y with the local behavior of the sequence being conserved in y.

The current Subsection will provide the definitions and the standard results on the two-scale convergence from [2] (see also [59, 74]), slightly modified for the case of homogenization with a parameter t (like for example in [60]). However, we point out that these modifications do not affect the proofs from [2] in any essential way.

We start by introducing the notation. Ω will denote an open subset of $\mathbb{R}^d$, the time interval of interest is $]0,T[$ and $\Omega_T = \Omega\times]0,T[$. The reference cell is $Y =]0,1[^d$. In the sequel the following spaces of Y-periodic functions will be used: $C_p^\infty(Y)$ is the space of infinitely differentiable functions in $\mathbb{R}^d$ which are periodic of period Y; $L^2_p(Y)$ consists of functions in $L^2_{loc}(\mathbb{R}^d)$ which are periodic of period Y; $H^1_p(Y)$ is the space of functions in $H^1_{loc}(\mathbb{R}^d)$, Y-periodic, and $\mathcal{D}(\Omega_T; C_p^\infty(Y))$ denotes the space of infinitely smooth and compactly supported functions in Ω_T with values in the space $C_p^\infty(Y)$.

We recall now the definition of the two-scale convergence and the key results concerning this notion.

Definition 1 *A sequence of functions (v^ε) in $L^2(\Omega_T)$* ***two-scale converges*** *to a limit $v_0 \in L^2(\Omega_T \times Y)$, denoted by $v^\varepsilon(x,t) \overset{2s}{\rightharpoonup} v_0(x,y,t)$, if for any function $\varphi(x,y,t) \in \mathcal{D}(\Omega_T; C_p^\infty(Y))$, one has*

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} v^\varepsilon(x,t)\varphi(x,\frac{x}{\varepsilon},t)dxdt = \int_{\Omega_T\times Y} v_0(x,y,t)\varphi(x,y,t)dydxdt. \tag{3.2}$$

If, in addition,

$$\lim_{\varepsilon\to 0} \|v^\varepsilon\|_{L^2(\Omega_T)} = \|v\|_{L^2(\Omega_T\times Y)}, \tag{3.3}$$

the sequence v^ε is said to ***strongly two-scale converge*** *to v.*

Loosely speaking, the two-scale convergence of a sequence $v^\varepsilon(x,t)$ to a function $v(x,y,t)$ can be interpreted as $v^\varepsilon(x,t)$ being close to $v(x,\frac{x}{\varepsilon},t)$ for small values of $\varepsilon > 0$.

Remark 2 *Any sequence (v^ε) which converges strongly in $L^2(\Omega_T)$ to a limit $v(x,t)$, two-scale converges to the same limit.*

The two-scale convergence implies the weak convergence: if a test function $\varphi(x,t)$ independent of y is taken in (3.2), it follows directly that

$$v^\varepsilon \rightharpoonup v(x,t) = \int_Y v_0(x,y,t)dy.$$

Moreover,

$$\lim_{\varepsilon\to 0} \|v^\varepsilon\|_{L^2(\Omega_T)} \geq \|v_0\|_{L^2(\Omega_T\times Y)} \geq \|v\|_{L^2(\Omega_T}.$$

For any smooth function $a(x,y,t)$ which is Y-periodic in y, the associated sequence $a^\varepsilon(x,t) = a(x,\frac{x}{\varepsilon},t)$ two-scale converges to $a(x,y,t)$.

We see that the two-scale limit contains more information on the behavior of a sequence than its weak limit as it takes into account its oscillations.

The fundamental result concerning the new notion of two-scale convergence is the following compactness theorem ([2], also [60, 81]).

Theorem 1 *From each bounded sequence (v^ε) in $L^2(\Omega_T)$ we can extract a subsequence, and there exists a limit $v_0(x,y,t) \in L^2(\Omega_T \times Y)$ such that this subsequence two-scale converges to v_0.*

If uniform bounds on derivatives of the functions (v^ε) hold true in addition, more information on the two-scale limit of the sequence (v^ε) is given by the next theorem ([2, 60, 81]). Set $H = \{u \in H^1_p(Y) : \int_Y u dy = 0\} = H^1_p(Y)/\mathbb{R}$.

Theorem 2 *i) Let (v^ε) be a bounded sequence in $L^2(0,T;H^1(\Omega))$ with a subsequence that converges weakly to a limit v in $L^2(0,T;H^1(\Omega))$. Then, along this subsequence, v^ε two-scale converges to $v(x,t)$. Also, there exists a function v_1 in $L^2(\Omega_T;H)$ such that, up to a subsequence, ∇v^ε two-scale converges to $\nabla_x v + \nabla_y v_1$.*

ii) Let (v^ε) and $(\varepsilon \nabla_x v^\varepsilon)$ be two bounded sequences in $L^2(\Omega_T)$. Then there exists a function $V(x,t,y)$ in $L^2(\Omega_T;H)$ such that, up to a subsequence, v^ε and $\varepsilon\nabla_x v^\varepsilon$ two-scale converge to $V(x,t,y)$ and to $\nabla_y V(x,t,y)$, respectively.

Let us point out that the two-scale convergence results can justify a posteriori the homogenization results obtained formally by the multiple-scale method. More precisely, the two-scale limit of the sequence (v^ε) is essentially the first term in the multiple scales expansion of v^ε. In general it will depend on the oscillations through the auxiliary variable y. Furthermore, for the functions v^ε, v and v_1 from Theorem 2 it holds

$$v^\varepsilon(x,t) = v(x, \frac{x}{\varepsilon}, t) + \varepsilon v_1(x, \frac{x}{\varepsilon}, t) + \cdots$$

In other words, a uniform bound on the gradient of v^ε is enough to justify the second term in the multiple scales expansion of v^ε.

A special shape of text functions will be used when passing to the two-scale limit in Chapter 5, which is allowed by the next result.

Theorem 3 *([81, Theorem 9]) Let $1 < p, q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let (u^ε) be a sequence in $L^p(\Omega_T)$ which two-scale converges to u. Then*

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} u^\varepsilon(x,t)\psi(x, \frac{x}{\varepsilon}, t) dx dt = \int_{\Omega_T} \int_Y u(x,y,t)\psi(x,y,t) dy dx dt,$$

for every ψ of the form $\psi(x,y,t) = \psi_1(x,t)\psi_2(y)$, $\psi_1 \in L^{rq}(\Omega_T)$, $\psi_2 \in L^{sq}_p(Y)$ with $1 \leq r, s \leq +\infty$ and such that $\frac{1}{r} + \frac{1}{s} = 1$.

Finally, an additional condition (3.3) posed for the strong two-scale convergence yields in return a kind of strong convergence and allows to pass to the limit in some products of two weakly convergent sequences in $L^2(\Omega_T)$. The precise assertions are contained in the following Theorem [2].

Theorem 4 *Let $v^\varepsilon \in L^2(\Omega_T)$ be a sequence that strongly two-scale converges to $v_0 \in L^2(\Omega_T \times Y)$. Then, for any sequence $w^\varepsilon \in L^2(\Omega_T)$ which two-scale converges to a limit $w_0 \in L^2(\Omega_T \times Y)$, it holds*

$$v^\varepsilon(x,t) w^\varepsilon(x,t) \rightharpoonup \int_Y v_0(x,y,t) w_0(x,y,t) dy \text{ in } \mathcal{D}'(\Omega).$$

Moreover, if the Y-periodic extension of $v_0(x,y,t)$ belongs to $L^2(\Omega_T; C_p(Y))$, we have

$$\lim_{\varepsilon \to 0} \|v^\varepsilon(x,t) - v_0(x, \frac{x}{\varepsilon}, t)\|_{L^2(\Omega_T)} = 0.$$

The second result of Theorem 4 is referred to as a *corrector type result* which corresponds to the prior remark on the relation of the two-scale limit and two-scale asymptotic expansion.

3.3 The concept of double porosity

In this Section we present the double porosity model which has been introduced on physical grounds by engineers and thereafter studied from mathematical point of view to describe the flow of one or more fluids in a naturally fractured reservoir. The new result concerning the double porosity model for immiscible compressible two-phase flow in a fractured reservoir is obtained rigorously by homogenization in Chapter 6.

A naturally fractured reservoir is the one containing many interconnected fracture planes throughout its extent. The fractures are formed in response to stress which may originate from high fluid pressure, thermal loading, the movements of the Earth's crust or formation of land folds, over millions of years. Accordingly, a fractured reservoir consists of layers of materials of very different petrographic characteristics. This type of porous medium is frequently encountered in hydrology and petroleum applications, for instance the sedimentary rock that composes a hydrocarbon reservoir. Actually, fractured reservoirs make up a large and increasing percentage of the world's hydrocarbon reserves [85]. The drawback of such reservoirs is their extreme complexity due to the vast number of variables and their interactions which makes them much more difficult to deal with than with unfractured ones.

It has long been known that the fluid flow mechanism in such reservoirs is significantly different from that of an ordinary, unfractured reservoirs. Specifically, the flow occurs as if the reservoir possessed two porous structures, one associated to the porous rock, and the other one to the system of fractures. The former being considered as a porous medium has been justified in the petroleum engineering literature by assuming that the cracks are partially filled with rock debris.

In the scope of this work we study the *totally fractured reservoir* that arises for instance in the modelling of granular materials. Namely, this is a fractured reservoir in which the system of fractures is so well developed that the matrix is broken into isolated individual blocks or cells. Consequently, in such a setting no flow takes place directly from block to block, but only an exchange of fluid between the cell and the surrounding fracture system is possible. There are also more general situations of *partially fissured reservoirs* where not only the fracture system but also the matrix of cells may be connected, so there is some flow directly within the cell matrix (see, e.g., [56, 60, 74]).

Accordingly, a naturally fractured reservoir is considered as a porous medium consisting of two superimposed continua, a discontinuous system of periodically repeating medium-sized matrix blocks interlaced on a fine scale by a connected system of thin fissures. It is assumed that the width of the fissures is considerably greater than the characteristic size of the pores, so that the fractures are notably more permeable than the system of pores in the porous matrix blocks. Hence, the reservoir's effective permeability is increased with respect to the permeability of the merely rock matrix. The transport of fluids through the reservoir primarily takes place within the fracture system where the flow is much readier than in the porous rock. On the other hand, the matrix stores most of the fluid. These contrasts cause great difficulties in modelling such mediums. Neither fractures nor matrix, or their interactions must not be neglected in a model of the flow. A discrete approach in modelling a fractured reservoir is not feasible since the fractures are typically too small compared to the size of the reservoir, and too numerous. The idea is therefore to homogenize the reservoir. However, a straightforward homogenization of the entire reservoir yields a single porosity model with some averaged overall permeability and porosity of the reservoir [32, 82]. This approach is not adequate since it ignores the contrasts in the properties of the two very distinct porous structures present in the reservoir, as well as their interaction which has a strong influence on the flow. This exchange between matrix blocks and fractures is a microscale process whose effects only must be incorporated in a large scale description, while the process itself must be withheld on a microscopic level.

A more appropriate way of modelling a naturally fractured reservoir turned out to be the so-called ***double porosity model*** (also referred to as the double porosity/permeability model or the dual-porosity model). In this model the whole medium is replaced by an equivalent imaginary coarse grained porous medium for which the fractures play the role of the pores, while the matrix blocks could be seen as fictive grains. At the macroscopic level, the flow occurs in the fractures - the more permeable part of the reservoir, and the matrix block system plays the role of a global source distributed over the entire medium. The matrix-fracture coupling on a fine scale is expressed by an additional memory or source term in the macroscopic fracture equations. At the same time, as ε tends to 0, infinitely many matrix blocks are obtained. In that sense, in the limit model for each point of the domain there is an associated matrix block, congruent to a standard matrix block, in which the equations are given that capture the flow at the medium-size scale. In summary, the double porosity model justifies its name since it preserves two scales: a macroscopic scale, corresponding to the high porosity of fractures, and a microscopic one, corresponding to the low porosity of the matrix blocks.

The double porosity/permeability concept was first derived experimentally as a physical notion and described by several authors in the engineering literature ([27,85,91,100]). Since then it has been used in a wide range of engineering applications related to geohydrology, petroleum reservoir engineering, civil engineering or soil science. The model of [27] assumes that the typical dimension of the fracture is far larger than that of the pores, that the permeability in the fractures is much greater than that of the matrix blocks and that the ratio of the volume of the fractures to the total volume is less than the porosity of each block, since the fractures occupy a smaller volume than the pores. It treats the interaction between fractures and matrix as a transfer function proportional to the difference between the pressures of the matrix blocks and fissures. Moreover, quasistationary exchanges between matrix and fractures are assumed. The supplementary memory terms are caused by not including such an assumption. Rigorous mathematical proofs for the dual-porosity model have not been available until [25] where the linearized single phase flow was considered.

Let us finish the introductory part by few remarks concerning the principles and tools of homogenization that we will use subsequently.

In a naturally fractured porous medium there are three distinct length scales: the thickness of the fractures is about 10^{-4}m, the average distance between fractures, i.e. the size of the matrix blocks is about 0.1 to 1m, and the size of the reservoir may be about $10^3 - 10^4$m [50]. The basic assumption for the model that we obtain is the existence of a representative elementary volume that is very small with respect to the size of the domain, but large enough to capture the interchanges between the matrix and the fractures.

The fracture planes in a fractured reservoir often form a fairly regular geometric pattern and we will work with the matrix blocks represented by identical squares or cubes. However, cracks generally originate from geological phenomena (shear, folding) which usually have predominant fracturing directions so for example parallelepiped could also have been chosen for the reference cell. An underlying assumption is that the flow is uniform at the surface of each matrix block. Besides, we are supposing a periodic structure of the medium which is standard hypothesis (there are studies in a probabilistic framework [38]). In this work we study the double porosity model assuming that the width of the fractures is of the same order as the blocks sizes, in other words, the volume fraction of the fissured part and the non-fissured part are kept positive constants of the same order. The various cases of the ratio between the fractures and matrix block sizes have been considered for instance in [16].

The microscopic model consists of the usual equations describing Darcy flow in a reservoir with the specific feature of a highly discontinuous porosity and permeability coefficients. Over the matrix domain the permeability is scaled by a parameter ε^2, where ε represents the size of a matrix block in regard to the size of the whole domain. This scaling conserves the flow between the matrix and the fractures from degenerating or blowing up as the block size tends to zero [26]. From other point of view, the characteristic time scale for any parabolic evolution in a single matrix block is of order ε^{-2} [39]. Anyhow, by scaling by ε^2 the form of the matrix equations on the standard cell is maintained independently of the value of ε and in the limit a double porosity model is obtained. The matrix gravitational term is compensated additionally by ε^{-1} for the same reason. In [103] the author studies other scaling factors for the matrix permeability. Indeed, if the ratio of the permeability for the matrix and the fractures is smaller than of order ε^2, the flow in the matrix blocks contributes very little to the fracture system and the microscopic models converge to the equations for the fracture flow in the entire domain as ε tends to 0. On the other hand, if the ratio is greater than ε^2, in the matrix blocks the flow is very fast and saturations are almost constant. In this case, the macroscopic model is of a special single-porosity type.

The homogenization procedure consists of letting a characteristic size of each block, ε, to zero.

Now we set up notation and terminology for the description of the fractured porous medium.

A fractured reservoir $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is considered to be a bounded, two-connected domain with periodic structure. Let $Y =]0,1[^d = Y_m \cup \Gamma \cup Y_f$ be the unit cell, where Y_m is a matrix block domain surrounded by a fracture domain Y_f and Γ is a smooth internal boundary between two parts. A small parameter $\varepsilon > 0$ is used to describe the ratio of a matrix block size to the size of Ω, and the fracture thickness is considered to be of order ε. We assume that the reservoir Ω is covered by the disjoint copies of εY shifted for the translations from $\varepsilon\mathcal{A}$, where $\mathcal{A}$ is an infinite lattice. For each $\varepsilon > 0$ and each $x \in \Omega$, $c^\varepsilon(x)$ stands for the lattice translation point of the ε-cell domain containing

x, that is, $c^\varepsilon : \Omega \to \varepsilon\mathcal{A}$ is defined by considering $x \in \varepsilon Y + c^\varepsilon(x)$. More precisely, if $x \in \varepsilon(Y + k)$ for some $k \in \mathbb{Z}^d$, then $c^\varepsilon(x) = \varepsilon k$. Further, χ_r are the characteristic functions of Y_r, $r \in \{f, m\}$, extended by Y-periodicity to $\mathbb{R}^d$. The system of matrix blocks, the system of the fractures and the matrix-fracture interface will be denoted by $\Omega_m^\varepsilon = \{x \in \Omega;\ \chi_m(\frac{x}{\varepsilon}) = 1\}$, $\Omega_f^\varepsilon = \{x \in \Omega;\ \chi_f(\frac{x}{\varepsilon}) = 1\}$ and Γ^ε, respectively. Here $\chi_r^\varepsilon(x) = \chi_r(\frac{x}{\varepsilon})$, $r \in \{f, m\}$. It is $\overline{\Omega} = \Omega_m^\varepsilon \cup \Gamma^\varepsilon \cup \Omega_f^\varepsilon \cup \partial\Omega$.

3.3.1 Dilation operator

In Chapter 6 we will present a homogenization result for the model of the immiscible compressible two-phase flow in a fractured porous medium. Using the two-scale convergence, one obtains the effective equations which contain a source-like term modelling the matrix-fracture interaction. This term is in a non-explicit form due to the non-identified two-scale limit that it involves. This difficulty is a consequence of the nonlinearity and the strong coupling in the problem. In order to provide the explicit form for this term, we will employ the suitable *dilation operator* which was introduced in [25] and afterwards used in [14, 39, 54, 103]. The term *periodic modulation* is used for the same concept [39]. In this Subsection we provide the definitions and the basic properties of the dilation operator which we are going to need in Chapter 6.

Definition 2 *For given $\varepsilon > 0$ we define a* ***dilation operator*** *D^ε mapping a measurable function $\psi \in L^2(\Omega_m^\varepsilon \times]0,T[)$ to a measurable function $D^\varepsilon\psi \in L^2(\Omega \times Y_m \times]0,T[)$ by*

$$(D^\varepsilon\psi)(x, y, t) = \psi(\varepsilon y + c^\varepsilon(x), t) \quad \text{for } x \in \Omega,\ y \in Y_m,\ t \in]0,T[. \tag{3.4}$$

Remark 3 *For given $\varepsilon > 0$ and for given function ψ which is defined on the matrix part Ω_m^ε of the reservoir, the dilated function $D^\varepsilon\psi$ is defined on the fixed domain Ω.*

$D^\varepsilon\psi$ is constant in x on a fixed ε-block in Ω, that is, on any $\varepsilon(Y + k)$, $k \in \mathbb{Z}^d$.

For a fixed $x \in \Omega$, $D^\varepsilon\psi(x, Y_m) = \varepsilon Y_m + c^\varepsilon(x)$.

One can extend D^ε by periodicity to allow $y \in \cup_{k\in\mathbb{Z}^d}(Y_m + k)$ and consequently $D^\varepsilon\psi$ is regarded as Y-periodic function in its second argument.

Essentially, the dilation operator provides the link between the macroscopic domain scaled by ε and the microscopic level of the standard cell. Namely, it transforms a macroscopic (slow) variable $x = \varepsilon y + c^\varepsilon(x) \in \varepsilon(Y + k)$ into the microscopic variable $y \in Y$.

The main properties of the dilation operator are given by the following Lemma ([25], Lemma 2):

Lemma 1 *For $v, w \in L^2(0,T;H^1(\Omega_m^\varepsilon))$,*

$$\|D^\varepsilon v\|_{L^2(\Omega\times Y_m\times]0,T[)} = \|v\|_{L^2(\Omega_m^\varepsilon\times]0,T[)},$$
$$\nabla_y(D^\varepsilon v) = \varepsilon D^\varepsilon(\nabla_x v) \text{ a.e. in } \Omega\times Y_m\times]0,T[,$$
$$\|\nabla_y(D^\varepsilon v)\|_{(L^2(\Omega\times Y_m\times]0,T[))^d} = \varepsilon\|D^\varepsilon(\nabla_x v)\|_{(L^2(\Omega\times Y_m\times]0,T[))^d} = \varepsilon\|\nabla_x v\|_{(L^2(\Omega_m^\varepsilon\times]0,T[))^d},$$
$$(D^\varepsilon v, D^\varepsilon w)_{L^2(\Omega\times Y_m\times]0,T[)} = (v,w)_{L^2(\Omega_m^\varepsilon\times]0,T[)},$$
$$(D^\varepsilon v, w)_{L^2(\Omega\times Y_m\times]0,T[)} = (v, D^\varepsilon w)_{L^2(\Omega\times Y_m\times]0,T[)}.$$

Furthermore, if we consider $g \in L^2(\Omega_T)$ as an element of $L^2(\Omega\times Y_m\times]0,T[)$ which does not depend on y, then

$$D^\varepsilon g \to g \text{ strongly in } L^2(\Omega\times Y_m\times]0,T[) \text{ as } \varepsilon\to 0. \tag{3.5}$$

The two-scale convergence is related to the weak convergence of dilated sequences in a manner described by the undermentioned result (the proof can be found in [39]).

Lemma 2 *Let (u^ε) be a uniformly bounded sequence in $L^2(\Omega_m^\varepsilon\times]0,T[)$ which satisfies the conditions*

$$D^\varepsilon u^\varepsilon \rightharpoonup u^0 \text{ weakly in } L^2(\Omega_T; L^2_p(Y_m))$$

and

$$\chi_m^\varepsilon u^\varepsilon \overset{2s}{\rightharpoonup} u^* \in L^2(\Omega_T; L^2_p(Y)),$$

then

$$u^0 = u^* \text{ a.e. in } \Omega\times Y_m\times]0,T[.$$

In addition, we present another result concerning the dilations (see [103]).

Lemma 3 *If $u^\varepsilon \in L^2(\Omega_m^\varepsilon\times]0,T[)$ and $\chi_m^\varepsilon u^\varepsilon \overset{2s}{\rightharpoonup} u \in L^2(\Omega_T; L^2_p(Y_m))$ strongly, then $D^\varepsilon u^\varepsilon \to u$ strongly in $L^2(\Omega\times Y_m\times]0,T[)$.*

Remark 4 *The concept of the dilation operator is closely related to the notion of the* unfolding operator *which was introduced in [57]. The definition and the properties of the unfolding operator can be found in [58] which contains all the proofs for this approach with some extensions and applications.*

3.4 Review of homogenization results for immiscible two-phase flow

This Section contains a review of the references studying the problems of homogenization for the two-phase immiscible flow in porous media, both for ordinary porous media and for the fractured porous media. We also mention the contributions concerning the homogenization of the single phase flow, and the miscible displacement of one compressible fluid by another.

3.4.1 Single porosity

An extensive reviews on upscaling methods for flow simulation models are given in [62, 66]. More information on the homogenization of one phase flow in the framework of the geological disposal of radioactive waste can be found in [33,34,40,41,44]. Many authors have studied the homogenization and upscaling of incompressible immiscible two-phase flow in porous media, see for instance [30,35,37,38,42,43,73,74,83,89,90,94,99]. Homogenization results for compressible miscible two-phase flow in porous media were rigorously obtained in [17, 56], and a solute transport in a highly heterogeneous aquifer was upscaled by means of the asymptotic expansion in [7]. On the other hand, for immiscible compressible two-phase flow in porous media, the first result was only recently established in [6]. In that work homogenization results were obtained for water-gas flow in porous media using the phase formulation, under the assumption that the capillary pressure function is bounded which is too restrictive for some applications.

A new result which extends such results to immiscible flow of two compressible phases in porous media in the global pressure formulation including the case of unbounded capillary pressure function is presented in Chapter 5.

3.4.2 Double porosity

A general form of the double porosity model for a single phase flow in a naturally fractured reservoir has been first described in [22]. The model was derived by explicitly considering fluid flow in individual matrix blocks. In presence of gravity, a linearized approximation of the density function is considered. In [25], this general model is rigorously justified from the point of view of homogenization theory, using the dilation operator that the authors introduce in this paper. One-phase flow model in a fractured porous medium is studied also in [15], where the authors rigorously obtain the effective models for two types of fractured porous medium: the medium characterized by the asymptotically vanishing volume fraction of fractures, and the case when the width of the fractures and the porous block size are of the same order. In [16] the global behavior of single phase incompressible flow in fractured media is discussed with respect to different parameters such as the fracture thickness, the size of blocks and the ratio of the block permeability and the permeability of fissures, and oscillating source terms.

The first contribution on the derivation of the double porosity model for two-phase flow in a fractured medium is [26], where the effective equations of the double porosity model are established by formal technique of asymptotic expansion for the cases of completely miscible incompressible flow, and immiscible incompressible two-phase flow. For the case of immiscible incompressible two-phase flow in a reduced pressure formulation, the double porosity model is rigorously justified by periodic homogenization in [39]. Another result on the two-phase incompressible immiscible flow in fractured media is established in [103]. In this work three different situations are considered: when the ratio of the permeabilities in the matrix blocks and in the fractures is of order ε^2, smaller than ε^2 and greater than ε^2, respectively. For the first case, the limit model is of a dual porosity type. The second case leads to a single-porosity model for the fracture flow, while the last one yields another type of single-porosity model for the fractures, with the addition of a source term from matrix blocks. For the displacement of one compressible miscible fluid by another in a naturally fractured reservoir, the double porosity model was rigorously derived in [54]. A dual porosity model for compositional three-phase flow was established by the formal asymptotic expansion in [24, 50]. Furthermore, [101] studies the existence of weak solutions for the model of the immiscible two-phase flow in fractured porous media. More precisely, four relations for the phase mobilities and capillary pressures are presented and the corresponding problems are shown to have an appropriately formulated weak solution when any of these relationships are satisfied. Finally, the double porosity model for the compressible flow of two fluids in an approximate form with the global pressure has only lately been established in [14].

Chapter 6 in this thesis presents a new result on the rigorous justification of the homogenization process for a double porosity model of immiscible compressible two-phase flow through a fractured porous medium in a fully equivalent global pressure formulation.

Chapter 4

An existence result for water-gas immiscible flow in global pressure formulation

4.1 Introduction

The objective of this Chapter is to present proof of the existence of the weak solutions for the fully equivalent global pressure formulation of immiscible, compressible two-phase flow in porous media, under physically relevant assumptions and allowing the non homogenous Dirichlet and Neumann boundary conditions. The system under study consists of incompressible wetting phase and compressible non-wetting phase, such as water and hydrogen in the context of gas migration through engineered and geological barriers for a deep repository of nuclear waste. The difficulties in dealing with this type of equations are generated by the nonlinearities and the coupling of the equations as well as by the degeneracy of the diffusion term in the saturation equation and the degeneracy of the time derivative term in the global pressure equation, where both weaken the energy estimates and make a proof of compactness results more involved. The results of this Section are contained in [9].

Section 4.2 begins with the formulation of the mathematical and physical model under consideration, then the assumptions on the data are stated and the main result on the existence of weak solutions of the problem is presented. This result is proved in three steps. In Section 4.3 we define the adequate regularized system by introducing a small regularization parameter $\eta > 0$ and state the existence result for weak solutions of the regularized problem. Section 4.4 provides a construction of the approximate solutions to the regularized system by replacing the time derivatives with finite differences with a small time step $h > 0$ and the existence result for the corresponding system, as well as a maximum principle for the saturation. In Section 4.5 we establish uniform estimates with respect to h and η using suitable test functions. This allows us to pass to the limit when $h \to 0$ which gives the existence of a weak solution for the regularized problem; this is performed in Section 4.6. Finally, in Section 4.7 we pass to the limit as $\eta \to 0$ using an adapted compactness result, as in [12, 76, 95], and prove the existence of weak solutions of the problem defined in Section 4.2.

4.2 Mathematical model and the main result

Here we study the model of water-gas immiscible flow in the fully equivalent global pressure formulation which is presented in Subsection 2.4.2 of Chapter 2. Throughout the current Chapter, the wetting phase (water) and the non-wetting phase (gas) will be indicated by subscripts w and g, respectively. The system is formulated with the non-wetting phase saturation $S := S_g$ and the global pressure P as primary variables, and the phase pressures P_w, P_g are expressed through S and P in (2.23)-(2.24). Accordingly, recall the problem in question:

$$\begin{aligned}\Phi \frac{\partial}{\partial t}(\rho_w(S,P)(1-S)) - \operatorname{div}(\Lambda_w(S,P)\mathbb{K}\nabla P) + \operatorname{div}(a(S,P)\mathbb{K}\nabla S)\\ + \operatorname{div}(\lambda_w(S)\rho_g(S,P)^2\mathbb{K}\mathbf{g}) = F_w,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Phi \frac{\partial}{\partial t}(\rho_g(S,P)S) - \operatorname{div}(\Lambda_g(S,P)\mathbb{K}\nabla P) - \operatorname{div}(a(S,P)\mathbb{K}\nabla S)\\ + \operatorname{div}(\lambda_g(S)\rho_g(S,P)^2\mathbb{K}\mathbf{g}) = F_g.\end{aligned} \tag{4.2}$$

Here $\Phi = \Phi(x)$ is the porosity, $\mathbb{K} = \mathbb{K}(x)$ is the absolute permeability tensor of the porous medium, F_w, F_g are known source terms and the gravity vector is denoted by $\mathbf{g}$. It is assumed that the wetting phase is incompressible ($\rho_w = \text{const.}$) and the non-wetting phase (gas) is compressible, $\rho_g = \rho_g(P_g)$.

As mentioned in Remark 1, one of the advantages of inducting the global pressure is that from the uniform estimates on the degenerate quadratic terms $\lambda_j(S_j)\mathbb{K}\nabla P_j \cdot \nabla P_j$, $j \in \{w, g\}$, one obtains the uniform bound on the global pressure and the degenerate capillary term. In order to remove the degeneracy of the capillary term in the a priori estimates, the non-wetting phase saturation S_g will be replaced by a new variable θ [12] (the saturation potential, cf. [4]) which is defined by

$$\theta = \beta(S) = \int_0^S \sqrt{\lambda_w(s)\lambda_g(s)}P_c'(s)ds. \tag{4.3}$$

Since β is strictly increasing, the transformation is well defined and we set $\mathcal{S} = (\beta)^{-1}$. Still, this change of variables does not eliminate the degeneracy from the diffusion term in the equations.

By introducing the coefficient $A(S,P)$ as

$$A(S,P) = \rho_w \rho_g(S,P) \frac{\sqrt{\lambda_w(S)\lambda_g(S)}}{\lambda(S,P)}, \tag{4.4}$$

and rewriting the system (4.1)-(4.2), we obtain the following equations describing the flow of water and gas in an equivalent formulation by the concept of the global pressure (as in [8]):

$$-\rho_w \Phi \frac{\partial S}{\partial t} - \operatorname{div}(\Lambda_w(S,P)\mathbb{K}\nabla P) + \operatorname{div}(A(S,P)\mathbb{K}\nabla\theta) + \rho_w^2 \operatorname{div}(\lambda_w(S)\mathbb{K}\mathbf{g}) = F_w, \tag{4.5}$$

$$\Phi \frac{\partial}{\partial t}(\rho_g(S,P)S) - \operatorname{div}(\Lambda_g(S,P)\mathbb{K}\nabla P) - \operatorname{div}(A(S,P)\mathbb{K}\nabla\theta) + \operatorname{div}(\lambda_g(S)\rho_g(S,P)^2\mathbb{K}\mathbf{g}) = F_g. \tag{4.6}$$

The system (4.5)-(4.6) is completed with the boundary and initial conditions as follows. Let a porous domain $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded, connected, Lipschitz domain. Throughout the whole thesis, we maintain the following notation: the domain boundary is considered to be decomposed as $\partial\Omega = \Gamma_D \cup \Gamma_N$, the time interval of interest is $]0,T[$ and we denote $Q = \Omega\times]0,T[$, $\Gamma_i^T = \Gamma_i\times]0,T[$, $i \in \{D,N\}$. The boundary conditions for the system in consideration are imposed in this way:

$$\theta = \theta_D, \quad P = P_D \quad \text{on } \Gamma_D^T, \tag{4.7}$$

$$\mathbf{Q}_w \cdot \mathbf{n} = G_w, \quad \mathbf{Q}_n \cdot \mathbf{n} = G_g \quad \text{on } \Gamma_N^T. \tag{4.8}$$

Here P_D, θ_D, G_w and G_g are given functions, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ and

$$\begin{aligned} \mathbf{Q}_w &= \rho_w \mathbf{q}_w = -\Lambda_w(S,P)\mathbb{K}\nabla P + A(S,P)\mathbb{K}\nabla\theta + \rho_w^2 \lambda_w(S)\mathbb{K}\mathbf{g}, \\ \mathbf{Q}_g &= \rho_g(P_g)\mathbf{q}_g = -\Lambda_g(S,P)\mathbb{K}\nabla P - A(S,P)\mathbb{K}\nabla\theta + \rho_g(S,P)^2\lambda_g(S)\mathbb{K}\mathbf{g} \end{aligned}$$

are the phase mass fluxes with $\mathbf{q}_j$ being the volumetric velocity of the j-phase, $j \in \{w,g\}$. The Dirichlet boundary data P_D, θ_D are assumed to be defined in the whole domain Q. In order to express their regularity the following space and the corresponding norm are introduced:

$$W = \{\varphi \in L^2(0,T;H^1(\Omega)) \colon \varphi \in L^\infty(0,T;L^1(\Omega)), \partial_t\varphi \in L^1(Q)\}, \tag{4.9}$$

$$|||\varphi||| = \|\varphi\|_{L^2(0,T;H^1(\Omega))} + \|\varphi\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_t\varphi\|_{L^1(Q)}.$$

The values of the non-wetting saturation and the phase pressures on the Dirichlet boundary are defined by

$$S_D = \mathcal{S}(\theta_D), \quad P_{wD} = P_w(S_D, P_D), \quad P_{gD} = P_g(S_D, P_D),$$

where $\mathcal{S} = \beta^{-1}$.

The initial conditions are

$$\theta(x,0) = \theta_0(x), \quad P(x,0) = p_0(x) \quad \text{in } \Omega. \tag{4.10}$$

We are going to prove the existence of weak solutions of the coupled system (4.5), (4.6) with the boundary and initial conditions (4.7), (4.8) and (4.10) under the following assumptions:

(A.1) The porosity Φ belongs to $L^\infty(\Omega)$, and there exist constants, $0 < \phi_m \le \phi_M < +\infty$, such that $0 < \phi_m \le \Phi(x) \le \phi_M$ a.e. in Ω.

(A.2) The permeability tensor $\mathbb{K}$ belongs to $(L^\infty(\Omega))^{d\times d}$, and there exist constants $0 < k_m \le k_M < +\infty$, such that for almost all $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^d$ it holds:

$$k_m|\boldsymbol{\xi}|^2 \le \mathbb{K}(x)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \le k_M|\boldsymbol{\xi}|^2.$$

(A.3) The relative mobilities satisfy $\lambda_w, \lambda_g \in C([0,1];\mathbb{R}^+)$, $\lambda_w(S_w = 0) = 0$ and $\lambda_g(S_g = 0) = 0$; λ_j is an increasing function of S_j. Moreover, there exist constants $\lambda_M \ge \lambda_m > 0$ such that for all $S \in [0,1]$

$$0 < \lambda_m \le \lambda_w(S) + \lambda_g(S) \le \lambda_M.$$

(A.4) There exist constants $p_{c,min} > 0$ and $M > 0$ such that the capillary pressure function $S \mapsto P_c(S)$, $P_c \in C([0,1[;\mathbb{R}^+) \cap C^1(]0,1[;\mathbb{R}^+)$, for all $S \in]0,1[$ satisfy

$$P_c'(S) \ge p_{c,min} > 0, \tag{4.11}$$

$$\int_0^1 P_c(s)\,ds + \sqrt{\lambda_g(S)\lambda_w(S)}P_c'(S) \le M. \tag{4.12}$$

Moreover, there exist $S^{\#} \in]0,1[$ and $\gamma > 0$ such that for all $S \in]0, S^{\#}]$

$$S^{2-\gamma} P_c'(S) \leq M, \tag{4.13}$$

$$P_c(S) - P_c(0) \leq M S P_c'(S). \tag{4.14}$$

(A.5) There exist $0 < \tau < 1$ and $C > 0$ such that for all $S_1, S_2 \in [0,1]$

$$C \left| \int_{S_1}^{S_2} \sqrt{\lambda_g(s)\lambda_w(s)}\, ds \right|^{\tau} \geq |S_1 - S_2|.$$

(A.6) $\rho_w > 0$, ρ_g is a $C^1(\mathbb{R})$ increasing function, and there exist $\rho_m, \rho_M > 0$ such that for all $p \in \mathbb{R}$ it holds

$$\rho_m \leq \rho_g(p) \leq \rho_M, \quad 0 < \rho_g'(p) \leq \rho_M.$$

(A.7) $F_w, F_g \in L^2(Q)$; $F_w \geq 0$ a.e. in Q.

(A.8) The boundary and initial data satisfy:

$$\begin{aligned}
&P_D, P_c(S_D) \in W,\ 0 \leq S_D \leq 1 \text{ a.e. in } Q;\\
&G_w, G_g \in L^2(\Gamma_N),\ G_w \leq 0;\\
&p_0, \theta_0 \in L^2(\Omega),\ 0 \leq \theta_0 \leq \beta(1) \text{ a.e. in } \Omega.
\end{aligned}$$

Remark 5 *Assumptions (A.1) – (A.3) are classical for porous media. The strength of singularities in the capillary pressure and its derivative at the end points $S = 0, 1$ is controlled by (A.4), which is together with (A.5) used to prove the Hölder continuity of the functions $\mathcal{S} = \beta^{-1}$ and $(S, P) \mapsto \rho_g(S, P)S$ in the proofs of Lemma 8 and Lemma 9. Let us point that, as a consequence of incompressibility of the wetting phase, the restrictions on the capillary pressure P_c in (4.13), (4.14) are given only at $S = 0$, which is less strict compared to the corresponding assumptions in [12], where both phases are compressible.*

The requirements on the sign of the boundary data in (A.7) and (A.8) are necessary only if the capillary pressure curve is unbounded at $S = S_g = 1$. In that case the restrictions $F_w \geq 0$ and $G_w \leq 0$ do not allow extraction of the wetting phase from the domain, since otherwise we can not control the growth of the wetting phase pressure to $-\infty$.

From the assumptions $P_D, P_c(S_D) \in W$ in (A.8) it easily follows that the functions $P_{wD} = P_w(S_D, P_D)$ and $P_{gD} = P_g(S_D, P_D)$ also belong to the space W. These are the con-

ditions on boundary data that allow us to obtain uniform a priori estimates in Section 4.5. Furthermore, due to (4.11) *in* (*A.4*) *it is also* $S_D \in W$.

Remark 6 *The boundedness of the phase mobilities and the phase densities in* (*A.3*) *and* (*A.6*) *imply the following bound for the gas pressure:*

$$|P_g(S,P)| \leq C(|P|+1), \tag{4.15}$$

while for the wetting phase pressure we have

$$P_w(S,P) \leq P, \tag{4.16}$$

due to $S \mapsto P_c(S)$ *being an increasing function. These bounds are going to be used to obtain the uniform a priori estimates in Chapter 6. Indeed, while the non-wetting pressure* P_g *is bounded, the wetting phase pressure* P_w *is unbounded when* $S \to 1$ *([12, 104]; cf. Remark 5).*

Remark 7 *It can be seen using* (*A.3*) *and* (*A.6*) *(see [8]) that* ω *is a positive smooth function for which there is a constant* C *such that*

$$e^{-CS} \leq \omega(S,P) \leq 1 \text{ in } [0,1] \times \mathbb{R}. \tag{4.17}$$

It also follows from (4.11) *and* (*A.5*) *that* $\mathcal{S} = \beta^{-1}$ *is Hölder continuous with exponent* τ*. More precisely,*

$$\frac{p_{c,min}^{\tau}}{C}|S_2 - S_1| \leq |\beta(S_2) - \beta(S_1)|^{\tau}. \tag{4.18}$$

In order to incorporate the Dirichlet boundary condition, the following space is introduced:

$$V = \{u \in H^1(\Omega); u|_{\Gamma_D} = 0\}.$$

The existence result for weak solutions of the system (4.5)-(4.6) with the boundary and initial conditions (4.7)-(4.10) is stated in the following theorem.

Theorem 5 *Let* (*A.1*)–(*A.8*) *hold. Denote* $S = \mathcal{S}(\theta)$*. Then there exists* (P,θ) *such that*

$$P \in L^2(0,T;V) + P_D,\ \theta \in L^2(0,T;V) + \theta_D,\ 0 \leq \theta \leq \beta(1) \text{ a.e. in } Q,$$
$$\partial_t(\Phi S) \in L^2(0,T;V'),\ \partial_t(\Phi\rho_g(S,P)S) \in L^2(0,T;V');$$

for all $\varphi, \psi \in L^2(0,T;V)$

$$
\begin{aligned}
-\rho_w \int_0^T \langle \partial_t(\Phi S), \varphi\rangle dt + \int_Q [\Lambda_w(S,P)\mathbb{K}\nabla P \cdot \nabla\varphi - A(S,P)\mathbb{K}\nabla\theta \cdot \nabla\varphi] dxdt \\
- \int_Q \lambda_w(S)\rho_w^2 \mathbb{K}\boldsymbol{g} \cdot \nabla\varphi dxdt = \int_Q F_w \varphi dxdt - \int_{\Gamma_N^T} G_w \varphi d\sigma dt,
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
\int_0^T \langle \partial_t(\Phi \rho_g(S,P)S), \psi\rangle dt + \int_Q [\Lambda_g(S,P)\mathbb{K}\nabla P \cdot \nabla\psi + A(S,P)\mathbb{K}\nabla\theta \cdot \nabla\psi] dxdt \\
- \int_Q \lambda_g(S)\rho_g(S,P)^2 \mathbb{K}\boldsymbol{g} \cdot \nabla\psi dxdt = \int_Q F_g \psi dxdt - \int_{\Gamma_N^T} G_g \psi d\sigma dt.
\end{aligned}
\tag{4.20}
$$

Furthermore, for all $\psi \in V$ *the functions*

$$
t \mapsto \int_\Omega \Phi S \psi dx, \quad t \mapsto \int_\Omega \Phi \rho_g(P_g(S,P)) S \psi dx
$$

are continuous in $[0,T]$ *and the initial conditions are satisfied in the following sense:*

$$
\left(\int_\Omega \Phi S \psi dx\right)(0) = \int_\Omega \Phi s_0 \psi dx,
$$

$$
\left(\int_\Omega \Phi \rho_g(P_g(S,P)) S \psi dx\right)(0) = \int_\Omega \Phi \rho_g(P_g(s_0,p_0)) s_0 \psi dx,
$$

where $s_0 = \mathcal{S}(\theta_0)$*.*

The proof of Theorem 5 is complicated primarily by the degeneracy of the equations caused by vanishing of the diffusion coefficient $A(S,P)$ at both ends $S = 0$ and $S = 1$. Therefore, we will introduce a regularized problem with a strictly positive coefficient $A(S,P)$ by adding a small positive constant η to it. In the same time we will regularize the unbounded capillary pressure function and prove Theorem 5 by passing to the limit as $\eta \to 0$ in the regularized problem. The other difficulty in proving Theorem 5 is vanishing of the time derivative term in equation (4.20) in the region where the gas phase is not present since therein the gas density can not be determined by its evolution. This trouble will be treated in Section 4.7 with appropriate compactness theorem.

4.3 Regularized problem

In this Section we construct the regularized problem which is formulated with the global pressure P and the non-wetting phase saturation S as primary variables. Introduction of this non-degenerate approximate problem is motivated by the following. A priori estimates, uniform with respect to the regularization parameter η, will be developed in Section 4.5 by using the phase pressures $P_w(S,P)$ and $P_g(S,P)$ as test functions in the variational formulation of the problem. This use of the phase pressures generates a new problem since, under the hypothesis (A.4) on the capillary pressure, the wetting and the non-wetting phase pressure partial derivatives with respect to S can be unbounded at $S = 1$ and $S = 0$ and thus for $P, S \in L^2(0,T;H^1(\Omega))$, $P_w(S,P)$ and $P_g(S,P)$ may not be valid test functions. Therefore, following the idea of [12], we will correct the capillary pressure function by introducing a regularized capillary pressure derivative, a regularized capillary pressure and regularized phase pressures as follows:

$$R_\eta(P_c'(S)) = \begin{cases} 2(1-\frac{S}{\eta})\frac{P_c(\eta)-P_c(0)}{\eta} + (2\frac{S}{\eta}-1)P_c'(\eta) & \text{for } S \le \eta \\ P_c'(S) & \text{for } \eta \le S \le 1-\eta\ , \\ P_c'(1-\eta) & \text{for } 1-\eta \le S \le 1 \end{cases} \tag{4.21}$$

$$P_c^\eta(S) = P_c(0) + \int_0^S R_\eta(P_c'(s))\,ds, \tag{4.22}$$

$$P_g^\eta(S,P) = P + P_c(0) + \int_0^S f_w(s,P)R_\eta(P_c'(s))\,ds, \tag{4.23}$$

$$P_w^\eta(S,P) = P - \int_0^S f_g(s,P)R_\eta(P_c'(s))\,ds. \tag{4.24}$$

It is clear that $P_g^\eta(S,P) - P_w^\eta(S,P) = P_c^\eta(S)$. Some properties of the regularized capillary pressure are listed below (the details can be found in [12, 104]).

For any $\eta > 0$, $P_c^\eta(S)$ is a bounded, monotone, $C^1([0,1])$ function, and $P_c^\eta(S) = P_c(S)$ for $S \in [\eta, 1-\eta]$. For sufficiently small η from (A.4) it follows that

$$\frac{d}{dS}P_c^\eta(S) \ge p_{c,min}/2 > 0. \tag{4.25}$$

Also, $|R_\eta(P_c'(S))| \le p_{c,max}^\eta < +\infty$ for some constant $p_{c,max}^\eta$.

Further, there is a constant $M \ge 1$ such that

$$R_\eta(P_c'(S)) \le MP_c'(S), \quad \text{for } S \in]0,1[. \tag{4.26}$$

If $S \ge \eta$, it is easy to check (4.26) with $M = 1$. To establish (4.26) for $S < \eta$ we use (4.14) in (A.4) to get:

$$
\begin{aligned}
R_\eta(P_c'(S)) &= 2(1-\frac{S}{\eta})\frac{P_c(\eta)-P_c(0)}{\eta} + (2\frac{S}{\eta}-1)P_c'(\eta) \\
&\le 2(1-\frac{S}{\eta})MP_c'(\eta) + (2\frac{S}{\eta}-1)P_c'(\eta) \le (2M+1)P_c'(\eta) \le (2M+1)P_c'(S),
\end{aligned}
$$

since the capillary pressure function is concave near $S=0$. Let us point out that (4.14) is assumed only to obtain (4.26), and moreover, (4.26) will be utilized solely in order to establish the estimates, uniform in η, on the regularized boundary data (see Remark 8).

The derivatives of the regularized phase pressures are mutually equal, as in the non-regularized case and it holds

$$
\frac{\partial P_g^\eta}{\partial P} = \frac{\partial P_w^\eta}{\partial P} = \omega^\eta(S,P),
$$

where

$$
\omega^\eta(S,P) = \exp\left(-\int_0^S \nu_g(s,P)\frac{\rho_w\rho_g(s,P)\lambda_w(s)\lambda_g(s)R_\eta(P_c'(s))}{(\rho_w\lambda_w(s)+\rho_g(s,P)\lambda_g(s))^2}\,ds\right).
$$

It is easily seen that

$$
\nabla P_w^\eta = \omega^\eta(S,P)\nabla P - f_g(S,P)R_\eta(P_c'(S))\nabla S, \tag{4.27}
$$

$$
\nabla P_g^\eta = \omega^\eta(S,P)\nabla P + f_w(S,P)R_\eta(P_c'(S))\nabla S, \tag{4.28}
$$

so that $P_w^\eta, P_g^\eta \in L^2(0,T;H^1(\Omega))$ for $P,S\in L^2(0,T;H^1(\Omega))$, as intended.

Another step in defining the regularized problem is substituting the function $A(S,P)$ by $A^\eta(S,P)$, for $\eta>0$, defined by

$$
A^\eta(S,P) = \frac{\rho_w\rho_g(S,P)}{\lambda(S,P)}\lambda_w(S)\lambda_g(S)R_\eta(P_c'(S)) + \eta > 0. \tag{4.29}
$$

At last, we are going to consider the regularized version of the system (4.5), (4.6) in which we will replace $\rho_g(S,P)$ by

$$
\rho_g^\eta(S,P) := \rho_g(P_g^\eta(S,P)). \tag{4.30}
$$

Now the regularized system is defined as

$$
\begin{aligned}
-\rho_w\Phi\frac{\partial S^\eta}{\partial t} - \mathrm{div}(\Lambda_w^\eta(S^\eta,P^\eta)\mathbb{K}\nabla P^\eta) + \mathrm{div}(A^\eta(S^\eta,P^\eta)\mathbb{K}\nabla S^\eta) \\
+ \rho_w^2\mathrm{div}(\lambda_w(S^\eta)\mathbb{K}\mathbf{g}) = F_w,
\end{aligned} \tag{4.31}
$$

$$\Phi\frac{\partial}{\partial t}(\rho_g^\eta(S^\eta,P^\eta)S^\eta) - \mathrm{div}(\Lambda_g^\eta(S^\eta,P^\eta)\mathbb{K}\nabla P^\eta) - \mathrm{div}(A^\eta(S^\eta,P^\eta)\mathbb{K}\nabla S^\eta) \\ + \mathrm{div}(\lambda_g(S^\eta)\rho_g^\eta(S^\eta,P^\eta)^2\mathbb{K}\mathbf{g}) = F_g, \tag{4.32}$$

where we define

$$\Lambda_w^\eta(S,P) = \rho_w\lambda_w(S)\omega^\eta(S,P), \quad \Lambda_g^\eta(S,P) = \rho_g(S,P)\lambda_g(S)\omega^\eta(S,P) \tag{4.33}$$

and introduce the regularized total mobility

$$\Lambda^\eta(S,P) = \Lambda_w^\eta(S,P) + \Lambda_g^\eta(S,P), \tag{4.34}$$

and the regularized function β:

$$\beta^\eta(S) = \int_0^S \sqrt{\lambda_w(s)\lambda_g(s)}R_\eta(P_c'(s))\,ds. \tag{4.35}$$

We will denote $\mathcal{S}^\eta = (\beta^\eta)^{-1}$.

Now we quote some uniform estimates and limits for the regularized coefficients, proved in [12], Lemma 1.

Lemma 4 *Assume (A.4) and (A.6). Then there exists a constant $C > 0$, independent of η, such that*

$$|P_g^\eta(S,P)| \le C(|P|+1), \tag{4.36}$$
$$P_w^\eta(S,P) \le P, \tag{4.37}$$
$$|\lambda_w(S)P_w^\eta(S,P)| \le C(|P|+1), \tag{4.38}$$
$$e^{-CS} \le \omega^\eta(S,P) \le 1, \tag{4.39}$$

and the following sequences converge uniformly in $[0,1]\times\mathbb{R}$ as $\eta \to 0$:

$$P_g^\eta(S,P) \to P_g(S,P), \tag{4.40}$$
$$\omega^\eta(S,P) \to \omega(S,P), \tag{4.41}$$
$$\Lambda_j^\eta(S,P) \to \Lambda_j(S,P),\ \ j\in\{w,g\}, \tag{4.42}$$
$$\beta^\eta(S) \to \beta(S) \ \ \textit{uniformly in } [0,1]. \tag{4.43}$$

Remark 8 *From the assumption on the boundary data $P_D, P_c(S_D) \in W$ in (A.8) it is easy to show, as mentioned in Remark 5, that $P_{wD}, P_{gD}, \beta(S_D) \in W$. We define the regularized* *phase pressure boundary values by* $P^\eta_{wD} = P^\eta_w(S_D, P_D)$ *and* $P^\eta_{gD} = P^\eta_g(S_D, P_D)$. *Now using the estimate* (4.26) *we can also show that the norms* $|||P^\eta_{wD}|||$, $|||P^\eta_{gD}|||$ *and* $|||\beta^\eta(S_D)|||$ *are uniformly bounded with respect to the parameter* η. *For example,*

$$\nabla P^\eta_{wD} = \omega^\eta(S_D, P_D)\nabla P_D - f_g(S_D, P_D)R_\eta(P_c'(S_D))\nabla S_D,$$

which, by (4.26) *and* (4.39), *gives the estimate*

$$|\nabla P^\eta_{wD}| \leq |\nabla P_D| + MP_c'(S_D)|\nabla S_D|,$$

leading to

$$\|P^\eta_{wD}\|_{L^2(0,T;H^1(\Omega))} \leq C(1 + \|P_D\|_{L^2(0,T;H^1(\Omega))} + \|P_c(S_D)\|_{L^2(0,T;H^1(\Omega))}).$$

The boundedness for the other two norms defining the norm $||| \cdot |||$ *and for* P_{gD} *can be obtained analogously. Also, due to uniform convergence in* (4.43) *we have*

$$\beta^\eta(S_D) \rightharpoonup \theta_D = \beta(S_D) \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ as } \eta \to 0. \tag{4.44}$$

The variational formulation of the regularized problem as well as the result on the existence of its weak solutions is stated in the following theorem.

Theorem 6 *Assume* (*A.1*)–(*A.4*), (*A.6*)–(*A.8*) *hold and* $p_0, s_0 \in H^1(\Omega)$. *For all* $\eta > 0$ *sufficiently small there exists* (P^η, S^η) *satisfying*

$$\begin{aligned}
&P^\eta \in L^2(0,T;V) + P_D,\ S^\eta \in L^2(0,T;V) + S_D,\ 0 \leq S^\eta \leq 1 \text{ a.e. in } Q,\\
&\partial_t(\Phi S^\eta), \partial_t(\Phi \rho^\eta_g(S^\eta, P^\eta)S^\eta) \in L^2(0,T;V');
\end{aligned}$$

for all $\varphi, \psi \in L^2(0,T;V)$

$$\begin{aligned}
-\rho_w \int_0^T \langle \partial_t(\Phi S^\eta), \varphi\rangle dt + \int_Q [\Lambda^\eta_w(S^\eta, P^\eta)\mathbb{K}\nabla P^\eta \cdot \nabla\varphi - A^\eta(S^\eta, P^\eta)\mathbb{K}\nabla S^\eta \cdot \nabla\varphi] dxdt\\
- \int_Q \lambda_w(S^\eta)\rho_w^2 \mathbb{K}\boldsymbol{g} \cdot \nabla\varphi dxdt = \int_Q F_w \varphi dxdt - \int_{\Gamma^T_N} G_w \varphi d\sigma dt,
\end{aligned} \tag{4.45}$$

$$
\begin{aligned}
&\int_0^T \langle \partial_t(\Phi\rho_g^\eta(S^\eta,P^\eta)S^\eta),\psi\rangle dt \\
&\quad + \int_Q [\Lambda_g^\eta(S^\eta,P^\eta)\mathbb{K}\nabla P^\eta\cdot\nabla\psi + A^\eta(S^\eta,P^\eta)\mathbb{K}\nabla S^\eta\cdot\nabla\psi]dxdt \\
&\quad - \int_Q \lambda_g(S^\eta)\rho_g^\eta(S^\eta,P^\eta)^2\mathbb{K}\boldsymbol{g}\cdot\nabla\psi dxdt = \int_Q F_g\psi dxdt - \int_{\Gamma_N^T} G_g\psi d\sigma dt.
\end{aligned} \tag{4.46}
$$

Furthermore, $S^\eta, \rho_g^\eta(S^\eta,P^\eta)S^\eta \in C([0,T];L^2(\Omega))$ *and*

$$
S^\eta(\cdot,0) = s_0,\ \rho_g^\eta(S^\eta,P^\eta)S^\eta(\cdot,0) = \rho_g^\eta(s_0,p_0)s_0 \ \textit{a.e. in}\ \Omega. \tag{4.47}
$$

Theorem 6 will be proved by performing the following steps: the discretization of the time derivatives with a small parameter $h > 0$, establishing uniform estimates for the solutions of the discretized problem, and passing to the limit as $h \to 0$. These are presented in three following sections, respectively.

4.4 Time discretization

In this Section we deal with the regularized problem (4.45)-(4.47) for a fixed $\eta > 0$ and for simplicity of the notation we skip the dependence of the saturation and the global pressure on the small parameter η in writing.

In order to discretize the regularized system (4.45)-(4.46), the time derivative is approximated by a backward difference. Namely, for each positive integer N the interval $[0,T]$ is divided into N subintervals, each of length $h = T/N$. Let $t_n = nh$ and $J_n =]t_{n-1}, t_n]$ for $1 \le n \le N$, and for any $h > 0$ denote the time difference operator by

$$
\partial^h v(t) = \frac{v(t+h) - v(t)}{h}.
$$

Next, for any Hilbert space $\mathcal{H}$, let

$$
l_h(\mathcal{H}) = \{v \in L^\infty(0,T;\mathcal{H}) \colon v \text{ is constant in time on each subinterval } J_n \subset [0,T]\}.
$$

Besides, for any $v^h \in l_h(\mathcal{H})$ we set $v^n = (v^h)^n = v^h|_{J_n}$ and assign to v^h a piecewise linear in time function

$$
\tilde{v}^h = \sum_{n=1}^N \left(\frac{t_n - t}{h}v^{n-1} + \frac{t - t_{n-1}}{h}v^n\right)\chi_{J_n}(t), \quad \tilde{v}^h(0) = v^h(0) = v^0 \tag{4.48}
$$

which satisfies

$$\partial_t \tilde{v}^h(t) = \partial^{-h} v^h(t), \quad \text{for } t \neq nh, n = 0, 1, \ldots, N.$$

Lastly, for any function $f \in L^1(0,T;\mathcal{H})$ we define $f^h \in l_h(\mathcal{H})$ by

$$f^h(t) = \frac{1}{h} \int_{J_n} f(\tau) d\tau, \quad t \in J_n.$$

The discrete problem is defined as follows: find $P^h \in l_h(V) + P_D^h$ and $S^h \in l_h(V) + S_D^h$ such that for all $\varphi \in l_h(V)$,

$$\begin{aligned} -\rho_w \int_Q \Phi \partial^{-h} S^h \varphi dx dt + \int_Q [\Lambda_w^\eta(S^h, P^h) \mathbb{K} \nabla P^h \cdot \nabla \varphi - A^\eta(S^h, P^h) \mathbb{K} \nabla S^h \cdot \nabla \varphi] dx dt \\ - \int_Q \lambda_w(S^h) \rho_w^2 \mathbb{K} \mathbf{g} \cdot \nabla \varphi dx dt = \int_Q F_w^h \varphi dx dt - \int_{\Gamma_N^T} G_w^h \varphi d\sigma dt, \end{aligned} \tag{4.49}$$

for all $\psi \in l_h(V)$,

$$\begin{aligned} &\int_Q \Phi \partial^{-h} (\rho_g^\eta(S^h, P^h) S^h) \psi dx dt \\ &\quad + \int_Q [\Lambda_g^\eta(S^h, P^h) \mathbb{K} \nabla P^h \cdot \nabla \psi + A^\eta(S^h, P^h) \mathbb{K} \nabla S^h \cdot \nabla \psi] dx dt \\ &\quad - \int_Q \lambda_g(S^h) \rho_g^\eta(S^h, P^h)^2 \mathbb{K} \mathbf{g} \cdot \nabla \psi dx dt = \int_Q F_g^h \psi dx dt - \int_{\Gamma_N^T} G_g^h \psi d\sigma dt, \end{aligned} \tag{4.50}$$

and $S^h = s_0$, $P^h = p_0$ for $t = 0$.

The following Proposition gives the existence result for the discrete system (4.49), (4.50), (4.47).

Proposition 1 *Assume (A.1)–(A.8). Then there exists a solution $P^h \in l_h(V) + P_D^h$, $S^h \in l_h(V) + S_D^h$ of (4.49), (4.50); moreover, $0 \leq S^h \leq 1$ a.e. in Q.*

Proof. The proof is based on the Schauder fixed point theorem.

For fixed $1 \leq k \leq N$, it is enough to prove that for known $P^{k-1} \in V + P_D^{k-1}, S^{k-1} \in V + S_D^{k-1}$ such that $0 \leq S^{k-1} \leq 1$, the following problem has a unique solution $P^k \in V + P_D^k, S^k \in V + S_D^k$:

$$\begin{aligned} \frac{\rho_w}{h} \int_\Omega \Phi(S^{k-1} - S^k) \varphi dx + \int_\Omega [\Lambda_w^\eta(S^k, P^k) \mathbb{K} \nabla P^k \cdot \nabla \varphi - A^\eta(S^k, P^k) \mathbb{K} \nabla S^k \cdot \nabla \varphi] dx \\ - \int_\Omega \lambda_w(S^k) \rho_w^2 \mathbb{K} \mathbf{g} \cdot \nabla \varphi dx = \int_\Omega F_w^k \varphi dx - \int_{\Gamma_N} G_w^k \varphi d\sigma \end{aligned} \tag{4.51}$$

for all $\varphi \in V$, and

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \Phi(\rho_g^\eta(S^k,P^k)S^k - \rho_g^\eta(S^{k-1},P^{k-1})S^{k-1})\psi dx \\
&\qquad + \int_\Omega [\Lambda_g^\eta(S^k,P^k)\mathbb{K}\nabla P^k \cdot \nabla\psi + A^\eta(S^k,P^k)\mathbb{K}\nabla S^k \cdot \nabla\psi]dx \\
&\qquad - \int_\Omega \lambda_g(S^k)\rho_g^\eta(S^k,P^k)^2\mathbb{K}\mathbf{g}\cdot\nabla\psi dx = \int_\Omega F_g^k\psi dx - \int_{\Gamma_N} G_g^k\psi d\sigma
\end{aligned} \tag{4.52}$$

for all $\psi \in V$. Alternatively, one can show the existence of a unique solution of the equivalent system which consists of (4.52) and the sum of the equations (4.51) and (4.52). This is achieved by applying the Schauder theorem to the mapping $\mathcal{T}\colon L^2(\Omega)\times L^2(\Omega) \to L^2(\Omega)\times L^2(\Omega)$ defined by $\mathcal{T}(\overline{S},\overline{P}) = (S,P)$. Here (S,P) is a solution of the following linear system: for all $\varphi \in V$

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \Phi(H^\eta(Z(\overline{S}),\overline{P}) - H^\eta(S^{k-1},P^{k-1}))\varphi dx + \int_\Omega \Lambda^\eta(Z(\overline{S}),\overline{P})\mathbb{K}\nabla P\cdot\nabla\varphi dx \\
&\qquad - \int_\Omega [H_1^\eta(Z(\overline{S}),\overline{P})\mathbb{K}\mathbf{g}\cdot\nabla\varphi - H_2^\eta(Z(\overline{S}),\overline{P})F_P^k\varphi]dx \\
&\qquad = \int_\Omega (F_w^k + F_g^k)\varphi dx - \int_{\Gamma_N}(G_w^k + G_g^k)\varphi d\sigma,
\end{aligned} \tag{4.53}$$

and for all $\psi \in V$

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \Phi(\rho_g^\eta(Z(\overline{S}),P)Z(\overline{S}) - \rho_g^\eta(S^{k-1},P^{k-1})S^{k-1})\psi dx \\
&\qquad + \int_\Omega [\Lambda_g^\eta(Z(\overline{S}),P)\mathbb{K}\nabla P\cdot\nabla\psi + A^\eta(Z(\overline{S}),P)\mathbb{K}\nabla S\cdot\nabla\psi]dx \\
&\qquad - \int_\Omega [\lambda_g(Z(\overline{S}))\rho_g^\eta(Z(\overline{S}),P)^2\mathbb{K}\mathbf{g}\cdot\nabla\psi - \rho_g^\eta(Z(\overline{S}),P)f_g(Z(\overline{S}),P)F_P^k\psi]dx \\
&\qquad = \int_\Omega F_g^k\psi dx - \int_{\Gamma_N} G_g^k\psi d\sigma,
\end{aligned} \tag{4.54}$$

where H^η, H_1^η and H_2^η are certain nonlinear functions of S and P, and $Z(S)$ is an appropriate cut-off function for $S \in [0,1]$ (see [12]). In here, (4.53) is a linear elliptic equation for the pressure P, and for given P, (4.54) is an elliptic problem for the saturation S. The uniform ellipticity is guaranteed by (4.29) and the solutions of these elliptic equations exist from the Lax-Milgram lemma.

□

4.5 Uniform estimates

In order to pass to the limit as $h \to 0$ in (4.49)-(4.50), we need a priori estimates uniform with respect to h. Recall that the problem is considered for fixed $\eta > 0$ which is skipped in writing. We will establish in this section the estimates that are uniform in h and also in η. For $k \in \{1, \dots, N\}$ we set

$$r_g^k = \rho_g(P_g^\eta(S^k, P^k))S^k$$

and denote by r_g^h and $\tilde{r}_g^h$ the corresponding functions which are piecewise constant in time and the associated piecewise linear in time functions, respectively.

Proposition 2 *Suppose that the assumptions of Proposition 1 hold. Let* $(P^h)_h$ *and* $(S^h)_h$ *be the sequences of solutions to* (4.49)-(4.50). *Then the following bounds hold, uniform with respect to* h*:*

$$\|P^h\|_{L^2(0,T;H^1(\Omega))} + \|S^h\|_{L^2(0,T;H^1(\Omega))} + \|\beta^\eta(S^h)\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{4.55}$$

$$\|\tilde{S}^h\|_{L^2(0,T;H^1(\Omega))} + \|r_g^h\|_{L^2(0,T;H^1(\Omega))} + \|\tilde{r}_g^h\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{4.56}$$

$$\|\partial_t(\Phi\tilde{S}^h)\|_{L^2(0,T;V')} + \|\partial_t(\Phi\tilde{r}_g^h)\|_{L^2(0,T;V')} \le C. \tag{4.57}$$

Proof. First, we quote some identities that are going to be used throughout the proof. From the relations (4.27), (4.28) and the definitions of the functions A^η and β^η we can obtain the following representations of the regularized wetting and non-wetting phase fluxes (without gravity term)

$$\Lambda_w^\eta(S,P)\mathbb{K}\nabla P - A^\eta(S,P)\mathbb{K}\nabla S = \rho_w\lambda_w(S)\mathbb{K}\nabla P_w^\eta - \eta\mathbb{K}\nabla S, \tag{4.58}$$

$$\Lambda_g^\eta(S,P)\mathbb{K}\nabla P + A^\eta(S,P)\mathbb{K}\nabla S = \rho_g(S,P)\lambda_g(S)\mathbb{K}\nabla P_g^\eta + \eta\mathbb{K}\nabla S, \tag{4.59}$$

as well as the equality

$$\begin{aligned} &\rho_w\lambda_w(S)\mathbb{K}\nabla P_w^\eta \cdot \nabla P_w^\eta + \rho_g(S,P)\lambda_g(S)\mathbb{K}\nabla P_g^\eta \cdot \nabla P_g^\eta \\ &= \Lambda^\eta(S,P)\omega^\eta(S,P)\mathbb{K}\nabla P \cdot \nabla P + \frac{\rho_w\rho_g(S,P)}{\lambda(S,P)}\mathbb{K}\nabla\beta^\eta(S) \cdot \nabla\beta^\eta(S). \end{aligned} \tag{4.60}$$

In this section, for simplicity, we assume that $P_c(0) = 0$. From now on, $C, C_1, \dots$ denote generic constants that do not depend on h or η.

Let us consider the discrete problem taken at a time level k, that is, the variational equations (4.51), (4.52). Similarly as in [72, 75–77] we use the following test functions in

(4.51), (4.52):

$$\varphi = \varphi(P_w^{\eta,k}) = \frac{1}{\rho_w}(P_w^{\eta,k} - P_{wD}^{\eta,k}) \quad \text{and } \psi = \psi(P_g^{\eta,k}) = \int_{P_{gD}^{\eta,k}}^{P_g^{\eta,k}} \frac{dp}{\rho_g(p)},$$

respectively, where we write $P_j^{\eta,k} = P_j^{\eta}(S^k, P^k)$ and $P_{jD}^{\eta,k} = P_{jD}^{\eta}(S_D^k, P_D^k)$, $j \in \{w, g\}$. Note that $\varphi(P_w^{\eta,k})$ and $\psi(P_g^{\eta,k})$ are admissible test functions for the system in consideration due to (4.27), (4.28). The sum of the equations (4.51) and (4.52) with the chosen test functions reads by taking into account the relations (4.58) and (4.59):

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \Phi\big[(S^{k-1} - S^k)P_w^{\eta,k} + (\rho_g(P_g^{\eta,k})S^k - \rho_g(P_g^{\eta,k-1})S^{k-1}) \int_0^{P_g^{\eta,k}} \frac{dp}{\rho_g(p)}\big]dx \\
&\quad + \frac{1}{\rho_w}\int_\Omega [\lambda_w(S^k)\rho_w \mathbb{K}\nabla P_w^{\eta,k} - \eta\mathbb{K}\nabla S^k] \cdot \nabla P_w^{\eta,k} dx \\
&\quad + \int_\Omega \frac{1}{\rho_g(P_g^{\eta,k})}[\lambda_g(S^k)\rho_g(S^k, P^k)\mathbb{K}\nabla P_g^{\eta,k} + \eta\mathbb{K}\nabla S^k] \cdot \nabla P_g^{\eta,k} dx \\
&\quad = \frac{1}{h}\int_\Omega \Phi\big[(S^{k-1} - S^k)P_{wD}^{\eta,k} + (\rho_g(P_g^{\eta,k})S^k - \rho_g(P_g^{\eta,k-1})S^{k-1}) \int_0^{P_{gD}^{\eta,k}} \frac{dp}{\rho_g(p)}\big]dx \\
&\quad + \frac{1}{\rho_w}\int_\Omega [\lambda_w(S^k)\rho_w \mathbb{K}\nabla P_w^{\eta,k} - \eta\mathbb{K}\nabla S^k] \cdot \nabla P_{wD}^{\eta,k} dx \\
&\quad + \int_\Omega \frac{1}{\rho_g(P_{gD}^{\eta,k})}[\lambda_g(S^k)\rho_g(S^k, P^k)\mathbb{K}\nabla P_g^{\eta,k} + \eta\mathbb{K}\nabla S^k] \cdot \nabla P_{gD}^{\eta,k} dx \\
&\quad + \int_\Omega [\lambda_w(S^k)\rho_w \mathbb{K}\mathbf{g} \cdot \nabla P_w^{\eta,k} + \lambda_g(S^k)\rho_g(P_g^{\eta,k})\mathbb{K}\mathbf{g} \cdot \nabla P_g^{\eta,k}]dx \\
&\quad - \int_\Omega [\lambda_w(S^k)\rho_w \mathbb{K}\mathbf{g} \cdot \nabla P_{wD}^{\eta,k} + \lambda_g(S^k)\frac{\rho_g^2(P_g^{\eta,k})}{\rho_g(P_{gD}^{\eta,k})}\mathbb{K}\mathbf{g} \cdot \nabla P_{gD}^{\eta,k}]dx \\
&\quad + \int_\Omega [\frac{1}{\rho_w}F_w^k(P_w^{\eta,k} - P_{wD}^{\eta,k}) + F_g^k \int_{P_{gD}^{\eta,k}}^{P_g^{\eta,k}} \frac{dp}{\rho_g(p)}]dx \\
&\quad - \int_{\Gamma_N} [\frac{1}{\rho_w}G_w^k(P_w^{\eta,k} - P_{wD}^{\eta,k}) + G_g^k \int_{P_{gD}^{\eta,k}}^{P_g^{\eta,k}} \frac{dp}{\rho_g(p)}]dx.
\end{aligned} \tag{4.61}$$

Let us denote the integral terms in the expression (4.61) by $Z_1, Z_2, \ldots, Z_{10}$, respectively. Denote the discrete time derivative terms as

$$Z_1 = \frac{1}{h}\int_\Omega \Phi\big[(S^{k-1} - S^k)P_w^{\eta,k} + (\rho_g(P_g^{\eta,k})S^k - \rho_g(P_g^{\eta,k-1})S^{k-1}) \int_0^{P_g^{\eta,k}} \frac{dp}{\rho_g(p)}\big]dx = \frac{1}{h}\int_\Omega \Phi X_1^k dx,$$

$$Z_4 = \frac{1}{h}\int_\Omega \Phi\big[(S^{k-1} - S^k)P_{wD}^{\eta,k} + (\rho_g(P_g^{\eta,k})S^k - \rho_g(P_g^{\eta,k-1})S^{k-1}) \int_0^{P_{gD}^{\eta,k}} \frac{dp}{\rho_g(p)}\big]dx = \frac{1}{h}\int_\Omega \Phi X_4^k dx.$$

We can obtain as in [12,75], using the monotonicity of the non-wetting phase mass density and the monotonicity of the capillary pressure,

$$
\begin{aligned}
X_1^k &= P_w^{\eta,k}(S^{k-1} - S^k) + \rho_g(P_g^{\eta,k})S^k \int_0^{P_g^{\eta,k}} \frac{dp}{\rho_g(p)} - \rho_g(P_g^{\eta,k-1})S^{k-1} \int_0^{P_g^{\eta,k-1}} \frac{dp}{\rho_g(p)} \\
&+ \rho_g(P_g^{\eta,k-1})S^{k-1} \int_{P_g^{\eta,k}}^{P_g^{\eta,k-1}} \frac{dp}{\rho_g(p)} \geq H^\eta(S^k, P^k) - H^\eta(S^{k-1}, P^{k-1}),
\end{aligned}
\tag{4.62}
$$

where the function H^η is introduced by

$$
H^\eta(S,P) = \left(\rho_g(P_g^\eta) \int_0^{P_g^\eta} \frac{dp}{\rho_g(p)} - P_g^\eta \right) S + \int_0^S P_c^\eta(z)dz.
$$

Since $S \geq 0$ a.e. in Q and $P_c \geq 0$, we then have

$$
H^\eta(S,P) \geq \left(\rho_g(P_g^\eta) \int_0^{P_g^\eta} \frac{dp}{\rho_g(p)} - P_g^\eta \right) S.
$$

Using the monotonicity and the boundedness of the gas density in (A.6), it is easy to show that $\nu_g(p) = \rho_g(p) \int_0^p \frac{dz}{\rho_g(z)} - p \geq 0$, and consequently

$$
H^\eta(S,P) \geq 0. \tag{4.63}
$$

The second discrete time derivative term can be transformed as follows:

$$
\begin{aligned}
Z_4 = \sum_{k=1}^N X_4^k &= \sum_{k=1}^N \left[(S^{k-1} - S^k)P_{wD}^{\eta,k} + (\rho_g(P_g^{\eta,k})S^k - \rho_g(P_g^{\eta,k-1})S^{k-1}) \int_0^{P_{gD}^{\eta,k}} \frac{dp}{\rho_g(p)}\right] \\
&= s_0 P_{wD}^{\eta,0} - S^N P_{wD}^{\eta,N} + \sum_{k=1}^N S^{k-1}(P_{wD}^{\eta,k} - P_{wD}^{\eta,k-1}) \\
&- \rho_g(P_g^\eta(s_0,p_0))s_0 \int_0^{P_{gD}^{\eta,0}} \frac{dp}{\rho_g(p)} + \rho_g(P_g^\eta(S^N,P^N))S^N \int_0^{P_{gD}^{\eta,N}} \frac{dp}{\rho_g(p)} \\
&- \sum_{k=1}^N \rho_g(P_g^{\eta,k-1})S^{k-1} \int_{P_{gD}^{\eta,k-1}}^{P_{gD}^{\eta,k}} \frac{dp}{\rho_g(p)},
\end{aligned}
$$

and by using (A.1) and (A.6)

$$
\begin{aligned}
&|\sum_{k=1}^{N}\int_{\Omega}\Phi X_4^k dx| \\
&\leq \frac{2\phi_M\rho_M}{\rho_m}\big(\sup_t\int_{\Omega}|P_{wD}^{\eta}|dx+\sup_t\int_{\Omega}|P_{gD}^{\eta}|dx+\int_Q(|\partial^{-h}P_{wD}^{\eta,h}|+|\partial^{-h}P_{gD}^{\eta,h}|)dxdt\big) \qquad (4.64)\\
&\leq C\big(\|P_{wD}^{\eta}\|_{L^\infty(0,T;L^1(\Omega))}+\|P_{gD}^{\eta}\|_{L^\infty(0,T;L^1(\Omega))}+\|\partial_t P_{wD}^{\eta}\|_{L^1(Q)}+\|\partial_t P_{gD}^{\eta}\|_{L^1(Q)}\big).
\end{aligned}
$$

Next, for the terms Z_2 and Z_3, we apply the equality (4.60), relations (4.27) and (4.28), and use (A.2), (A.3), (A.6) and the bounds (4.25), (4.39) to obtain

$$
\begin{aligned}
Z_2+Z_3 \geq\ & \frac{\lambda_m\rho_m\omega_m^2 k_m}{\rho_M}\int_{\Omega}|\nabla P^k|^2dx+\frac{\rho_m^2 k_m}{\lambda_M\rho_M^2}\int_{\Omega}|\nabla\beta^{\eta}(S^k)|^2dx\\
&+\eta\frac{k_m p_{c,min}}{2\rho_M}\int_{\Omega}|\nabla S^k|^2dx-\eta\frac{k_M\omega_M}{2\rho_m}\int_{\Omega}|\nabla P^k|\cdot|\nabla S^k|dx.
\end{aligned}
$$

It follows that one can find a constant C_1 and a constant η_0, such that for all $0<\eta\leq\eta_0$,

$$
Z_2+Z_3\geq C_1\int_{\Omega}(|\nabla P^k|^2+|\nabla\beta^{\eta}(S^k)|^2+\eta|\nabla S^k|^2)dx.
$$

Using the relations (4.27), (4.28) we first get

$$
\begin{aligned}
Z_5+Z_6 =\ & \int_{\Omega}\lambda_w(S^k)\mathbb{K}(\omega^{\eta}(S^k,P^k)\nabla P^k - f_g(S^k,P^k)R_{\eta}(P_c'(S^k))\nabla S^k)\cdot\nabla P_{wD}^{\eta,k}dx\\
&+\int_{\Omega}\frac{1}{\rho_g(P_{gD}^{\eta,k})}\lambda_g(S^k)\rho_g(S^k,P^k)\mathbb{K}(\omega^{\eta}(S^k,P^k)\nabla P^k+f_w(S^k,P^k)R_{\eta}(P_c'(S^k))\nabla S^k)\cdot\nabla P_{gD}^{\eta,k}dx\\
&-\eta\int_{\Omega}\frac{1}{\rho_w}\mathbb{K}\nabla S^k\cdot\nabla P_{wD}^{\eta,k}dx+\eta\int_{\Omega}\frac{1}{\rho_g(P_{gD}^{\eta,k})}\mathbb{K}\nabla S^k\cdot\nabla P_{gD}^{\eta,k}dx.
\end{aligned}
$$

By applying the definition of β^{η} given by (4.35), and by using (A.2), (A.3) and (A.6) we then obtain

$$
\begin{aligned}
|Z_5+Z_6|\leq\ & \frac{\lambda_M k_M\omega_M\rho_M}{\rho_m}\int_{\Omega}|\nabla P^k|(|\nabla P_{wD}^{\eta,k}|+|\nabla P_{gD}^{\eta,k}|)dx\\
&+\frac{\rho_M^2\lambda_M k_M}{\rho_m^2\lambda_m}\int_{\Omega}|\nabla\beta^{\eta}(S^k)|(|\nabla P_{wD}^{\eta,k}|+|\nabla P_{gD}^{\eta,k}|)dx\\
&+\eta\frac{k_M}{\rho_m}\int_{\Omega}|\nabla S^k|(|\nabla P_{wD}^{\eta,k}|+|\nabla P_{gD}^{\eta,k}|)dx,
\end{aligned}
$$

and finally we get

$$|Z_5 + Z_6| \le C_2 \int_\Omega (|\nabla P^k|^2 + |\nabla \beta^\eta(S^k)|^2 + \eta |\nabla S^k|^2) dx + C_3 \int_\Omega (|\nabla P_{wD}^{\eta,k}|^2 + |\nabla P_{gD}^{\eta,k}|^2) dx,$$

where $C_2 > 0$ can be chosen arbitrary small.

Similarly, using (4.27), (4.28) and (A.2), (A.3) and (A.6) yields

$$|Z_7 + Z_8| \le 2\lambda_M \rho_M k_M \omega_m |\mathbf{g}| \int_\Omega |\nabla P^k|^2 dx + \frac{2\rho_M^2 \lambda_M k_M |\mathbf{g}|}{\rho_m \lambda_m} \int_\Omega |\nabla \beta^\eta(S^k)|^2 dx + \frac{\lambda_M \rho_M^2 k_M |\mathbf{g}|}{\rho_m} \int_\Omega (|\nabla P_{wD}^{\eta,k}| + |\nabla P_{gD}^{\eta,k}|) dx.$$

Hence it can be seen that for any $C_4 > 0$,

$$|Z_7 + Z_8| \le C_4 \int_\Omega (|\nabla P^k|^2 + |\nabla \beta^\eta(S^k)|^2) dx + C_5 \int_\Omega (1 + |\nabla P_{wD}^{\eta,k}|^2 + |\nabla P_{gD}^{\eta,k}|^2) dx.$$

In order to estimate Z_9, we use the uniform bounds on the regularized phase pressures (4.36) and (4.37), the assumption (A.6) and the nonnegativity of the source term F_w in (A.7) to obtain

$$|Z_9| \le \frac{1}{\rho_m} \int_\Omega \left((|F_w^k| + |F_g^k|)(|P^k| + 1) + |F_w^k||P_{wD}^{\eta,k}| + |F_g^k||P_{gD}^{\eta,k}| \right) dx,$$

which leads to

$$|Z_9| \le C_6 \int_\Omega |P^k|^2 dx + C_7 \int_\Omega (1 + |F_w^k|^2 + |F_g^k|^2 + |P_{wD}^{\eta,k}|^2 + |P_{gD}^{\eta,k}|^2) dx,$$

for arbitrary $C_6 > 0$.

Finally, in a similar manner, (4.36), (4.37), (A.6), the condition $G_w \le 0$ in (A.8) and the trace theorem imply that for any $C_8 > 0$ it holds

$$|Z_{10}| \le C_8 \|P^k\|_{H^1(\Omega)}^2 + C_9 (1 + \|P_{wD}^{\eta,k}\|_{H^1(\Omega)}^2 + \|P_{gD}^{\eta,k}\|_{H^1(\Omega)}^2 + \|G_w^k\|_{L^2(\Gamma_N)}^2 + \|G_g^k\|_{L^2(\Gamma_N)}^2).$$

Collecting the estimates for Z_j, $j = 1, \dots, 10$ we get for arbitrary small $C_2 > 0$, $C_3 > 0$

$$
\begin{aligned}
&\frac{1}{h}\int_\Omega \Phi(H^\eta(S^k, P^k) - H^\eta(S^{k-1}, P^{k-1}))dx \\
&+ C_1 \int_\Omega (|\nabla P^k|^2 + |\nabla \beta^\eta(S^k)|^2 + \eta |\nabla S^k|^2)dx \\
&\le \frac{1}{h}\int_\Omega \Phi X_4^k dx + C_2(\int_\Omega |\nabla P^k|^2 dx + \int_\Omega |\nabla \beta^\eta(S^k)|^2 dx + \eta \int_\Omega |\nabla S^k|^2 dx) \\
&+ C_3 \int_\Omega |P^k|^2 dx + C_4(1 + \|P_{wD}^{\eta,k}\|^2_{L^2(0,T;H^1(\Omega))} + \|P_{gD}^{\eta,k}\|^2_{L^2(0,T;H^1(\Omega))} \\
&+ \|F_w^k\|^2_{L^2(\Omega)} + \|F_g^k\|^2_{L^2(\Omega)} + \|G_w^k\|^2_{L^2(\Gamma_N)} + \|G_g^k\|^2_{L^2(\Gamma_N)}).
\end{aligned}
$$

We multiply this inequality by h, sum it for $k = 1, \dots, N$, take into account (4.64) and use the Poincaré inequality for P to find

$$
\begin{aligned}
&\int_\Omega \Phi H^\eta(S^h, P^h)(T)dx + C_1 \int_Q (|\nabla P^h|^2 + |\nabla \beta^\eta(S^h)|^2 + \eta |\nabla S^h|^2)dxdt \\
&\le C_2(1 + \|F_w\|^2_{L^2(Q)} + \|F_g\|^2_{L^2(Q)} + \|P_D\|^2_{L^2(0,T;H^1(\Omega))} \\
&+ |||P_{wD}^\eta|||^2 + |||P_{gD}^\eta|||^2 + \|G_w\|^2_{L^2(\Gamma_N^T)} + \|G_g\|^2_{L^2(\Gamma_N^T)}) + \int_\Omega \Phi H^\eta(s^0, p^0)dx.
\end{aligned}
\tag{4.65}
$$

The first term in (4.65) is nonnegative due to (4.63), and the last term is uniformly bounded with respect to η which can be easily seen from the estimate (4.36). Further, all the other terms on the right-hand side of the inequality (4.65) are bounded, uniformly in η, which follows from (A.7), and from the uniform boundedness of the regularized phase pressure boundary data which is a consequence of (A.8), as commented in Remark 8. Now we employ the Poincaré inequality and the fact that $P_D, S_D \in L^2(0,T;H^1(\Omega))$, as well as $\beta^\eta(S_D)$ being bounded in $L^2(0,T;H^1(\Omega))$ independently of η, which is assured by (4.44) (see Remark 8). Therefore, the uniform estimate (4.55) is established.

To prove (4.56), we first note that the functions S^h, $\tilde{S}^h$, r_g^h and $\tilde{r}_g^h$ are uniformly bounded in $L^\infty(Q)$. Next, using the relations (4.27) and (4.28) we easily obtain

$$
|\nabla r_g^h| \le C_\eta(|\nabla P^h| + |\nabla S^h|), \quad |\nabla \tilde{S}^h| \le C|\nabla S^h|, \quad |\nabla \tilde{r}_g^h| \le C_\eta(|\nabla P^h| + |\nabla S^h|).
$$

These estimates brought together with the uniform bound (4.55) yield the estimate (4.56). Eventually, for the time derivative of $\Phi \tilde{S}^h$ we obtain from the variational equation (4.49), for any $\varphi \in l_h(V)$:

$$-\rho_w \int_Q \Phi\partial_t(\Phi\tilde{S}^h)\varphi dxdt = -\int_Q [\Lambda_w^\eta(S^h, P^h)\mathbb{K}\nabla P^h \cdot \nabla\varphi + A^\eta(S^h, P^h)\mathbb{K}\nabla S^h \cdot \nabla\varphi]dxdt$$
$$+ \int_Q \lambda_w(S^h)\rho_w^2\mathbb{K}\mathbf{g} \cdot \nabla\varphi dxdt + \int_Q F_w^h\varphi dxdt - \int_{\Gamma_N^T} G_w^h\varphi d\sigma dt. \tag{4.66}$$

By using the estimate (4.55), the boundedness of the coefficients independently of h and η, and the density of $\cup_{h>0} l_h(V)$ in $L^2(0,T;V)$, one establishes the first estimate in (4.57). The estimate on the time derivative of $\Phi\tilde{r}_g^h$ is obtained analogously, using (4.50). This completes the proof of Proposition 2. □

Let us re-emphasize that the uniform bounds obtained in Proposition 2 are also independent of η, which will be the key point in establishing uniform a priori estimates for the solutions of the non-degenerate problem in Proposition 4.

4.6 Proof of Theorem 6

In this Section we will perform passage to the limit as $h \to 0$ in the discrete problem. Since one does not establish the almost everywhere convergence of the global pressure in the whole domain Q, in order to identify the limit of the function $\rho_g(P_g^\eta(S^h, P^h))S^h$, we need to use an auxiliary result that is now presented. This result covers also an analogous problem which occurs when passing to the limit as $\eta \to 0$ in Section 4.7.

Lemma 5 *Let $\eta > 0$ be fixed and let $(S^\varepsilon)_\varepsilon$, $(P^\varepsilon)_\varepsilon$ be sequences satisfying as $\varepsilon \to 0$:*

i) $S^\varepsilon \to S$ a.e. in Q; $0 \le S^\varepsilon \le 1$ a.e. in Q;

ii) $P^\varepsilon \rightharpoonup P$ in $L^2(Q)$;

iii) $\rho_g(P_g^\eta(S^\varepsilon, P^\varepsilon))S^\varepsilon \to r_g^\eta$ a.e. in Q.

Then $r_g^\eta = \rho_g(P_g^\eta(S, P))S$. The same is true if $P_g^\eta(S, P)$ is replaced by $P_g(S, P)$, and r_g^η by r_g.

Proof. *Let us denote $Q^+ = \{(x,t) \in Q : S(x,t) > 0\}$ and $Q^0 = \{(x,t) \in Q : S(x,t) = 0\}$. Consider first the case $S > 0$. From i) and iii) we conclude that, as $\varepsilon \to 0$,*

$$\rho_g(P_g^\eta(S^\varepsilon, P^\varepsilon)) \to \frac{r_g}{S} \text{ a.e. in } Q^+$$

and

$$P_g^\eta(S^\varepsilon, P^\varepsilon) \to (\rho_g)^{-1}(\frac{r_g}{S}) \ a.e. \ in \ Q^+,$$

due to the smoothness and monotonicity of ρ_g. Using the boundedness of the function $\lambda_w(S)P_c'(S)$ in neighborhood of $S=1$ (which is a consequence of the assumption (4.12) *in* (*A.4*)*) and* (*A.3*)*,* (*A.6*)*, for any S_1, S_2 we obtain*

$$|P_g^\eta(S_1, P) - P_g^\eta(S_2, P)| \le C(|P_c^\eta(S_1) - P_c^\eta(S_2)| + |S_1 - S_2|)$$

and therefore

$$P_g^\eta(S, P^\varepsilon) \to (\rho_g)^{-1}(\frac{r_g}{S}) \ a.e. \ in \ Q^+.$$

Since $P \mapsto P_g^\eta(S, P)$ is invertible (see (4.39)*), we have $P^\varepsilon \to X$ a.e. in Q^+, for some X. From ii) we have $X = P$ so $r_g = \rho_g(P_g^\eta(S, P))S$ a.e. in Q^+. On the other hand, if $S = 0$, then the boundedness of ρ_g and i) imply*

$$\rho_g(P_g^\eta(S^\varepsilon, P^\varepsilon))S^\varepsilon \to r_g = 0 = \rho_g(P_g^\eta(S, P))S \ a.e. \ in \ Q^0.$$

The same argument holds if P_g^η is replaced by P_g. □

Before stating the results of convergence for the weak solutions of the discrete problem, we cite a compactness lemma from [12, Lemma 7] which is going to be applied to establish the strong convergence of the sequences S^h and r_g^h as $h \to 0$. Indeed, [12, Lemma 7] is a generalization of a classical compactness result from [95] (cf. [71]), and its proof is based on the simple modification of Lemma 8 in [95] (see [11, Lemma 8]).

Lemma 6 *(Lemma 7, [12]) Let Ω be a bounded open set and $Q = \Omega \times]0, T[$. Let $(r^h)_{h>0}$ be a family of functions in $L^2(Q)$ and let $\Phi \in L^\infty(\Omega)$ be such that $0 < \phi_m \le \Phi(x) \le \phi_M < +\infty$. Let $V \subseteq H^1(\Omega)$, dense in $L^2(\Omega)$ and $0 < \sigma < 1$, $p \ge 2$. Assume that $(r^h)_{h>0}$ satisfies:*

- *$(r^h)_{h>0}$ is uniformly bounded in $L^2(0, T; W^{\sigma,p}(\Omega))$;*
- *$(\partial_t(\Phi r^h))_{h>0}$ is uniformly bounded in $L^2(0, T; V')$.*

Then $(r^h)_{h>0}$ is relatively compact in $L^2(Q)$.

Now we obtain the convergence results holding as $h \to 0$ as well as the maximum principle for the saturation.

Proposition 3 *If the assumptions of Theorem 6 are satisfied then the following convergence results hold true as $h \to 0$, up to a subsequence:*

$$\|S^h - \tilde{S}^h\|_{L^2(Q)} + \|r_g^h - \tilde{r}_g^h\|_{L^2(Q)} \to 0, \tag{4.67}$$

$$S^h \to S \in L^2(0,T;V) + S_D \text{ weakly in } L^2(0,T;H^1(\Omega)), \text{ strongly in } L^2(Q), \tag{4.68}$$

$$r_g^h \to r_g = \rho_g(P_g^\eta(S,P))S \text{ strongly in } L^2(Q), \tag{4.69}$$

$$\beta^\eta(S^h) \rightharpoonup \beta^\eta(S) \in L^2(0,T;V) + \beta^\eta(S_D) \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and a.e. in } Q, \tag{4.70}$$

$$P^h \rightharpoonup P \in L^2(0,T;V) + P_D \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{4.71}$$

Moreover, $0 \leq S \leq 1$ *a.e. in* Q *and*

$$\partial_t(\Phi\tilde{S}^h) \rightharpoonup \partial_t(\Phi S) \quad \text{weakly in } L^2(0,T;V'), \tag{4.72}$$

$$\partial_t(\Phi\tilde{r}_g^h) \rightharpoonup \partial_t(\Phi r_g) \quad \text{weakly in } L^2(0,T;V'). \tag{4.73}$$

Proof. In order to prove (4.67) we employ the test functions $\varphi = (S^{k-1} - S^k)\zeta$ in (4.51) and $\psi = (r_g^k - r_g^{k-1})\xi$ in (4.52), where ζ and ξ are arbitrary positive functions in $C_0^1(\overline{\Omega})$, used to eliminate the non-homogenous boundary conditions (cf. [12, Proposition 4] and also [75, Proposition 3.3]). From (4.51) we hence obtain

$$\begin{aligned}
&\frac{\rho_w}{h}\int_\Omega \Phi(S^{k-1} - S^k)^2\zeta dx \\
&\qquad = -\int_\Omega [\Lambda_w^\eta(S^k,P^k)\mathbb{K}\nabla P^k - A^\eta(S^k,P^k)\mathbb{K}\nabla S^k]\cdot\nabla((S^{k-1} - S^k)\zeta)dx \\
&\qquad + \int_\Omega [\lambda_w(S^k)\rho_w^2\mathbb{K}\mathbf{g}\cdot\nabla((S^{k-1} - S^k)\zeta) + F_w^k(S^{k-1} - S^k)\zeta]dx - \int_{\Gamma_N} G_w^k(S^{k-1} - S^k)\zeta d\sigma.
\end{aligned} \tag{4.74}$$

Since the coefficients are bounded, one proceeds from (4.74) as follows:

$$\begin{aligned}
\frac{\phi_m\rho_w}{h}\int_\Omega (S^{k-1} - S^k)^2\zeta dx &\leq C(1 + \|\nabla P^k\|_{L^2(\Omega)} + \|\nabla S^k\|_{L^2(\Omega)})\|\nabla((S^{k-1} - S^k)\zeta)\|_{L^2(\Omega)} \\
&\quad + C(\|F_w^k\|_{L^2(\Omega)} + \|G_w^k\|_{L^2(\Gamma_N)})\,\|(S^{k-1} - S^k)\zeta\|_{L^2(\Omega)}.
\end{aligned}$$

This leads to the further estimate

$$\begin{aligned}
&\frac{\phi_m\rho_m}{h}\|(S^{k-1} - S^k)\sqrt{\zeta}\|_{L^2(\Omega)}^2 \\
&\leq C(1 + \|\nabla P^k\|_{L^2(\Omega)} + \|\nabla S^k\|_{L^2(\Omega)})(\|\nabla S^k\|_{L^2(\Omega)} + \|\nabla S^{k-1}\|_{L^2(\Omega)}) \\
&+ C(\|F_w^k\|_{L^2(\Omega)}^2 + \|G_w^k\|_{L^2(\Gamma_N)}^2),
\end{aligned}$$

and, after multiplying by h and summing for k from 1 to N, to

$$\phi_m\rho_m\sum_{k=1}^{N}\|(S^{k-1}-S^k)\sqrt{\zeta}\|^2_{L^2(\Omega)} \le C\big(1+\|\nabla P^h\|^2_{L^2(Q)}+2\|S^h\|^2_{L^2(Q)}+2\|\nabla S^h\|^2_{L^2(Q)}$$
$$+\|s_0\|^2_{L^2(\Omega)}+\|\nabla s_0\|^2_{L^2(\Omega)}+\|F^h_w\|^2_{L^2(Q)}+\|G^h_w\|^2_{L^2(\Gamma^T_N)}\big).$$

By applying a priori estimates from Proposition 2 and using $s_0 \in H^1(\Omega)$, we conclude that, for a constant C independent of h and η,

$$\sum_{k=1}^{N}\|(S^{k-1}-S^k)\sqrt{\zeta}\|^2_{L^2(\Omega)} \le C.$$

Furthermore, it is easy to compute that for bounded ζ it holds

$$\|(S^h-\tilde{S}^h)\zeta\|^2_{L^2(Q)} = \frac{h}{3}\sum_{k=1}^{N}\|(S^k-S^{k-1})\zeta\|^2_{L^2(\Omega)},$$

which means that

$$\|(S^h-\tilde{S}^h)\sqrt{\zeta}\|_{L^2(Q)} \to 0 \text{ as } h \to 0.$$

Due to the positivity of ζ in Ω we finally obtain, possibly along a subsequence, employing the Lebesgue thorem,

$$\|S^h-\tilde{S}^h\|_{L^2(Q)} \to 0 \text{ as } h \to 0.$$

The second estimate in (4.67) can be derived analogously.

The weak convergence results for the sequences P^h, S^h and $\beta^\eta(S^h)$ in $L^2(0,T;H^1(\Omega))$ follow from the a priori estimates in Proposition 2. Since $P^h_D \rightharpoonup P_D$ in $L^2(0,T;H^1(\Omega))$, we can conclude $P \in L^2(0,T;V)+P_D$. In the same manner it follows that $S \in L^2(0,T;V)+S_D$ and $\theta \in L^2(0,T;V)+\theta_D$, where θ is the weak limit of $\beta^\eta(S^h)$.

Next, the uniform estimates (4.56) and (4.55) in Proposition 2 allow us to apply Lemma 6 to a sequence $\tilde{S}^h$ and to obtain the relative compactness of $\tilde{S}^h$ in $L^2(Q)$. From (4.67) it follows that S^h is relatively compact in $L^2(Q)$. Combined with the weak convergence of S^h to S, this completes the proof of (4.68). The fact that $0 \le S^h \le 1$ a.e. in Q implies that $0 \le S \le 1$ a.e. in Q.

The limit of $\beta^\eta(S^h)$ is identified using a.e. convergence of S^h.

Using the same arguments as for $\tilde{S}^h$, we obtain that $\tilde{r}^h_g \to r_g$ strongly in $L^2(Q)$ and also $r^h_g \to r_g$ strongly in $L^2(Q)$. Moreover, the limit r_g is identified by Lemma 5 as

$r_g = \rho_g(P_g^\eta(S, P))S$.

Finally, the existence of the weak limits (4.72) and (4.73) is a consequence of the estimate (4.57) and the limits are identified in a standard way. Therefore Proposition 3 is proved.

□

The final step in proving the existence of weak solutions of the regularized system (4.31)-(4.32) is to pass to the limit as $h \to 0$ in the discrete system (4.49)-(4.50). Since we have established the pointwise convergence of the global pressure P^h only on the set where the limit of saturation is strictly positive, in order to pass to the limit as $h \to 0$ in the nonlinear functions of S and P we are going to utilize their particular form which is recognized in the following lemma. Let us remark that this result will be used for passing to the limit as $\eta \to 0$ as well.

Lemma 7 *Let $F \in C([0, 1] \times \mathbb{R})$ and let there exist functions $F_1, F_2 \in C([0, 1])$ such that $F_1(S) \leq F(S, P) \leq F_2(S)$ and $F_1(0) = F_2(0)$. Denote $Q^+ = \{(x, t) \in Q : S(x, t) > 0\}$. Then for any two sequences $(S^\varepsilon)_\varepsilon$, $(P^\varepsilon)_\varepsilon$, such that, as $\varepsilon \to 0$, it holds*

i) $S^\varepsilon \to S$ a.e. in Q;

ii) $P^\varepsilon \to P$ a.e. in Q^+,

we have

$$F(S^\varepsilon, P^\varepsilon) \to F(S, P) \text{ a.e. in } Q. \tag{4.75}$$

Proof. For a.e. $(x, t) \in Q^+$, $F(S^\varepsilon, P^\varepsilon) \to F(S, P)$ because of the continuity of F and the pointwise convergences of its arguments S^ε and P^ε. When $S = 0$, one has due to the boundedness and the continuity of F_1 and F_2

$$F(S^\varepsilon, X) \to L := F_1(0) = F_2(0) = F(0, X) \text{ for any } X \in \mathbb{R},$$

and hence (4.75) is established. □

Remark 9 *It is easy to verify that Lemma 7 can be applied to all (nonlinear) coefficients in* (4.49)-(4.50) *and that these coefficients converge to the following limits a.e. in Q as*

$h \to 0:$

$$\Lambda_w^\eta(S^h,P^h) \to \Lambda_w^\eta(S,P), \quad A^\eta(S^h,P^h) \to A^\eta(S,P), \quad f_w(S^h,P^h) \to f_w(S,P),$$
$$\Lambda_g^\eta(S^h,P^h) \to \Lambda_g^\eta(S,P), \quad \lambda_g(S^h)\rho_g^\eta(S^h,P^h)^2 \to \lambda_g(S)\rho_g^\eta(S,P)^2,$$
$$\rho_g^\eta(S^h,P^h)f_g(S^h,P^h) \to \rho_g^\eta(S,P)f_g(S,P).$$

Now we employ the convergence results in Proposition 3 and Remark 9 to pass to the limit as $h \to 0$ in the discrete system (4.49)-(4.50) and obtain (4.45)-(4.46). Next, we conclude in a standard way that $\rho_g(P_g^\eta(S,P))S, S \in C([0,T];L^2(\Omega))$ and that the initial conditions are satisfied (see [12]). This completes the proof of Theorem 6.

4.7 Proof of Theorem 5

In this section we prove the existence of weak solutions for the degenerate problem. From now on we express again the dependence of the solution of the regularized problem on the parameter η. Our final step is passing to the limit as $\eta \to 0$ in the regularized problem (4.45), (4.46). Note that Theorem 6 on the existence of the weak solutions for the regularized problem holds assuming that the initial data p_0, s_0 belong to $H^1(\Omega)$. Therefore, in order to be able to apply Theorem 6, we will replace the initial conditions s_0 and p_0 by the regularized initial conditions s_0^η and p_0^η from $H^1(\Omega)$ such that $s_0^\eta \to s_0$ and $p_0^\eta \to p_0$ in $L^2(\Omega)$ and a.e. in Ω when η tends to zero.

Proposition 4 *For sufficiently small η, let $(P^\eta,S^\eta)_\eta$ be the sequence of solutions given by Theorem 6. Denote $P_g^\eta = P_g^\eta(S^\eta,P^\eta)$. The following bounds are valid, uniform with respect to η:*

$$\|P^\eta\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{4.76}$$
$$\|\beta^\eta(S^\eta)\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{4.77}$$
$$\|\sqrt{\eta}\nabla S^\eta\|_{L^2(Q)^d} \le C, \tag{4.78}$$
$$\|\partial_t(\Phi S^\eta)\|_{L^2(0,T;V')} + \|\partial_t(\Phi\rho_g(P_g^\eta)S^\eta)\|_{L^2(0,T;V')} \le C. \tag{4.79}$$

Proof. We pass to the limit as $h \to 0$ in the estimate (4.65) which is uniform in η, and then make use of the weak lower semicontinuity of the seminorms $f \mapsto \int_Q |\nabla f|^2 dxdt$ to obtain

$$
\begin{aligned}
&\int_Q (|\nabla P^\eta|^2 + |\nabla \beta^\eta(S^\eta)|^2)\,dxdt + \eta \int_Q |\nabla S^\eta|^2 dxdt \\
&\le C(1 + \|F_w\|^2_{L^2(Q)} + \|F_g\|^2_{L^2(Q)} + \|P_D\|^2_{L^2(0,T;H^1(\Omega))} + |||P^\eta_{wD}|||^2 + |||P^\eta_{gD}|||^2 \\
&+ \|G_w\|^2_{L^2(\Gamma^T_N)} + \|G_g\|^2_{L^2(\Gamma^T_N)}).
\end{aligned}
$$

Using Remark 8 and the Poincaré inequality, (4.76), (4.77) and (4.78) follow immediately. The uniform estimates for the time derivatives of the functions $(\Phi S^\eta)_\eta$ and $(\Phi\rho_g(P^\eta_g)S^\eta)_\eta$ follow in a standard way from the estimates (4.76)-(4.78) by setting an arbitrary $\varphi \in L^2(0,T;V)$ in the weak formulation (4.45)-(4.46). This completes the proof of Proposition 4. $\square$

The compactness results for the families $(S^\eta)_\eta$ and $(\rho_g(P^\eta_g(S^\eta, P^\eta))S^\eta)_\eta$ will follow from the two following Lemmas, which can be considered as special cases of Lemma 5 in [12] (see also [75], Lemma 4.3) and therefore we will not give the proofs.

Lemma 8 *For any $c > 0$ and for any $\eta_0 > 0$, the set*

$$
A^{c,\eta_0} = \{S\colon 0 < \eta \le \eta_0, \quad \|\beta^\eta(S)\|_{L^2(0,T;H^1(\Omega))} \le c, \quad \|\partial_t(\Phi S)\|_{L^2(0,T;V')} \le c\}
$$

is relatively compact in $L^2(Q)$.

Lemma 9 *For any $c > 0$ and for any $\eta_0 > 0$, the set*

$$
\begin{aligned}
B^{c,\eta_0} = \{&\rho_g(P^\eta_g(S,P))S\colon 0 < \eta \le \eta_0, \quad \|P\|_{L^2(0,T;H^1(\Omega))} \le c, \\
&\|\beta^\eta(S)\|_{L^2(0,T;H^1(\Omega))} \le c, \quad \|\partial_t(\Phi\rho_g(P^\eta_g(S,P))S)\|_{L^2(0,T;V')} \le c\}
\end{aligned}
$$

is relatively compact in $L^2(Q)$.

The limit behavior as $\eta \to 0$ of the solutions to the regularized problem given by Theorem 6, as well as the maximum principle for the saturation, is described by the following result.

Lemma 10 *Let $\theta^\eta = \beta^\eta(S^\eta)$. The following convergence results hold, up to a subsequence,* *as* $\eta \to 0$.

$$P^\eta \rightharpoonup P \in L^2(0,T;V) + P_D \ \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{4.80}$$

$$\theta^\eta \rightharpoonup \theta \in L^2(0,T;V) + \theta_D \ \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and a.e. in } Q, \tag{4.81}$$

$$S^\eta \to \mathcal{S}(\theta) \ \text{ strongly in } L^2(Q), \tag{4.82}$$

$$\partial_t(\Phi S^\eta) \rightharpoonup \partial_t(\Phi \mathcal{S}(\theta)) \ \text{ weakly in } L^2(0,T;V'), \tag{4.83}$$

$$\partial_t(\Phi \rho_g(P_g^\eta(S^\eta, P^\eta))S^\eta) \rightharpoonup \partial_t(\Phi \rho_g(P_g(\mathcal{S}(\theta), P))\mathcal{S}(\theta)) \ \text{ weakly in } L^2(0,T;V'). \tag{4.84}$$

Moreover, $0 \le \theta \le \beta(1)$ *a.e. in* Q.

Proof. Weak convergences of (P^η), (θ^η), $\partial_t(\Phi S^\eta)$ and $\partial_t(\Phi \rho_g(P_g^\eta(S^\eta, P^\eta))S^\eta)$ follow from the uniform bounds (4.76), (4.77) and (4.79) in Proposition 4. Herein, the boundary values of the limits P and θ in Γ_D are kept equal to P_D and θ_D, respectively. Strong convergence of S^η to S in $L^2(Q)$ is a consequence of Lemma 8, and the limit can be identified as $S = \mathcal{S}(\theta)$. Moreover, Lemma 9 gives that

$$\rho_g(P_g^\eta(S^\eta, P^\eta))S^\eta \to L_g \ \text{ in } L^2(Q) \text{ and a.e. in } Q.$$

Taking into account the uniform convergence of P_g^η in (4.40), it follows

$$\rho_g(P_g(S^\eta, P^\eta))S^\eta \to L_g \ \text{ in } L^2(Q) \text{ and a.e. in } Q,$$

and the limit can be identified as $L_g = \rho_g(S,P)S$ by applying Lemma 5. Finally, this enables us to identify the limit given in (4.84) in a standard way. This completes the proof of Lemma 10.

□

Remark 10 *Using Lemma 7 we get the following convergence results a.e. in* Q, *as* $\eta \to 0$,

$$\Lambda_w(S^\eta, P^\eta) \to \Lambda_w(S,P), \quad A(S^\eta, P^\eta) \to A(S,P), \quad f_w(S^\eta, P^\eta) \to f_w(S,P),$$
$$\Lambda_g(S^\eta, P^\eta) \to \Lambda_g(S,P), \quad \lambda_g(S^\eta)\rho_g^\eta(S^\eta, P^\eta)^2 \to \lambda_g(S)\rho_g(S,P)^2,$$
$$\rho_g^\eta(S^\eta, P^\eta)f_g(S^\eta, P^\eta) \to \rho_g(S,P)f_g(S,P).$$

Finally, we insert a test function $\varphi \in C^1([0,T];V)$ such that $\varphi(T) = 0$ into (4.31). After the integration by parts in the time derivative term we obtain

$$\rho_w \int_Q \Phi S^\eta \partial_t \varphi dxdt + \int_Q [\Lambda_w^\eta(S^\eta, P^\eta)\mathbb{K}\nabla P^\eta \cdot \nabla\varphi - A(S^\eta, P^\eta)\mathbb{K}\nabla\theta^\eta \cdot \nabla\varphi]dxdt$$
$$- \int_Q \lambda_w(S^\eta)\rho_w^2 \mathbb{K}\mathbf{g} \cdot \nabla\varphi dxdt - \eta \int_Q \mathbb{K}\nabla S^\eta \cdot \nabla\varphi dxdt$$
$$= \int_Q F_w \varphi dxdt - \int_{\Gamma_N^T} G_w \varphi d\sigma dt - \rho_w \int_\Omega \Phi s_0^\eta \varphi(0) dx.$$

Here we take into account the definitions (4.29), (4.4) and (4.35) which give

$$A^\eta(S^\eta, P^\eta)\nabla S^\eta = A(S^\eta, P^\eta)\nabla\beta^\eta(S^\eta) + \eta\nabla S^\eta.$$

We can pass to the limit as $\eta \to 0$ in the nonlinear terms using pointwise convergence in Remark 10 and uniform convergence in Lemma 4. The penalisation term tends to zero due to (4.78). Thus we obtain

$$\rho_w \int_Q \Phi S \partial_t \varphi dxdt + \int_Q [\Lambda_w(S, P)\mathbb{K}\nabla P \cdot \nabla\varphi - A(S, P)\mathbb{K}\nabla\theta \cdot \nabla\varphi]dxdt$$
$$- \int_Q \lambda_w(S)\rho_w^2 \mathbb{K}\mathbf{g} \cdot \nabla\varphi dxdt$$
$$= \int_Q F_w \varphi dxdt - \int_{\Gamma_N^T} G_w \varphi d\sigma dt - \rho_w \int_\Omega \Phi s_0 \varphi(0) dx,$$

where $S = \mathcal{S}(\theta)$. In the same way, taking a test function $\psi \in C^1([0,T];V)$ with $\psi(T) = 0$ we get by integration by parts from (4.46)

$$- \int_Q \Phi \rho_g^\eta(S^\eta, P^\eta) S^\eta \partial_t \psi dxdt$$
$$+ \int_Q [\Lambda_g^\eta(S^\eta, P^\eta)\mathbb{K}\nabla P^\eta \cdot \nabla\psi + A(S^\eta, P^\eta)\mathbb{K}\nabla\theta^\eta \cdot \nabla\psi]dxdt$$
$$- \int_Q \lambda_g(S^\eta)\rho_g^\eta(S^\eta, P^\eta)^2 \mathbb{K}\mathbf{g} \cdot \nabla\psi dxdt + \eta \int_Q \mathbb{K}\nabla S^\eta \cdot \nabla\psi dxdt$$
$$= \int_Q F_g \psi dxdt - \int_{\Gamma_N^T} G_g \psi d\sigma dt + \int_\Omega \Phi \rho_g^\eta(s_0^\eta, p_0^\eta) s_0^\eta \psi(0) dx,$$

and after passing to the limit as $\eta \to 0$,

$$
\begin{aligned}
-\int_Q \Phi\rho_g(S,P)S\partial_t\psi dt &+ \int_Q [\Lambda_g(S,P)\mathbb{K}\nabla P\cdot\nabla\psi + A(S,P)\mathbb{K}\nabla\theta\cdot\nabla\psi]dxdt \\
&- \int_Q [\lambda_g(S)\rho_g(S,P)^2\mathbb{K}\mathbf{g}\cdot\nabla\psi - \rho_g(S,P)f_g(S,P)F_P\psi]dxdt \\
&= \int_Q F_g\psi dxdt - \int_{\Gamma_N^T} G_g\psi d\sigma dt + \int_\Omega \Phi\rho_g(s_0,p_0)s_0\psi(0)dx.
\end{aligned}
$$

Using the fact that the functions $\Phi\rho_g(S,P)S$ and ΦS belong to $C([0,T];V')$ and by an integration by parts, we easily conclude that the initial condition in Theorem 5 is satisfied and the proof of Theorem 5 is completed.

Chapter 5

Homogenization of immiscible compressible two-phase flow in a global pressure formulation

5.1 Introduction

In this Chapter we present a homogenization result for the model of immiscible compressible two-phase flow in porous media in the fully equivalent global pressure formulation, focusing our attention to a strongly heterogeneous porous media of a single-rock type with a periodic microstructure. A general case of two compressible fluids is considered, taking into account gravity and capillary effects. Under some realistic assumptions on the data, a nonlinear homogenized problem is obtained. The effective coefficients of the macroscopic model are described as solutions of a cell problem, and the upscaled model is rigorously mathematically derived by means of the two-scale convergence. This Chapter contains results from [10]. In Section 5.2 we formulate the problem describing the microscopic behavior for our model and we state the assumptions on data. For completeness, we quote the existence result for weak solutions of the microscopic problem obtained in [12]. The main result of the current Chapter is the homogenized model which is presented in Section 5.3. The a priori estimates with respect to the space and time variables of weak solutions of the microscopic problem are obtained in Section 5.4, and Section 5.5 is devoted to establishing the compactness results for weak solutions of the microscopic problem, which are needed to pass to the limit as $\varepsilon \to 0$ in the microscopic equations. Finally, the homogenization result is proven in Section 5.6, using the two-scale convergence technique.

5.2 Mathematical formulation

Our starting point is a microscopic model defined on a domain with a periodic microstructure. More precisely, we model a reservoir $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, as a bounded, connected, Lipschitz domain with a periodic structure. In such a geometrical configuration of the reservoir with strongly and quickly varying petrographic properties, we assume that the porosity and the absolute permeability tensor are rapidly oscillating functions of the microscopic scale $y = x/\varepsilon$, where x is the macroscopic scale and $\varepsilon > 0$ is a small parameter depicting the characteristic size of the periodicity blocks of the periodic reservoir. Namely, let $\Phi^\varepsilon(x) = \Phi(x/\varepsilon)$ and $\mathbb{K}^\varepsilon(x) = \mathbb{K}(x/\varepsilon)$ be the porosity and the absolute permeability of the porous medium, where Φ and $\mathbb{K}$ are Y-periodic functions of y and the unit cell is noted by $Y =]0, 1[^d$. Recall the notation $Q = \Omega \times]0, T[$, $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_i^T = \Gamma_i \times]0, T[$, $i \in \{D, N\}$.

The two compressible phases will be indicated by the subscripts w (wetting phase) and n (non-wetting phase) throughout the current Chapter. Let S_w^ε, $S^\varepsilon := S_n^\varepsilon$, P_w^ε, P_n^ε be the saturations of the wetting and the non-wetting phases, and the pressures of the wetting and the non-wetting phases, respectively. The global pressure is denoted by P^ε, and $\theta^\varepsilon = \beta(S^\varepsilon)$ is the saturation potential, where β is defined in (4.3). As presented in Subsection 2.4.2, the phase pressures are related to the global pressure and the non-wetting saturation by

$$P_n^\varepsilon := P_n^\varepsilon(S^\varepsilon, P^\varepsilon) = P^\varepsilon + P_c(0) + \int_0^{S^\varepsilon} f_w(s, P^\varepsilon) P_c'(s) ds, \tag{5.1}$$

$$P_w^\varepsilon := P_w^\varepsilon(S^\varepsilon, P^\varepsilon) = P_n^\varepsilon(S^\varepsilon, P^\varepsilon) - P_c(S^\varepsilon). \tag{5.2}$$

The microscopic equations for the compressible two-phase flow in a heterogenous porous medium in the global pressure formulation are given in Q by (see (2.34)-(2.35)):

$$\begin{aligned} \Phi^\varepsilon \frac{\partial}{\partial t}(\rho_w(S^\varepsilon, P^\varepsilon)(1 - S^\varepsilon)) - \operatorname{div}(\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon) + \operatorname{div}(A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon) \\ + \operatorname{div}(\lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \mathbf{g}) + \rho_w(S^\varepsilon, P^\varepsilon) f_w(S^\varepsilon, P^\varepsilon) F_P = \rho_w(S^\varepsilon, P^\varepsilon) S_w^* F_I, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \Phi^\varepsilon \frac{\partial}{\partial t}(\rho_n(S^\varepsilon, P^\varepsilon) S^\varepsilon) - \operatorname{div}(\Lambda_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon) - \operatorname{div}(A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon) \\ + \operatorname{div}(\lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \mathbf{g}) + \rho_n(S^\varepsilon, P^\varepsilon) f_n(S^\varepsilon, P^\varepsilon) F_P = \rho_n(S^\varepsilon, P^\varepsilon) S_n^* F_I, \end{aligned} \tag{5.4}$$

where the nonlinear coefficients $\Lambda_j(S, P)$, $A(S, P)$ and $f_j(S, P)$ ($j \in \{w, n\}$) are given by (2.33), (4.4) and (2.28)-(2.27), respectively, and the source/sink terms are specified through the given injection and production rates F_I, $F_P \geq 0$ and a known saturation determining the composition of the injected fluid, denoted by S_j^* ($j \in \{w, n\}$). For convenience we are assuming that $S_n^* = 1$.

The boundary conditions for the system (5.3)-(5.4) are taken in the form

$$\theta^\varepsilon = 0, \quad P^\varepsilon = 0 \quad \text{on } \Gamma_D^T, \tag{5.5}$$

$$\mathbf{Q}_w^\varepsilon \cdot \mathbf{n} = \mathbf{Q}_n^\varepsilon \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N^T, \tag{5.6}$$

where $\mathbf{n}$ is the outward pointing unit normal to $\partial\Omega$ and

$$\mathbf{Q}_w^\varepsilon = \rho_w(P_w^\varepsilon)\mathbf{q}_w^\varepsilon = -\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla\theta^\varepsilon + \lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)^2\mathbb{K}^\varepsilon\mathbf{g},$$
$$\mathbf{Q}_n^\varepsilon = \rho_n(P_n^\varepsilon)\mathbf{q}_n^\varepsilon = -\Lambda_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla\theta^\varepsilon + \lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)^2\mathbb{K}^\varepsilon\mathbf{g}$$

are the phase mass fluxes with $\mathbf{q}_j^\varepsilon$ being the volumetric velocity of the j-phase, $j \in \{w, n\}$.

The initial conditions are given by

$$\theta^\varepsilon(x, 0) = \theta_0(x), \quad P^\varepsilon(x, 0) = p_0(x) \quad \text{in } \Omega. \tag{5.7}$$

Now we state the assumptions on the data which will assure the existence for weak solutions of the microscopic problem by the result of [12].

(H.1) The porosity $\Phi = \Phi(y)$ is an Y-periodic function which belongs to $L^\infty(Y)$, and there exist constants, $0 < \phi_m \le \phi_M < +\infty$, such that $0 < \phi_m \le \Phi(y) \le \phi_M$ a.e. in Y.

(H.2) The permeability tensor $\mathbb{K} = \mathbb{K}(y)$ is an Y-periodic function which belongs to $(L^\infty(Y))^{d\times d}$, and there exist constants $0 < k_m \le k_M < +\infty$, such that for almost all $y \in Y$ and all $\boldsymbol{\xi} \in \mathbb{R}^d$ it holds

$$k_m|\boldsymbol{\xi}|^2 \le \mathbb{K}(y)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \le k_M|\boldsymbol{\xi}|^2.$$

(H.3) The relative mobilities satisfy $\lambda_w, \lambda_n \in C([0, 1]; \mathbb{R}^+)$, $\lambda_w(S_w = 0) = 0$ and $\lambda_n(S = 0) = 0$; λ_j is a non decreasing function of S_j. Moreover, there exist constants $0 < \lambda_m \le \lambda_M < +\infty$ such that for all $S \in [0, 1]$

$$0 < \lambda_m \le \lambda_w(S) + \lambda_n(S) \le \lambda_M.$$

(H.4) There exist constants $p_{c,min} > 0$ and $M > 0$ such that the capillary pressure function

$S_n \mapsto P_c(S_n)$, $P_c \in C([0,1[;\mathbb{R}^+) \cap C^1(]0,1[;\mathbb{R}^+)$, for all $S \in]0,1[$ satisfy

$$P_c'(S) \geq p_{c,min} > 0, \tag{5.8}$$

$$P_c(S)(1-S) + \int_0^1 P_c(s)\,ds + \sqrt{\lambda_n(S)\lambda_w(S)}P_c'(S) \leq M. \tag{5.9}$$

(H.5) There exists $S^\# \in]0,1[$, $0 < \gamma$ and $M > 0$ such that for all $S \in]0, S^\#]$

$$S^{-\gamma}\lambda_n(S)(P_c(S) - P_c(0)) + S^{2-\gamma}P_c'(S) \leq M, \tag{5.10}$$

and for all $S \in [S^\#, 1[$

$$(1-S)^{2-\gamma}P_c'(S) \leq M. \tag{5.11}$$

(H.6) The functions ρ_w and ρ_n are non decreasing and belong to $C^1(\mathbb{R})$. Furthermore, there exist $\rho_m, \rho_M > 0$ such that for all $p \in \mathbb{R}$ it holds

$$\rho_m \leq \rho_w(p), \rho_n(p) \leq \rho_M, \quad 0 < \rho_w'(p), \rho_n'(p) \leq \rho_M.$$

(H.7) $F_I, F_P \in L^2(Q)$ and $F_I, F_P \geq 0$ a.e. in Q.

(H.8) There exist $0 < \tau < 1$ and $C > 0$ such that for all $S_1, S_2 \in [0,1]$

$$C\left|\int_{S_1}^{S_2} \sqrt{\lambda_n(s)\lambda_w(s)}\,ds\right|^\tau \geq |S_1 - S_2|.$$

(H.9) $\theta_0, p_0 \in L^2(\Omega)$, $0 \leq \theta_0 \leq \beta(1)$ a.e. in Ω.

Remark 11 *From (H.4) it easily follows (see [12]) that there exists a constant M, independent of ε, such that for all $S \in [0,1]$ and $P \in \mathbb{R}$,*

$$|P_n(S,P)| \leq |P| + M, \quad |\lambda_w(S)P_w(S,P)| \leq \lambda_w(S)|P| + M, \quad |(1-S)P_w(S,P)| \leq |P| + M. \tag{5.12}$$

As before, we set $V = \{u \in H^1(\Omega); u|_{\Gamma_D} = 0\}$.

According to [12], the existence result for weak solutions of the system (5.3)-(5.4) with the boundary and initial conditions (5.5)-(5.7) is given by the following theorem.

Theorem 7 *(Existence for fixed $\varepsilon > 0$, cf. [12]) Let (H.1)-(H.9) hold. Let $\varepsilon > 0$, then there exists $(P^\varepsilon, \theta^\varepsilon)$ satisfying*

$$P^\varepsilon \in L^2(0,T;V),\ \theta^\varepsilon \in L^2(0,T;V),\ 0 \le \theta^\varepsilon \le \beta(1)\ a.e.\ in\ Q,\ S^\varepsilon = \mathcal{S}(\theta^\varepsilon),$$
$$\Phi^\varepsilon \partial_t(\rho_w(S^\varepsilon, P^\varepsilon)(1 - S^\varepsilon)) \in L^2(0,T;V'), \quad \Phi^\varepsilon \partial_t(\rho_n(S^\varepsilon, P^\varepsilon)S^\varepsilon) \in L^2(0,T;V'),$$

for all $\varphi, \psi \in L^2(0,T;V)$

$$\int_0^T \langle \Phi^\varepsilon \partial_t(\rho_w(S^\varepsilon, P^\varepsilon)(1 - S^\varepsilon)), \varphi\rangle dt + \int_Q [\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon \cdot \nabla\varphi - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla\theta^\varepsilon \cdot \nabla\varphi] dxdt$$
$$- \int_Q [\lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \boldsymbol{g} \cdot \nabla\varphi - \rho_w(S^\varepsilon, P^\varepsilon) f_w(S^\varepsilon, P^\varepsilon) F_P \varphi] dxdt = 0, \tag{5.13}$$

$$\int_0^T \langle \Phi^\varepsilon \partial_t(\rho_n(S^\varepsilon, P^\varepsilon)S^\varepsilon), \psi\rangle dt + \int_Q [\Lambda_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon \cdot \nabla\psi + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla\theta^\varepsilon \cdot \nabla\psi] dxdt$$
$$- \int_Q [\lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \boldsymbol{g} \cdot \nabla\psi - \rho_n(S^\varepsilon, P^\varepsilon) f_n(S^\varepsilon, P^\varepsilon) F_P \psi] dxdt$$
$$= \int_Q \rho_n(S^\varepsilon, P^\varepsilon) F_I \psi dxdt. \tag{5.14}$$

Furthermore, for all $\psi \in V$ the functions

$$t \mapsto \int_\Omega \Phi^\varepsilon \rho_w(P_w(S^\varepsilon, P^\varepsilon))(1 - S^\varepsilon)\psi dx, \quad t \mapsto \int_\Omega \Phi^\varepsilon \rho_n(P_n(S^\varepsilon, P^\varepsilon))S^\varepsilon \psi dx$$

are continuous in $[0,T]$ and the initial conditions are satisfied in the following sense:

$$\left(\int_\Omega \Phi^\varepsilon \rho_w(P_w(S^\varepsilon, P^\varepsilon))(1 - S^\varepsilon)\psi dx\right)(0) = \int_\Omega \Phi^\varepsilon \rho_w(P_w(s_0, p_0))(1 - s_0)\psi dx,$$

$$\left(\int_\Omega \Phi^\varepsilon \rho_n(P_n(S^\varepsilon, P^\varepsilon))S^\varepsilon \psi dx\right)(0) = \int_\Omega \Phi^\varepsilon \rho_n(P_n(s_0, p_0))s_0 \psi dx,$$

where $s_0 = \mathcal{S}(\theta_0)$.

5.3 A homogenization result

We have already pointed out that we study the asymptotic behavior of the solution to the problem (5.3)-(5.4), (5.5)-(5.7) as $\varepsilon \to 0$. In particular, we are going to show that the effective model reads

$$\begin{aligned}\langle\Phi\rangle\frac{\partial}{\partial t}(\rho_w(S,P)(1-S)) - \operatorname{div}(\Lambda_w(S,P)\mathbb{K}^h\nabla P) + \operatorname{div}(A(S,P)\mathbb{K}^h\nabla\theta)\\ + \operatorname{div}(\lambda_w(S)\rho_w(S,P)^2\mathbb{K}^h\mathbf{g}) + \rho_w(S,P)f_w(S,P)F_P = 0;\end{aligned} \tag{5.15}$$

$$\begin{aligned}\langle\Phi\rangle\frac{\partial}{\partial t}(\rho_n(S,P)S) - \operatorname{div}(\Lambda_n(S,P)\mathbb{K}^h\nabla P) - \operatorname{div}(A(S,P)\mathbb{K}^h\nabla\theta)\\ + \operatorname{div}(\lambda_n(S)\rho_n(S,P)^2\mathbb{K}^h\mathbf{g}) + \rho_n(S,P)f_n(S,P)F_P = \rho_n(S,P)F_I,\end{aligned} \tag{5.16}$$

where $\langle u\rangle$ denotes the mean value of the function u over the unit cell Y and the constant homogenized tensor $\mathbb{K}^h$ is given by

$$\mathbb{K}^h_{ij} = \int_Y \mathbb{K}(y)\,(\nabla_y\chi_i(y)+\mathbf{e}_i)\,(\nabla_y\chi_j(y)+\mathbf{e}_j)\,dy. \tag{5.17}$$

Here, $\chi_i(y)$ (for $i=1,\dots,d$) is a solution of the cell problem

$$\begin{cases} -\quad \operatorname{div}_y\left(\mathbb{K}(y)(\nabla_y\chi_i(y)+\mathbf{e}_i)\right) = 0 \text{ in } Y,\\ \quad\ \chi_i(y)\ Y-\ \text{periodic}, \end{cases} \tag{5.18}$$

with $\mathbf{e}_i$ being the unit vector in the i-th direction.

The boundary conditions for the system (5.15)-(5.18) are

$$\theta = 0, \quad P = 0 \quad \text{on } \Gamma^T_D, \tag{5.19}$$

$$\mathbf{Q}_w\cdot\mathbf{n} = \mathbf{Q}_n\cdot\mathbf{n} = 0 \quad \text{on } \Gamma^T_N, \tag{5.20}$$

where

$$\begin{aligned}\mathbf{Q}_w &= -\Lambda_w(S,P)\mathbb{K}^h\nabla P + A(S,P)\mathbb{K}^h\nabla\theta + \lambda_w(S)\rho_w(S,P)^2\mathbb{K}^h\mathbf{g},\\ \mathbf{Q}_n &= -\Lambda_n(S,P)\mathbb{K}^h\nabla P - A(S,P)\mathbb{K}^h\nabla\theta + \lambda_n(S)\rho_n(S,P)^2\mathbb{K}^h\mathbf{g}.\end{aligned}$$

The initial conditions for the system (5.15)-(5.18) read

$$\theta(x,0) = \theta_0(x), \quad P(x,0) = p_0(x) \quad \text{in } \Omega. \tag{5.21}$$

Now we are ready to state the main result of this Chapter.

Theorem 8 *Let (H.1)-(H.9) hold. Let* $(P^\varepsilon,\theta^\varepsilon)$ *be a weak solution of the problem (5.3),* *(5.4), (5.5), (5.6), (5.7) and* $S^\varepsilon = \mathcal{S}^\varepsilon(\theta^\varepsilon)$*. Then, up to a subsequence, it holds*

$$\begin{aligned}
&P^\varepsilon \rightharpoonup P \text{ weakly in } L^2(0,T;V) \text{ and a.e. in } Q,\\
&\theta^\varepsilon \rightharpoonup \theta \text{ weakly in } L^2(0,T;V),\\
&S^\varepsilon \to S \text{ a.e. in } Q,\\
&\nabla P^\varepsilon(x,t) \overset{2s}{\rightharpoonup} \nabla P(x,t) + \nabla_y P_1(x,y,t),\\
&\nabla \theta^\varepsilon(x,t) \overset{2s}{\rightharpoonup} \nabla \theta(x,t) + \nabla_y \theta_1(x,y,t),
\end{aligned}$$

where

$$P_1(x,y,t) = \sum_{i=1}^{d} \left(\frac{\partial P}{\partial x_i}(x,t) - B(S,P) g_i \right) \chi_i(y), \tag{5.22}$$

$$\theta_1(x,y,t) = \sum_{i=1}^{d} \left(\frac{\partial \theta}{\partial x_i}(x,t) - E(S,P) g_i \right) \chi_i(y), \tag{5.23}$$

with $\chi_i(y)$ *being a solution of the cell problem (5.18), while the functions* B *and* E *are given by*

$$B(S,P) = \frac{\lambda_w(S)\rho_w(S,P)^2 + \lambda_n(S)\rho_n(S,P)^2}{\Lambda_w(S,P) + \Lambda_g(S,P)}, \tag{5.24}$$

$$E(S,P) = \frac{\lambda(S,P)\sqrt{\lambda_w(S)\lambda_n(S)}\omega(S,P)}{\Lambda_w(S,P) + \Lambda_g(S,P)} (\rho_n(S,P) - \rho_w(S,P)). \tag{5.25}$$

Finally, the pair (P,θ) *is a weak solution of the problem* (5.15)-(5.21) *and* $S = \mathcal{S}(\theta)$*.*

Here $\overset{2s}{\rightharpoonup}$ denotes the two-scale convergence which is defined in Section 3.2.1. Theorem 8 is proven in Section 5.6.

5.4 A priori estimates

In order to obtain the needed uniform estimates, we follow the choice of the test functions suggested in [72] (see also [75–77]). In this section, for simplicity, we assume that $P_c(0) = 0$. From now on, $C, C_1, \ldots$ denote generic constants that do not depend on ε. We will also use the notation

$$h_w(S^\varepsilon, P^\varepsilon) = \rho_w(S^\varepsilon, P^\varepsilon)(1 - S^\varepsilon) \text{ and } h_n(S^\varepsilon, P^\varepsilon) = \rho_n(S^\varepsilon, P^\varepsilon) S^\varepsilon.$$

Lemma 11 *Let* $(P^\varepsilon)_\varepsilon$ *and* $(\theta^\varepsilon)_\varepsilon$ *be the sequences of solutions to* (5.13)-(5.14) *and let* $S^\varepsilon =$

$\mathcal{S}(\theta^\varepsilon)$. *Then the following uniform bounds with respect to* ε *hold:*

$$\|P^\varepsilon\|_{L^2(0,T;V)} + \|\theta^\varepsilon\|_{L^2(0,T;V)} \le C, \tag{5.26}$$
$$\|\partial_t(\Phi^\varepsilon h_w(S^\varepsilon, P^\varepsilon))\|_{L^2(0,T;V')} + \|\partial_t(\Phi^\varepsilon h_n(S^\varepsilon, P^\varepsilon))\|_{L^2(0,T;V')} \le C. \tag{5.27}$$

Proof. In the same manner as in Section 4.5, we begin by quoting some relations that are going to be used throughout the proof. First, from the relations (5.1) and (5.2) and the definition of the function ω it follows that the gradients of the phase pressures and the gradient of the global pressure are related by (cf. (4.27)-(4.28)):

$$\omega(S^\varepsilon, P^\varepsilon)\nabla P^\varepsilon = \nabla P_w(S^\varepsilon, P^\varepsilon) + \frac{\lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)}{\lambda(S^\varepsilon, P^\varepsilon)}\nabla P_c(S^\varepsilon) \tag{5.28}$$
$$= \nabla P_n(S^\varepsilon, P^\varepsilon) - \frac{\lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)}{\lambda(S^\varepsilon, P^\varepsilon)}\nabla P_c(S^\varepsilon). \tag{5.29}$$

Next, using the relations (5.28) and (5.29) and the definition of the function β, we can obtain the following representations of the phase fluxes, without gravity term, (cf. (4.58)-(4.59)):

$$\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla\theta^\varepsilon = \lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P_w(S^\varepsilon, P^\varepsilon), \tag{5.30}$$
$$\Lambda_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla\theta^\varepsilon = \lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P_n(S^\varepsilon, P^\varepsilon), \tag{5.31}$$

as well as the equality (cf. (2.32), (4.60))

$$\begin{gathered}\lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P_w^\varepsilon \cdot \nabla P_w^\varepsilon + \lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P_n^\varepsilon \cdot \nabla P_n^\varepsilon = \\ \frac{\rho_w(S^\varepsilon, P^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)}{\lambda(S^\varepsilon, P^\varepsilon)}\mathbb{K}^\varepsilon\nabla\theta^\varepsilon \cdot \nabla\theta^\varepsilon + \omega(S^\varepsilon, P^\varepsilon)^2\lambda(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon \cdot \nabla P^\varepsilon,\end{gathered} \tag{5.32}$$

Following [72], as test-functions in the weak formulation (5.13)-(5.14) we choose

$$\varphi^\varepsilon = \int_0^{P_w^\varepsilon} \frac{dp}{\rho_w(p)}, \quad \psi^\varepsilon = \int_0^{P_n^\varepsilon} \frac{dp}{\rho_n(p)}.$$

After summing the obtained equations we get

$$
\begin{aligned}
&\int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} h_w(S^\varepsilon, P^\varepsilon)\varphi dx + \int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} h_n(S^\varepsilon, P^\varepsilon)\psi dx \\
&+ \int_\Omega \left(\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon\right) \frac{1}{\rho_w(S^\varepsilon, P^\varepsilon)} \cdot \nabla P_w^\varepsilon dx \\
&+ \int_\Omega \left(\Lambda_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon\right) \frac{1}{\rho_n(S^\varepsilon, P^\varepsilon)} \cdot \nabla P_n^\varepsilon dx \\
&= \int_\Omega \lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \mathbf{g} \cdot \nabla P_w^\varepsilon dx + \int_\Omega \lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \mathbf{g} \cdot \nabla P_n^\varepsilon dx \\
&- \int_\Omega \rho_w(S^\varepsilon, P^\varepsilon) f_w(S^\varepsilon, P^\varepsilon) F_P \varphi dx - \int_\Omega \rho_n(S^\varepsilon, P^\varepsilon) f_n(S^\varepsilon, P^\varepsilon) F_P \psi dx \\
&+ \int_\Omega \rho_n(S^\varepsilon, P^\varepsilon) F_I \psi dx.
\end{aligned}
\tag{5.33}
$$

Let us denote the integral terms in the expression (5.33) by $Z_1,\ Z_2, \ldots,\ Z_9$, respectively.

First, as in (4.62), an easy computation shows that (cf. [72])

$$
Z_1 + Z_2 = \int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} \mathcal{H}(S^\varepsilon, P^\varepsilon) dx,
$$

where

$$
\mathcal{H}(S^\varepsilon, P^\varepsilon) = h_w(S^\varepsilon, P^\varepsilon)\varphi + h_n(S^\varepsilon, P^\varepsilon)\psi - (1 - S^\varepsilon)P_w^\varepsilon - S^\varepsilon P_n^\varepsilon + \int_0^{S^\varepsilon} P_c(s) ds.
$$

Since $S^\varepsilon \geq 0$ a.e. in Q and $P_c \geq 0$, we have

$$
\begin{aligned}
\mathcal{H}(S^\varepsilon, P^\varepsilon) \geq & \left(h_w(S^\varepsilon, P^\varepsilon)\varphi - (1 - S^\varepsilon)P_w^\varepsilon\right) + \left(h_n(S^\varepsilon, P^\varepsilon)\psi - S^\varepsilon P_n^\varepsilon\right) \\
=& (1 - S^\varepsilon)\left(\rho_w(S^\varepsilon, P^\varepsilon) \int_0^{P_w^\varepsilon} \frac{dp}{\rho_w(p)} - P_w^\varepsilon\right) + S^\varepsilon \left(\rho_n(S^\varepsilon, P^\varepsilon) \int_0^{P_n^\varepsilon} \frac{dp}{\rho_n(p)} - P_n^\varepsilon\right).
\end{aligned}
$$

Using (H.6), it is easy to show that $\nu_j(p) = \rho_j(p) \int_0^p \frac{dz}{\rho_j(z)} - p \geq 0$, $j \in \{w, n\}$, and consequently

$$
\mathcal{H}(S^\varepsilon, P^\varepsilon) \geq 0 \text{ a.e. in } Q. \tag{5.34}
$$

Next, by applying the identities (5.30) and (5.31) and using (H.6) it follows that

$$
Z_3 + Z_4 \geq \frac{1}{\rho_M} \int_\Omega \left(\lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P_w^\varepsilon \cdot \nabla P_w^\varepsilon + \lambda_n(S^\varepsilon)\rho_n(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P_n^\varepsilon \cdot \nabla P_n^\varepsilon\right) dx.
$$

Combining this bound with the expression (5.32) and the assumptions (H.3) and (H.6) yields the following estimate:

$$Z_3 + Z_4 \geq C_1 \left(\int_\Omega |\nabla \theta^\varepsilon|^2 dx + \int_\Omega |\nabla P^\varepsilon|^2 dx \right),$$

where $C_1 = \min(\frac{\rho_m^2 k_m}{\rho_M^2 \lambda_M}, \frac{\rho_m \lambda_m \omega_m^2 k_m}{\rho_M})$.

From relations (5.28) and (5.29) it follows that the sum of the terms Z_5 and Z_6 can be written as

$$|Z_5 + Z_6| = |\int_\Omega \lambda(S^\varepsilon, P^\varepsilon)\omega(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \mathbf{g} \cdot \nabla P^\varepsilon dx| \leq \frac{C_1}{2} \int_\Omega |\nabla P^\varepsilon|^2 dx + C_2,$$

with $C_2 = \frac{(\lambda_M \rho_M \omega_M k_M |\mathbf{g}|)^2 |\Omega|}{2C_1}$.

In order to estimate the integrals Z_7, Z_8 and Z_9, we use (5.12). In this way we get

$$\begin{aligned} |Z_7 + Z_8| + |Z_9| \leq & \frac{\rho_M^2}{\lambda_m \rho_m^2} \int_\Omega F_P \left(\lambda_w(S^\varepsilon)|P_w(S^\varepsilon, P^\varepsilon)| + \lambda_n(S^\varepsilon)|P_n(S^\varepsilon, P^\varepsilon) - P_c(0)| \right) dx \\ & + \frac{\rho_M}{\rho_m} \int_\Omega F_I |P_n(S^\varepsilon, P^\varepsilon)| dx \leq C_3 \int_\Omega (F_P + F_I)(|P^\varepsilon| + 1) dx, \end{aligned}$$

with $C_3 = \max(\frac{\lambda_M \rho_M^2}{\lambda_m \rho_m^2}, \frac{\rho_M^2}{\lambda_m \rho_m^2}(M(1+\lambda_M) + \lambda_M P_c(0)), \frac{\rho_M}{\rho_m}, \frac{\rho_M}{\rho_m}(M + P_c(0)))$.

Finally, we collect all obtained estimates for Z_k, $k = 1, \ldots, 9$. Thus for a.e. $t \in]0, T[$ it follows that

$$\begin{aligned} \int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} \mathcal{H}(S^\varepsilon, P^\varepsilon) dx + C_1 \int_\Omega |\nabla \theta^\varepsilon|^2 dx + \frac{C_1}{2} \int_\Omega |\nabla P^\varepsilon|^2 dx \\ \leq C_2 + C_3 \int_\Omega (F_P + F_I)(|P^\varepsilon| + 1) dx. \end{aligned} \tag{5.35}$$

From (H.4) and the relation (5.12) it follows that $0 \leq \mathcal{H}(S, P) \leq C(|P| + 1)$ and hence $\int_\Omega \Phi^\varepsilon \mathcal{H}(s_0, p_0) dx \leq C$. Therefore, by integrating (5.35) we obtain

$$C_1 \int_0^T \int_\Omega |\nabla \theta^\varepsilon|^2 dx d\tau + \frac{C_1}{2} \int_0^T \int_\Omega |\nabla P^\varepsilon|^2 dx d\tau \leq C_4 \|P^\varepsilon\|_{L^2(Q)} + C_5,$$

which implies the estimate (5.26).

The uniform estimates for the time derivatives of the functions $\Phi^\varepsilon h_j(S^\varepsilon, P^\varepsilon)$ ($j \in \{w, n\}$) follow directly from the estimates (5.26) by setting an arbitrary $\varphi \in L^2(0, T; V)$ in the weak formulation (5.13)-(5.14). This completes the proof of Lemma 11. □

5.5 A compactness result

In this section we obtain compactness results that will be used in passing to the limit as $\varepsilon \to 0$ in the weak formulation (5.13)-(5.14). Compactness of the family $\{(h_w(S^\varepsilon, P^\varepsilon), h_n(S^\varepsilon, P^\varepsilon))\}_{\varepsilon>0}$ in $L^2(Q)$ is achieved in two steps: first we prove that the functions h_w and h_n are Hölder continuous with respect to θ and P, where we write $h_j(\theta, P) = h_j(\mathcal{S}(\theta), P)$, $j \in \{w, n\}$. Then we show that this allows us to apply the compactness result from [6] (Lemma 4.2)

Lemma 12 *There exists* $0 < \sigma < 1$ *such that for any* $\theta_i \in [0, \beta(1)]$*,* $P_i \in \mathbb{R}$*,* $i = 1, 2$*,*

$$|h_w(\theta_2, P_2) - h_w(\theta_1, P_1)| + |h_n(\theta_2, P_2) - h_n(\theta_1, P_1)| \leq C(|P_2 - P_1|^\sigma + |\theta_2 - \theta_1|^\sigma). \quad (5.36)$$

Proof. Following the idea from [12, Lemma 5], we choose arbitrary $P_i \in \mathbb{R}$, $S_i \in [0, 1]$ and denote $\theta_i = \beta(S_i)$, $i = 1, 2$. For the case $S_1 < S_2$ (and the opposite case is treated in the same way) we can get by using (H.6) and (4.17):

$$\begin{aligned}
&|h_w(\theta_2, P_2) - h_w(\theta_1, P_1)| \\
\leq &|(\rho_w(S_2, P_2) - \rho_w(S_2, P_1))(1 - S_2)| + |(\rho_w(S_2, P_1) - \rho_w(S_1, P_1))(1 - S_2)| \\
+ &|\rho_w(S_1, P_1)(S_2 - S_1)| \\
\leq &\rho_M \left(3 \min(1, |P_2 - P_1|) + (1 - S_2)| \int_{S_1}^{S_2} f_n(s, P_1) P_c'(s) ds| + |S_2 - S_1| \right).
\end{aligned}$$

For the second summand we have by applying (H.3), (H.4) and (H.6), for γ given by (H.5),

$$(1 - S_2)| \int_{S_1}^{S_2} f_n(s, P_1) P_c'(s) ds| \leq \begin{cases} C_1 |S_2 - S_1| & \text{if } S_1 < S_2 \leq S^\# \\ C_2 |S_2 - S_1|^\gamma & \text{if } S^\# \leq S_1 < S_2 \\ C_1 |S_2 - S_1| + C_2 |S_2 - S_1|^\gamma & \text{if } S_1 < S^\# < S_2. \end{cases} \quad (5.37)$$

In the summary we obtain the bound

$$|h_w(\theta_2, P_2) - h_w(\theta_1, P_1)| \leq C \left(\min(1, |P_2 - P_1|) + |S_2 - S_1| + |S_2 - S_1|^\gamma \right),$$

and finally it can be seen that

$$|h_w(\theta_2, P_2) - h_w(\theta_1, P_1)| \leq C \left(|P_2 - P_1|^\sigma + |\theta_2 - \theta_1|^\sigma \right),$$

where $\sigma = \tau\gamma$ if $\gamma \leq 1$, $\sigma = \tau$ if $\gamma > 1$, and $0 < \tau < 1$ is given in (H.8). The proof for h_n is similar. Thus (5.36) is proven.

□

Next we claim that the functions P^ε, S^ε converge a.e. in Q. We start the proof by obtaining the following compactness result for the sets $\{h_j(S^\varepsilon, P^\varepsilon) : \varepsilon > 0\}$, $j \in \{w, n\}$.

Lemma 13 *The set $\{h_j(S^\varepsilon, P^\varepsilon) : \varepsilon > 0\}$ is compact in $L^2(Q)$, $j \in \{w, n\}$.*

The proof of Lemma 13 is based on the application of a new compactness result obtained in [6, Lemma 4.2]. Let us recall this result.

Lemma 14 *(Lemma 4.2, [6]) Let the function $\Phi = \Phi(y)$ be a Y-periodic function, $\Phi \in L^\infty(Y)$, and there are positive constants ϕ_m, ϕ_M such that $0 < \phi_m \leq \Phi(y) \leq \phi_M < 1$ a.e. in Y, $\Phi^\varepsilon(x) = \Phi(\frac{x}{\varepsilon})$ and let $\{v^\varepsilon\}_{\varepsilon>0} \subset L^2(Q)$ be a family of functions satisfying the properties:*

1. *the function v^ε is uniformly bounded in the space $L^\infty(Q)$, i.e. $0 \leq v^\varepsilon \leq C$;*
2. *there exists a function ϖ such that $\varpi(\xi) \to 0$ as $\xi \to 0$ and the following inequality holds true:*
$$\int_Q |v^\varepsilon(x+h,t) - v^\varepsilon(x,t)|^2 dxdt \leq C\varpi(|h|);$$
3. *the function v^ε is such that*
$$\|\frac{\partial}{\partial t}(\Phi^\varepsilon v^\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \leq C.$$

Then the family $\{v^\varepsilon\}_{\varepsilon>0}$ is a compact set in $L^2(Q)$.

Proof. [Proof of Lemma 13] We only need to verify the condition 2. of Lemma 14. For $P^\varepsilon \in L^2(0,T;V)$, by Lemma 11, we have

$$\int_Q |P^\varepsilon(x+h,t) - P^\varepsilon(x,t)|^2 dxdt \leq h^2 \|\nabla P^\varepsilon\|^2_{L^2(Q)} \leq Ch^2.$$

Next, using the Hölder inequality we get for any $0 < \alpha < 1$

$$\int_Q |P^\varepsilon(x+h,t) - P^\varepsilon(x,t)|^{2\alpha} dxdt \leq C \left(\int_Q |P^\varepsilon(x+h,t) - P^\varepsilon(x,t)|^2 dxdt \right)^\alpha \leq Ch^{2\alpha}.$$

An analogous conclusion can be drawn for the function θ^ε. Now we use the result of Lemma 12 to obtain for $j \in \{w, n\}$

$$\begin{aligned}
&\int_Q |h_j(\theta^\varepsilon(x+h,t), P^\varepsilon(x+h,t)) - h_j(\theta^\varepsilon(x,t), P^\varepsilon(x,t))|^2 dxdt \\
&\leq C\Big(\int_Q |\theta^\varepsilon(x+h,t) - \theta^\varepsilon(x,t)|^{2\sigma} dxdt + \int_Q |P^\varepsilon(x+h,t) - P^\varepsilon(x,t)|^{2\sigma} dxdt\Big) \\
&\leq Ch^{2\sigma}.
\end{aligned}$$

Thus the assumptions of Lemma 14 are satisfied and the result of Lemma 13 follows directly. □

Corollary 1 *There exist functions $P, S \in L^2(Q)$ such that, up to a subsequence,*

$$P^\varepsilon \to P \ \text{a.e. in } Q, \tag{5.38}$$

$$S^\varepsilon \to S \ \text{a.e. in } Q. \tag{5.39}$$

Proof. From Lemma 13 we conclude there are functions $h_w, \ h_n \in L^2(Q)$ such that, up to a subsequence,

$$\begin{aligned}
&h_w(\theta^\varepsilon, P^\varepsilon) \to h_w \text{ strongly in } L^2(Q) \text{ and a.e. in } Q, \\
&h_n(\theta^\varepsilon, P^\varepsilon) \to h_n \text{ strongly in } L^2(Q) \text{ and a.e. in } Q.
\end{aligned}$$

Now we exploit the fact that the map G defined by

$$G(S^\varepsilon, P^\varepsilon) = (h_w(\theta^\varepsilon, P^\varepsilon), h_n(\theta^\varepsilon, P^\varepsilon)), \ \theta^\varepsilon = \beta(S^\varepsilon)$$

is a diffeomorphism from $[0,1] \times \mathbb{R}$ to $G([0,1] \times \mathbb{R})$ (for details see [12, 104]) so it has a continuous inverse. Therefore, almost everywhere in Q convergence of $h_j(\theta^\varepsilon, P^\varepsilon)$ (for $j \in \{w, n\}$) implies the convergences (5.38) and (5.39), as claimed.

□

5.6 The proof of the homogenization result

The goal of this section is to rigorously justify the convergence results for the homogenized problem (5.15)-(5.21) given by Theorem 8. In order to pass to the limit as $\varepsilon \to 0$

in the weak formulation (5.13)-(5.14), we use the a priori estimates and the compactness results of the previous two Sections. Moreover, we use the two-scale convergence technique which is presented in Subsection 3.2.1.

By $\mathcal{D}(Q)$ we denote the space of infinitely smooth and compactly supported functions in Q with values in $\mathbb{R}$. Recall also the notation $H = \{u \in H_p^1(Y) : \int_Y u dy = 0\} = H_p^1(Y)/\mathbb{R}$.

Now the theory results on the two-scale convergence cited in Subsection 3.2.1 will be applied to the uniform estimates for the functions P^ε, θ^ε and S^ε which were obtained in the previous sections. Taking into account also the compactness results for these functions from the Section 5.5, we will establish the convergence results for the sequence of solutions depending on ε which will allow us to pass to the limit as $\varepsilon \to 0$ in the weak formulation (5.13)-(5.14).

Namely, the a priori estimate (5.26) with the convergence results (5.38) and (5.39) implies that there exist $P, \theta \in L^2(0,T;V)$ such that, up to a subsequence,

$$P^\varepsilon \rightharpoonup P \text{ in } L^2(0,T;V), \quad \theta^\varepsilon \rightharpoonup \theta \text{ in } L^2(0,T;V)$$

and we can identify $\theta = \beta(S)$, where S is given by (5.39). Moreover, due to the a priori estimate (5.26) and Theorem 2, there exist functions

$$P_1(x,y,t),\ \theta_1(x,y,t) \in L^2(Q;H)$$

such that

$$\nabla P^\varepsilon(x,t) \overset{2s}{\rightharpoonup} \nabla P(x,t) + \nabla_y P_1(x,y,t), \tag{5.40}$$

$$\nabla \theta^\varepsilon(x,t) \overset{2s}{\rightharpoonup} \nabla \theta(x,t) + \nabla_y \theta_1(x,y,t). \tag{5.41}$$

We proceed by passing to the limit in the weak formulation (5.13)-(5.14). First, let us consider the equation (5.13). We take the test function of the form

$$\varphi_w(x, \frac{x}{\varepsilon}, t) = \varphi(x,t) + \varepsilon\zeta(x, \frac{x}{\varepsilon}, t) = \varphi(x,t) + \varepsilon\zeta_1(x,t)\zeta_2(\frac{x}{\varepsilon}),$$

in (5.13), where $\varphi \in \mathcal{D}(Q),\ \zeta_1 \in \mathcal{D}(Q),\ \zeta_2 \in C_p^\infty(Y)$, which yields

$$\begin{aligned}
&-\int_Q \Phi(\frac{x}{\varepsilon})\rho_w(S^\varepsilon, P^\varepsilon)(1-S^\varepsilon)\left[\partial_t\varphi(x,t) + \varepsilon\partial_t\zeta(x,\frac{x}{\varepsilon},t)\right] dxdt \\
&+\int_Q \Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}(\frac{x}{\varepsilon})\nabla P^\varepsilon \cdot [\nabla\varphi + \varepsilon\nabla_x\zeta + \nabla_y\zeta]\, dxdt \\
&-\int_Q A(S^\varepsilon, P^\varepsilon)\mathbb{K}(\frac{x}{\varepsilon})\nabla\theta^\varepsilon \cdot [\nabla\varphi + \varepsilon\nabla_x\zeta + \nabla_y\zeta]\, dxdt \\
&-\int_Q \lambda_w(S^\varepsilon)\rho_w(S^\varepsilon, P^\varepsilon)^2\mathbb{K}(\frac{x}{\varepsilon})\mathbf{g} \cdot [\nabla\varphi + \varepsilon\nabla_x\zeta + \nabla_y\zeta]\, dxdt \\
&+\int_Q \rho_w(S^\varepsilon, P^\varepsilon) f_w(S^\varepsilon, P^\varepsilon) F_P\left[\varphi(x,t) + \varepsilon\zeta(x,\frac{x}{\varepsilon},t)\right] dxdt = 0
\end{aligned} \tag{5.42}$$

Passing to the limit as $\varepsilon \to 0$ in nonlinear terms of the equation (5.42) can be accomplished using the following facts: the a priori estimates (5.26), the strong convergence results from Lemma 13, the two-scale convergence results (5.40), (5.41) and Lebesgue's theorem, with the latter result being applicable due to the almost everywhere convergence from Corollary 1, the continuity and the boundedness of the coefficients.

Thus, taking the two-scale limit in the equation (5.42) gives

$$\begin{aligned}
&-\langle\Phi\rangle\int_Q \rho_w(S,P)(1-S)\partial_t\varphi(x,t)dxdt \\
&+\int_Q\int_Y \Lambda_w(S,P)\mathbb{K}(y)(\nabla P + \nabla_y P_1(x,y,t)) \cdot [\nabla\varphi(x,t) + \nabla_y\zeta(x,y,t)]dydxdt \\
&-\int_Q\int_Y A(S,P)\mathbb{K}(y)(\nabla\theta + \nabla_y\theta_1(x,y,t)) \cdot [\nabla\varphi(x,t) + \nabla_y\zeta(x,y,t)]dydxdt \\
&-\int_Q\int_Y \lambda_w(S)\rho_w(S,P)^2\mathbb{K}(y)\mathbf{g} \cdot [\nabla\varphi + \nabla_y\zeta]dydxdt \\
&+\int_Q \rho_w(S,P) f_w(S,P) F_P\varphi(x,t)dxdt = 0.
\end{aligned} \tag{5.43}$$

Next, we consider the equation (5.14) in the same way. After taking the appropriate test function, we pass to the limit when $\varepsilon \to 0$. One gets

$$
\begin{aligned}
&- \langle \Phi \rangle \int_Q \rho_n(S,P) S \partial_t \varphi(x,t) dxdt \\
&+ \int_Q \int_Y \Lambda_n(S,P) \mathbb{K}(y)(\nabla P + \nabla_y P_1(x,y,t)) \cdot [\nabla \varphi(x,t) + \nabla_y \zeta(x,y,t)] dydxdt \\
&+ \int_Q \int_Y A(S,P) \mathbb{K}(y)(\nabla \theta + \nabla_y \theta_1(x,y,t)) \cdot [\nabla \varphi(x,t) + \nabla_y \zeta(x,y,t)] dydxdt \\
&- \int_Q \int_Y \lambda_n(S) \rho_n(S,P)^2 \mathbb{K}(y) \mathbf{g} \cdot [\nabla \varphi + \nabla_y \zeta] dydxdt \\
&+ \int_Q \rho_n(S,P) f_n(S,P) F_P \varphi(x,t) dxdt - \int_Q \rho_n(S,P) F_I \varphi(x,t) dxdt = 0.
\end{aligned} \tag{5.44}
$$

Our next goal is a separation of variables for the functions P_1 and θ_1. To this end, we set $\varphi \equiv 0$ in the equations (5.43) and (5.44). After summing the two equations one gets

$$
\int_Y \mathbb{K}(y)(\nabla P(x,t) - B(S,P)\mathbf{g}) \cdot \nabla_y \zeta_2(y) dy = - \int_Y \mathbb{K}(y) \nabla_y P_1(x,y,t) \cdot \nabla_y \zeta_2(y) dy, \tag{5.45}
$$

where the coefficient B is given by (5.24). From this equation we obtain P_1 in form given by (5.22).

Finally, by setting $\varphi \equiv 0$ in (5.43) and by using (5.24) we get

$$
\begin{aligned}
&\big(\Lambda_w(S,P)B(S,P) - \lambda_w(S)\rho_w(S,P)^2\big) \int_Y \mathbb{K}(y)\mathbf{g} \cdot \nabla_y \zeta_2(y) dy \\
&= A(S,P) \left(\int_Y \mathbb{K}(y) \nabla \theta(x,t) \cdot \nabla_y \zeta_2(y) dy + \int_Y \mathbb{K}(y) \nabla_y \theta_1(x,y,t) \cdot \nabla_y \zeta_2(y) dy \right).
\end{aligned}
$$

Denoting $\Lambda(S,P) = \Lambda_w(S,P) + \Lambda_g(S,P)$ and using the fact that

$$
\Lambda_w(S,P)B(S,P) - \lambda_w(S)\rho_w(S,P)^2 = \frac{\Lambda_w(S,P)\Lambda_n(S,P)}{\omega(S,P)\Lambda(S,P)} (\rho_n(S,P) - \rho_w(S,P))
$$

and the expression (5.25), we can write

$$
\int_Y \mathbb{K}(y)(\nabla \theta(x,t) - E(S,P)\mathbf{g}) \cdot \nabla_y \zeta_2(y) dy = - \int_Y \mathbb{K}(y) \nabla_y \theta_1(x,y,t) \cdot \nabla_y \zeta_2(y) dy. \tag{5.46}
$$

This equation leads to the form of the function θ_1 given by the formula (5.23).

In our final step towards establishing the homogenized equations, we choose $\zeta_2 \equiv 0$ in (5.43) and (5.44), take into account the representations (5.22) and (5.23) and use the definition of the homogenized tensor $\mathbb{K}^h$ given by (5.17) and (5.18). In this way, for all

$\varphi \in \mathcal{D}(Q)$ we obtain

$$\langle\Phi\rangle \int_Q \partial_t(\rho_w(S,P)(1-S))\varphi(x,t)dxdt + \int_Q \Lambda_w(S,P)\mathbb{K}^h\nabla P(x,t)\cdot\nabla\varphi(x,t)dxdt$$
$$-\int_Q A(S,P)\mathbb{K}^h\nabla\theta(x,t)\cdot\nabla\varphi(x,t)dxdt - \int_Q \lambda_w(S)\rho_w(S,P)^2\mathbb{K}^h\mathbf{g}\cdot\nabla\varphi(x,t)dxdt$$
$$+\int_Q \rho_w(S,P)f_w(S,P)F_P(x,t)\varphi(x,t)dxdt = 0$$

and

$$\langle\Phi\rangle \int_Q \partial_t(\rho_n(S,P)S)\varphi(x,t)dxdt + \int_Q \Lambda_n(S,P)\mathbb{K}^h\nabla P(x,t)\cdot\nabla\varphi(x,t)dxdt$$
$$+\int_Q A(S,P)\mathbb{K}^h\nabla\theta(x,t)\cdot\nabla\varphi(x,t)dxdt - \int_Q \lambda_n(S)\rho_n(S,P)^2\mathbb{K}^h\mathbf{g}\cdot\nabla\varphi(x,t)dxdt$$
$$+\int_Q \rho_n(S,P)f_n(S,P)F_P(x,t)\varphi(x,t)dxdt - \int_Q \rho_n(S,P)F_I(x,t)\varphi(x,t)dxdt = 0.$$

This finishes the proof of Theorem 8.

□

Remark 12 *Combining the techniques of this Chapter and the Chapter 4, the homogenization results established in the current Chapter are also valid in the case where the wetting and the non-wetting phases are treated as incompressible and compressible, respectively. Namely, first we may derive the a priori estimates for the space derivatives of the functions P^ε and θ^ε, just as in the case of the two compressible fluids. On the other hand, the uniform estimates on time derivatives can be established for the functions $\Phi^\varepsilon S^\varepsilon$ and $\Phi^\varepsilon \rho_g(S^\varepsilon, P^\varepsilon)S^\varepsilon$. Thereafter, as in the Chapter 4, we can obtain a convergence of S^ε a.e. in Q, and a convergence of P^ε a.e. in the set where the limit of the saturation is strictly positive. This is enough to pass to the limit as $\varepsilon \to 0$ in the nonlinear terms due to their specific shape (see Lemma 7).*

Nevertheless, let us point out that in the case of the incompressible and compressible phases we could have also considered the non-homogenous Dirichlet and Neumann boundary data, due to the existence result in Chapter 4, Theorem 5.

Chapter 6

A double porosity model for immiscible compressible two-phase flow

6.1 Introduction

This Chapter contains a new homogenization result for the system modelling a fully equivalent global pressure formulation for immiscible compressible two-phase flow in a fractured porous medium. More precisely, the objective of the current Chapter is to rigorously justify the homogenization process for a double porosity model for the new formulation of immiscible compressible two-phase flow, by using the notion of the two-scale convergence. The considered system consists of an incompressible wetting phase and a compressible non-wetting phase. The Chapter is organized in the following way. In Section 6.2 we set up the problem which describes the microscopic model, we state the assumptions on data and recall an existence result for weak solutions of the microscopic problem obtained in Chapter 4. Section 6.3 exhibits the double porosity homogenized model whose derivation is a central result of the present Chapter. To that aim, in Section 6.4 first we introduce the extension operator operating from the fracture subdomain to the whole domain, as in [1]. Then we obtain the a priori estimates for the weak solutions of the microscopic problem in regard to the space and time variables. In Section 6.5 the modes of convergence of the microscopic solutions to the solutions of the effective problem are stated. The requisite compactness results for the extended fracture solutions are established in Section 6.6; the two-scale convergence results for the weak solutions are formulated as well. These convergence results are used to pass to the limit as $\varepsilon \to 0$ in the microscopic equations, which is accomplished in Subsection 6.7.1 by using two-scale convergence arguments. However, the obtained effective equations for the flow in the fractures contain a non-explicit limit term. In order to identify this term, in Subsection 6.7.2 we employ the dilation operator introduced in Section 3.3.1. At the same time the effective equations for the matrix flow are obtained.

6.2 Mathematical formulation

A naturally fractured reservoir is represented by a bounded, two-connected domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, with a periodic structure. More precisely, Ω is a union of disjoint cube cell domains congruent to a standard cell Y. We take the unit cell $Y =]0,1[^d$ to consist of a compactly contained domain Y_m completely surrounded by a connected fracture domain Y_f, with a smooth internal boundary Γ_{fm} between the two media. Therefore it is $Y = Y_m \cup \Gamma_{fm} \cup Y_f$. The outward unit normal vector to Y_m is denoted by ν.

The periodic microstructure of a reservoir is depicted by a small parameter $\varepsilon > 0$ representing the typical (linear) size of a matrix block with respect to the size of Ω, and the fracture thickness is considered to be of order ε. Accordingly, for $\varepsilon > 0$ the reservoir Ω is assumed to be covered by the disjoint copies of εY shifted for the translations from $\varepsilon\mathcal{A}$, where $\mathcal{A}$ is an appropriate infinite lattice. The system of the matrix blocks in a reservoir, the fractured part of a reservoir and the matrix-fracture interface are denoted by Ω_m^ε, Ω_f^ε and Γ_{fm}^ε, respectively. Hence we have

$$\begin{aligned}
\Omega_m^\varepsilon &= \Omega \cap \bigcup_{c\in\mathcal{A}} \varepsilon(Y_m + c),\\
\Omega_f^\varepsilon &= \Omega \cap \bigcup_{c\in\mathcal{A}} \varepsilon(Y_f + c) = \Omega \setminus \overline{\Omega_m^\varepsilon},\\
\Gamma_{fm}^\varepsilon &= \Omega \cap \bigcup_{c\in\mathcal{A}} \varepsilon(\Gamma_{fm} + c) = \partial\Omega_f^\varepsilon \cap \partial\Omega_m^\varepsilon \cap \Omega.
\end{aligned}$$

We denote the outward unit normal vector to Ω_m^ε by ν^ε. In order to avoid technical details in relation with the reservoir boundary, it is supposed that the family of parameters ε is such that $\partial\Omega \subset \partial\Omega_f^\varepsilon$.

For any $\varepsilon > 0$ and any $x \in \Omega$, $c^\varepsilon(x)$ stands for the lattice translation point of the ε-cell domain containing x, that is, $c^\varepsilon : \Omega \to \varepsilon\mathcal{A}$ is well defined by considering $x \in \varepsilon Y + c^\varepsilon(x)$. More precisely, if $x \in \varepsilon(Y + k)$ for some $k \in \mathbb{Z}^d$, then $c^\varepsilon(x) = \varepsilon k$. Further, χ_r is the characteristic function of Y_r, $r \in \{f, m\}$, extended by Y-periodicity to $\mathbb{R}^d$, and set $\chi_r^\varepsilon(x) = \chi_r(\frac{x}{\varepsilon})$, $r \in \{f, m\}$. Hence one has $\Omega_m^\varepsilon = \{x \in \Omega;\ \chi_m^\varepsilon(x) = 1\}$ and $\Omega_f^\varepsilon = \{x \in \Omega;\ \chi_f^\varepsilon(x) = 1\}$.

We keep the notation $Q = \Omega \times]0, T[$, $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_i^T = \Gamma_i \times]0, T[$, $i \in \{D, N\}$. Additionally, it will be denoted $\Omega_r^{\varepsilon,T} = \Omega_r^\varepsilon \times]0, T[$, $r \in \{f, m\}$ and $\Gamma_{fm}^{\varepsilon,T} = \Gamma_{fm}^\varepsilon \times]0, T[$.

We start from a microscopic model describing the two-phase flow in a fractured reservoir. In view of the periodicity microstructure assumption and since the identical properties of the matrix blocks are assumed (see Subsection 3.3), the porosity and the absolute permeability of the matrix are taken to be the functions $\phi^\varepsilon(x) = \phi(\frac{x}{\varepsilon})$ and $k^\varepsilon(x) = k(\frac{x}{\varepsilon})$, respectively, where ϕ and k are Y-periodic functions. In the fractures, the porosity and the absolute permeability are denoted by $\Phi(x)$ and $\mathbb{K}(x)$, respectively. As argued in Subsection 3.3, the permeability in the matrix part is scaled by ε^2 which preserves the form of the matrix equations in the effective model [25, 39]. Hence we denote

$$\Phi^\varepsilon(x) = \chi_f^\varepsilon(x)\Phi(x) + \chi_m^\varepsilon(x)\phi^\varepsilon(x), \tag{6.1}$$

$$\mathbb{K}^\varepsilon(x) = \chi_f^\varepsilon(x)\mathbb{K}(x) + \varepsilon^2\chi_m^\varepsilon(x)k^\varepsilon(x). \tag{6.2}$$

The porosity Φ^ε and the absolute permeability $\mathbb{K}^\varepsilon$ of the reservoir are highly discontinuous across the boundary Γ_{fm}^ε. We denote the gravity vector by $\mathbf{g}$ and write (cf. Subsection 3.3)

$$\mathbf{g}^\varepsilon(x) = \chi_f^\varepsilon(x)\mathbf{g} + \chi_m^\varepsilon(x)\varepsilon^{-1}\mathbf{g}. \tag{6.3}$$

The system studied in this Chapter consists of an incompressible wetting phase (marked by the subscript w) and a compressible non-wetting phase (marked by g), for instance the immiscible flow of gas and water. We maintain the notation of Chapter 5: by S_w^ε, $S^\varepsilon := S_g^\varepsilon$, P_w^ε, P_g^ε, P^ε and θ^ε we denote the saturations of the wetting and the non-wetting phases, the wetting and the non-wetting phase pressures, the global pressure and the saturation potential, respectively. The wetting phase (water) is assumed incompressible ($\rho_w = \text{const.}$) and the non-wetting phase (gas) is compressible, $\rho_g = \rho_g(P_g^\varepsilon)$. Let us also denote

$$V^\varepsilon = V^\varepsilon(S^\varepsilon, P^\varepsilon) = \rho_g(S^\varepsilon, P^\varepsilon)S^\varepsilon.$$

We assume that the fluids have a constant viscosity. The notation for the unknown functions in a fractured porous medium is introduced by using the subscript f for the fracture functions and m for the matrix functions. Namely, for given $\varepsilon > 0$, the saturation and the pressure functions in a fractured porous medium are expressed in the following way:

$$\gamma^\varepsilon(x, t) = \chi_f^\varepsilon(x)\gamma_f^\varepsilon(x, t) + \chi_m^\varepsilon(x)\gamma_m^\varepsilon(x, t),$$

where γ stands for S, P, P_w, P_g, θ, V.

We recall now the microscopic equations describing the immiscible water-gas flow in porous media in the global pressure formulation (see (2.34)-(2.35) and (4.5)-(4.6)):

$$\begin{aligned} -\rho_w \Phi^\varepsilon \frac{\partial S^\varepsilon}{\partial t} - \operatorname{div}(\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon) + \operatorname{div}(A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon) \\ + \rho_w^2 \operatorname{div}(\lambda_w(S^\varepsilon)\mathbb{K}^\varepsilon \mathbf{g}^\varepsilon) = F_w, \end{aligned} \tag{6.4}$$

$$\begin{aligned} \Phi^\varepsilon \frac{\partial}{\partial t}(\rho_g(S^\varepsilon, P^\varepsilon)S^\varepsilon) - \operatorname{div}(\Lambda_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon) - \operatorname{div}(A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon) \\ + \operatorname{div}(\lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \mathbf{g}^\varepsilon) = F_g. \end{aligned} \tag{6.5}$$

The boundary conditions for this system are taken in the form

$$\theta^\varepsilon = 0, \quad P^\varepsilon = 0 \quad \text{on } \Gamma_D^T, \tag{6.6}$$

$$\mathbf{Q}_w^\varepsilon \cdot \mathbf{n} = \mathbf{Q}_g^\varepsilon \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N^T, \tag{6.7}$$

where $\mathbf{n}$ is the outward pointing unit normal to $\partial\Omega$ and

$$\begin{aligned} \mathbf{Q}_w^\varepsilon &= \rho_w \mathbf{q}_w^\varepsilon = -\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon + \lambda_w(S^\varepsilon)\rho_w^2 \mathbb{K}^\varepsilon \mathbf{g}^\varepsilon, \\ \mathbf{Q}_g^\varepsilon &= \rho_g(P_g^\varepsilon)\mathbf{q}_g^\varepsilon = -\Lambda_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon + \lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \mathbf{g}^\varepsilon \end{aligned}$$

are the phase mass fluxes with $\mathbf{q}_j$ being the volumetric velocity of the j-phase, $j = w, g$.

The initial conditions are given by

$$\theta^\varepsilon(x, 0) = \theta_0(x), \quad P^\varepsilon(x, 0) = p_0(x) \quad \text{in } \Omega. \tag{6.8}$$

Now we formulate the assumptions on the data which are going to ensure the existence of weak solutions for the problem (6.4)-(6.5) with the boundary and initial conditions (6.6)-(6.8) by Theorem 5 of Chapter 4. Namely, in a periodically fractured porous medium setting, the assumptions (A.1) and (A.2) regarding the porosity and the absolute permeability functions are going to be substituted by the following hypothesis:

(A.1-d) The fracture porosity Φ belongs to $L^\infty(\Omega)$ and the matrix porosity $\phi = \phi(y)$ is an Y-periodic function which belongs to $L^\infty(Y)$. Furthermore, there exist constants, $0 < \phi_m \leq \phi_M < +\infty$, such that $0 < \phi_m \leq \Phi(x) \leq \phi_M$ a.e. in Ω, and $0 < \phi_m \leq \phi(y) \leq \phi_M$ a.e. in Y.

(A.2-d) The fracture permeability tensor $\mathbb{K}$ belongs to $(L^\infty(\Omega))^{d\times d}$ and the matrix permeability tensor $k = k(y)$ is an Y-periodic function which belongs to $(L^\infty(Y))^{d\times d}$. Furthermore, there exist constants $0 < k_m \leq k_M < +\infty$, such that for almost all

$x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^d$ it holds:

$$k_m|\boldsymbol{\xi}|^2 \leq \mathbb{K}(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq k_M|\boldsymbol{\xi}|^2,$$

and for almost all $y \in Y$ and all $\boldsymbol{\xi} \in \mathbb{R}^d$ it holds:

$$k_m|\boldsymbol{\xi}|^2 \leq k(y)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq k_M|\boldsymbol{\xi}|^2.$$

Also, since the homogenous boundary conditions are considered in this Chapter, we replace the assumption (A.8) by

(A.8-d) The initial data $p_0, \theta_0 \in L^2(\Omega)$, $0 \leq \theta_0 \leq \beta(1)$ a.e. in Ω.

According to Theorem 5, under the assumptions (A.1-d) - (A.2-d), (A.3) - (A.7) and (A.8-d) for each $\varepsilon > 0$ the problem (6.4)-(6.5) with the boundary and initial conditions (6.6)-(6.8) has at least one weak solution given by the following theorem.

Recall the notation $V = \{u \in H^1(\Omega); u|_{\Gamma_D} = 0\}$.

Theorem 9 *(Existence for fixed $\varepsilon > 0$) Let (A.1-d) - (A.2-d), (A.3) - (A.7) and (A.8-d) hold; let $\varepsilon > 0$. Denote $S^\varepsilon = \mathcal{S}(\theta^\varepsilon)$. Then there exists $(P^\varepsilon, \theta^\varepsilon)$ such that*

$$P^\varepsilon \in L^2(0,T;V),\ \theta^\varepsilon \in L^2(0,T;V),\ 0 \leq \theta^\varepsilon \leq \beta(1) \ a.e.\ in\ Q,$$
$$\partial_t(\Phi^\varepsilon S^\varepsilon) \in L^2(0,T;V'),\ \partial_t(\Phi^\varepsilon \rho_g(S^\varepsilon, P^\varepsilon)S^\varepsilon) \in L^2(0,T;V');$$

for all $\varphi, \psi \in L^2(0,T;V)$

$$\begin{aligned} -\rho_w \int_0^T \langle \partial_t(\Phi^\varepsilon S^\varepsilon), \varphi \rangle dt + \int_Q [\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon \cdot \nabla\varphi - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla\theta^\varepsilon \cdot \nabla\varphi] dxdt \\ - \int_Q \lambda_w(S^\varepsilon)\rho_w^2 \mathbb{K}^\varepsilon \boldsymbol{g}^\varepsilon \cdot \nabla\varphi dxdt = \int_Q F_w \varphi dxdt, \end{aligned} \tag{6.9}$$

$$\begin{aligned} \int_0^T \langle \partial_t(\Phi \rho_g(S^\varepsilon, P^\varepsilon)S^\varepsilon), \psi \rangle dt + \int_Q [\Lambda_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon \cdot \nabla\psi + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla\theta^\varepsilon \cdot \nabla\psi] dxdt \\ - \int_Q \lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \boldsymbol{g}^\varepsilon \cdot \nabla\psi dxdt = \int_Q F_g \psi dxdt. \end{aligned} \tag{6.10}$$

Furthermore, for all $\psi \in V$ the functions

$$t \mapsto \int_\Omega \Phi^\varepsilon S^\varepsilon \psi dx, \quad t \mapsto \int_\Omega \Phi^\varepsilon \rho_g(P_g(S^\varepsilon, P^\varepsilon))S^\varepsilon \psi dx$$

are continuous in $[0,T]$ *and the initial conditions are satisfied in the following sense:*

$$\left(\int_\Omega \Phi^\varepsilon S^\varepsilon \psi dx\right)(0) = \int_\Omega \Phi^\varepsilon s_0 \psi dx,$$

$$\left(\int_\Omega \Phi^\varepsilon \rho_g(P_g(S^\varepsilon, P^\varepsilon)) S^\varepsilon \psi dx\right)(0) = \int_\Omega \Phi^\varepsilon \rho_g(P_g(s_0, p_0)) s_0 \psi dx,$$

where $s_0 = \mathcal{S}(\theta_0)$.

6.3 A homogenization result

The principal aim of the current Chapter is to show that, as the scaling parameter ε tends to 0, the microscopic model tends in some sense to the effective model which we now present. More precisely, in Section 6.7 we will prove that the weak solutions of the problem (6.4)-(6.5), (6.6)-(6.8) converge as $\varepsilon \to 0$ to weak solutions of the following problem.

The macroscopic equations for the fracture system are given in Q by

$$\begin{aligned} -\rho_w \Phi^H \frac{\partial S}{\partial t} - \operatorname{div}(\Lambda_w(S,P)\mathbb{K}^H \nabla P) + \operatorname{div}(A(S,P)\mathbb{K}^H \nabla \theta) \\ + \rho_w^2 \operatorname{div}(\lambda_w(S)\mathbb{K}^H \mathbf{g}) = F_w + \mathcal{Q}_w, \end{aligned} \tag{6.11}$$

$$\begin{aligned} \Phi^H \frac{\partial}{\partial t}(\rho_g(S,P)S) - \operatorname{div}(\Lambda_g(S,P)\mathbb{K}^H \nabla P) - \operatorname{div}(A(S,P)\mathbb{K}^H \nabla \theta) \\ + \operatorname{div}(\lambda_g(S)\rho_g(S,P)^2 \mathbb{K}^H \mathbf{g}) = F_g + \mathcal{Q}_g, \end{aligned} \tag{6.12}$$

where the matrix source terms are given for $(x,t) \in Q$ by

$$\mathcal{Q}_w(x,t) = \rho_w \int_{Y_m} \phi(y) \frac{\partial s}{\partial t}(x,y,t) dy \tag{6.13}$$

and

$$\mathcal{Q}_g(x,t) = -\int_{Y_m} \phi(y) \frac{\partial}{\partial t}(\rho_g(s(x,y,t), p(x,y,t)) s(x,y,t)) dy; \tag{6.14}$$

the homogenized fracture porosity Φ^H is given by

$$\Phi^H(x) = |Y_f| \Phi^*(x), \tag{6.15}$$

and the homogenized fracture system permeability tensor $\mathbb{K}^H$ is defined by

$$\mathbb{K}^H_{ij}(x) = \int_{Y_f} \mathbb{K}(x) \left(\nabla_y \chi_i(x,y) + \mathbf{e}_i\right) \left(\nabla_y \chi_j(x,y) + \mathbf{e}_j\right) dy. \tag{6.16}$$

Here, $\chi_i(x,y)$ (for $i = 1, \ldots, d$) is a solution of the cell problem

$$\begin{cases} - & \operatorname{div}_y \left(\mathbb{K}(x)(\nabla_y \chi_i(x,y) + \mathbf{e}_i)\right) = 0 \text{ in } Y_f, \\ & \mathbb{K}(x)(\nabla_y \chi_i(x,y) + \mathbf{e}_i) \cdot \nu = 0 \text{ on } \Gamma_{fm}, \\ & y \mapsto \chi_i(x,y)\ Y- \text{ periodic}, \end{cases} \tag{6.17}$$

with $\mathbf{e}_i$ being the unit vector in the i-th direction.

The boundary conditions for the system (6.11)-(6.17) are

$$\theta = 0, \quad P = 0 \quad \text{on } \Gamma^T_D, \tag{6.18}$$

$$\mathbf{Q}_w \cdot \mathbf{n} = \mathbf{Q}_g \cdot \mathbf{n} = 0 \quad \text{on } \Gamma^T_N, \tag{6.19}$$

where

$$\mathbf{Q}_w = -\Lambda_w(S,P)\mathbb{K}^H \nabla P + A(S,P)\mathbb{K}^H \nabla \theta + \lambda_w(S)\rho_w^2 \mathbb{K}^H \mathbf{g},$$

$$\mathbf{Q}_g = -\Lambda_g(S,P)\mathbb{K}^H \nabla P - A(S,P)\mathbb{K}^H \nabla \theta + \lambda_g(S)\rho_g(S,P)^2 \mathbb{K}^H \mathbf{g}.$$

The initial conditions for the system (6.11)-(6.17) read

$$\theta(x,0) = \theta_0(x), \quad P(x,0) = p_0(x) \quad \text{in } \Omega. \tag{6.20}$$

On the other hand, to each $x \in \Omega$ there is an associated matrix block congruent to Y_m. The flow equations in $\Omega \times Y_m \times]0,T[$, which generate the new source terms, are as follows:

$$-\rho_w \phi(y) \frac{\partial s}{\partial t}(x,y,t) - \operatorname{div}_y(\Lambda_w(s,p)k(y)\nabla_y p) + \operatorname{div}_y(A(s,p)k(y)\nabla_y \vartheta) + \rho_w^2 \operatorname{div}(\lambda_w(s)k(y)\mathbf{g}) = F_w, \tag{6.21}$$

$$\phi(y) \frac{\partial v}{\partial t}(x,y,t) - \operatorname{div}_y(\Lambda_g(s,p)k(y)\nabla_y p) - \operatorname{div}_y(A(s,p)k(y)\nabla_y \vartheta) + \operatorname{div}_y(\lambda_g(s)\rho_g(s,p)^2 k(y)\mathbf{g}) = F_g. \tag{6.22}$$

The system (6.21)-(6.22) is completed with the following boundary conditions:

$$\vartheta(x,y,t) = \theta(x,t), \quad p(x,y,t) = P(x,t) \quad \text{in } \Omega \times \Gamma_{fm} \times]0,T[, \tag{6.23}$$

and the initial conditions:

$$\vartheta(x,y,0) = \theta_0(x), \quad p(x,y,0) = p_0(x) \quad \text{in } \Omega \times Y_m. \tag{6.24}$$

A mode of convergence of the weak solutions for the microscopic problem to weak solutions of the effective problem is going to be specified in Section 6.5.

6.4 A priori estimates

6.4.1 Extension of the fracture solutions

In Subsection 6.4.2 there will be obtained the a priori estimates for the solutions of the microscopic problem in the fractured part Ω_f^ε. In order to make use of these uniform estimates for establishing some additional a priori estimates as well as for deriving the compactness results for the fracture solutions, we will need to extend the functions P_f^ε, θ_f^ε, S_f^ε and V_f^ε to the whole fixed domain Ω. To that aim we are going to use the results of [1] which are presented in this Subsection.

In view of the setting introduced in Section 6.2, we note that for any $\varepsilon > 0$ the fracture domain $\Omega_f^\varepsilon = \Omega \cap \bigcup_{c \in \mathcal{A}} \varepsilon(Y_f + c)$, where

$$\bigcup_{c \in \mathcal{A}} (Y_f + c) \text{ is a periodic, connected and open subset of } \mathbb{R}^d. \tag{6.25}$$

Let us suppose additionally that

$$\bigcup_{c \in \mathcal{A}} (Y_f + c) \text{ has a Lipschitz boundary.} \tag{6.26}$$

For $a \in \mathbb{R}$, we denote

$$\Omega(\varepsilon a) = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon a\}.$$

According to Theorem 2.1 of [1], under the assumptions (6.25) and (6.26) there exists a linear and continuous extension operator $\Pi^\varepsilon : H^1(\Omega_f^\varepsilon) \to H^1_{loc}(\Omega)$ and constants

$k_0, k_1, k_2 > 0$ such that for all $u \in H^1(\Omega_f^\varepsilon)$ it holds

$$\Pi^\varepsilon u = u \text{ a.e. in } \Omega_f^\varepsilon,$$
$$\int_{\Omega(\varepsilon k_0)} |\Pi^\varepsilon u|^2 dx \le k_1 \int_{\Omega_f^\varepsilon} |u|^2 dx,$$
$$\int_{\Omega(\varepsilon k_0)} |\nabla(\Pi^\varepsilon u)|^2 dx \le k_2 \int_{\Omega_f^\varepsilon} |\nabla u|^2 dx.$$

The constants $k_i, i = 1, 2, 3$ depend on Y_f and d but are independent of ε and Ω.

For the sake of simplicity it will be assumed that there are no matrix blocks in an εk_0-neighborhood of $\partial\Omega$ and hence the extension results are valid in Ω [1]. Namely, for any $u \in H^1(\Omega_f^\varepsilon)$ we have

$$\int_\Omega |\Pi_\varepsilon u|^2 dx \le k_1 \int_{\Omega_f^\varepsilon} |u|^2 dx, \tag{6.27}$$
$$\int_\Omega |\nabla(\Pi_\varepsilon u)|^2 dx \le k_2 \int_{\Omega_f^\varepsilon} |\nabla u|^2 dx. \tag{6.28}$$

Moreover, if $C_1 < u < C_2$ a.e. in Ω_f^ε, then it is also $C_1 < \Pi_\varepsilon u < C_2$ a.e. in Ω [1].

Remark 13 *By Theorem 9, for any $\varepsilon > 0$ the solution $(P_f^\varepsilon, \theta_f^\varepsilon)$ to the microscopic problem belongs to $L^2(0, T; H^1(\Omega_f^\varepsilon))$ and satisfies the boundary condition* (6.6)*. Therefore, by the above arguments the functions P_f^ε and θ_f^ε can immediately be extended to*

$$\widetilde{P}_f^\varepsilon := \Pi_\varepsilon P_f^\varepsilon \in L^2(0, T; V), \tag{6.29}$$
$$\widetilde{\theta}_f^\varepsilon := \Pi_\varepsilon \theta_f^\varepsilon \in L^2(0, T; V). \tag{6.30}$$

However, since the gradients of the functions S_f^ε and V_f^ε do not belong to $L^2(\Omega_f^{\varepsilon,T})$, their extensions to Ω are defined indirectly by

$$\widetilde{S}_f^\varepsilon := \mathcal{S}(\widetilde{\theta}_f^\varepsilon) \tag{6.31}$$

and

$$\widetilde{V}_f^\varepsilon := \rho_g(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)\widetilde{S}_f^\varepsilon. \tag{6.32}$$

6.4.2 Uniform estimates

As in the corresponding parts of Sections 4.5 and 5.4, to obtain a priori estimates we employ suitable test functions, suggested in [72] (cf. [75–77]). In this Subsection, for simplicity, we assume that $P_c(0) = 0$. Throughout the proof, $C, C_1, \ldots$ denote generic constants that are independent of ε.

Proposition 5 *Let* $(P^\varepsilon, \theta^\varepsilon)_\varepsilon$ *be the sequence of solutions to* (6.9)-(6.10) *and let* $S^\varepsilon = \mathcal{S}(\theta^\varepsilon)$. *The following estimates, uniform with respect to* ε*, hold:*

$$\|P_f^\varepsilon\|_{L^2(0,T;H^1(\Omega_f^\varepsilon))} + \|\theta_f^\varepsilon\|_{L^2(0,T;H^1(\Omega_f^\varepsilon))} \le C, \tag{6.33}$$

$$\|P_m^\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} + \|\theta_m^\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} \le C, \tag{6.34}$$

$$\varepsilon\|\nabla P_m^\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} + \varepsilon\|\nabla\theta_m^\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} \le C, \tag{6.35}$$

$$\|\partial_t(\Phi^\varepsilon S^\varepsilon)\|_{L^2(0,T;V')} + \|\partial_t(\Phi^\varepsilon \rho_g(S^\varepsilon, P^\varepsilon)S^\varepsilon)\|_{L^2(0,T;V')} \le C. \tag{6.36}$$

Proof. At first we recall the relations (5.28) -(5.32) that were accentuated when obtaining the uniform estimates in Section 5.4, taking into account that herein the wetting phase mass density is constant, and the non-wetting (gas) phase is marked by the subscript g. In fact, the following relations are going to be employed in the sequel.

$$\omega(S^\varepsilon, P^\varepsilon)\nabla P^\varepsilon = \nabla P_w(S^\varepsilon, P^\varepsilon) + \frac{\lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)}{\lambda(S^\varepsilon, P^\varepsilon)}\nabla P_c(S^\varepsilon) \tag{6.37}$$

$$= \nabla P_g(S^\varepsilon, P^\varepsilon) - \frac{\lambda_w(S^\varepsilon)\rho_w}{\lambda(S^\varepsilon, P^\varepsilon)}\nabla P_c(S^\varepsilon), \tag{6.38}$$

$$\begin{aligned} &\lambda_w(S^\varepsilon)\rho_w\mathbb{K}^\varepsilon\nabla P_w^\varepsilon \cdot \nabla P_w^\varepsilon + \lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P_g^\varepsilon \cdot \nabla P_g^\varepsilon \\ &= \frac{\rho_w\rho_g(S^\varepsilon, P^\varepsilon)}{\lambda(S^\varepsilon, P^\varepsilon)}\mathbb{K}^\varepsilon\nabla\theta^\varepsilon \cdot \nabla\theta^\varepsilon + \omega(S^\varepsilon, P^\varepsilon)^2\lambda(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon \cdot \nabla P^\varepsilon, \end{aligned} \tag{6.39}$$

$$\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla\theta^\varepsilon = \lambda_w(S^\varepsilon)\rho_w\mathbb{K}^\varepsilon\nabla P_w(S^\varepsilon, P^\varepsilon), \tag{6.40}$$

$$\Lambda_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P^\varepsilon + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla\theta^\varepsilon = \lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon\nabla P_g(S^\varepsilon, P^\varepsilon). \tag{6.41}$$

As in [72], after inserting the test-functions

$$\varphi^\varepsilon = \frac{1}{\rho_w}P_w^\varepsilon, \quad \psi^\varepsilon = \int_0^{P_g^\varepsilon} \frac{dp}{\rho_g(p)}$$

in the weak formulation (6.9)-(6.10) and summing the obtained equations we get

$$
\begin{aligned}
&- \rho_w \int_\Omega \Phi^\varepsilon \frac{\partial S^\varepsilon}{\partial t} \varphi^\varepsilon dx + \int_\Omega \Phi^\varepsilon \frac{\partial V^\varepsilon}{\partial t} \psi^\varepsilon dx \\
&+ \int_\Omega (\Lambda_w(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon - A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon) \frac{1}{\rho_w} \cdot \nabla P_w^\varepsilon dx \\
&+ \int_\Omega (\Lambda_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon + A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla \theta^\varepsilon) \frac{1}{\rho_g(S^\varepsilon, P^\varepsilon)} \cdot \nabla P_g^\varepsilon dx \qquad (6.42) \\
&= \int_\Omega \lambda_w(S^\varepsilon)\rho_w \mathbb{K}^\varepsilon \mathbf{g} \cdot \nabla P_w^\varepsilon dx + \int_\Omega \lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \mathbf{g} \cdot \nabla P_g^\varepsilon dx \\
&+ \int_\Omega F_w \varphi^\varepsilon dx + \int_\Omega F_g \psi^\varepsilon dx.
\end{aligned}
$$

The integral terms in the equality (6.42) are denoted by $Z_1,\ Z_2, \ldots,\ Z_8$, respectively. We note that the forthcoming calculations are analogous to those from the proof of Lemma 11 which are still correct if only one of the phases is compressible.

In the same way as for the corresponding terms in the proof of Lemma 11, one can easily compute that

$$
Z_1 + Z_2 = \int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} \mathcal{G}(S^\varepsilon, P^\varepsilon) dx,
$$

where

$$
\mathcal{G}(S^\varepsilon, P^\varepsilon) = V^\varepsilon(S^\varepsilon, P^\varepsilon)\psi - P_g^\varepsilon S^\varepsilon + \int_0^{S^\varepsilon} P_c(s) ds,
$$

and

$$
\mathcal{G}(S^\varepsilon, P^\varepsilon) \geq 0 \text{ a.e. in } Q. \qquad (6.43)
$$

Then we apply the identities (6.40) and (6.41) and use (A.6) to see that

$$
Z_3 + Z_4 \geq \frac{1}{\rho_M} \int_\Omega \left(\lambda_w(S^\varepsilon)\rho_w \mathbb{K}^\varepsilon \nabla P_w^\varepsilon \cdot \nabla P_w^\varepsilon + \lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P_g^\varepsilon \cdot \nabla P_g^\varepsilon \right) dx.
$$

Combining this bound with the expression (6.39) and taking into account the assumption on the permeability tensor (6.2) yields

$$Z_3 + Z_4 \geq \frac{1}{\rho_M} \int_{\Omega_f^\varepsilon} \left(\frac{\rho_w \rho_g(S_f^\varepsilon, P_f^\varepsilon)}{\lambda(S_f^\varepsilon, P_f^\varepsilon)} \mathbb{K} \nabla \theta_f^\varepsilon \cdot \nabla \theta_f^\varepsilon + \omega(S_f^\varepsilon, P_f^\varepsilon)^2 \lambda(S_f^\varepsilon, P_f^\varepsilon) \mathbb{K} \nabla P_f^\varepsilon \cdot \nabla P_f^\varepsilon \right) dx$$
$$+ \varepsilon^2 \frac{1}{\rho_M} \int_{\Omega_m^\varepsilon} \left(\frac{\rho_w \rho_g(S_m^\varepsilon, P_m^\varepsilon)}{\lambda(S_m^\varepsilon, P_m^\varepsilon)} k^\varepsilon \nabla \theta_m^\varepsilon \cdot \nabla \theta_m^\varepsilon + \omega(S_m^\varepsilon, P_m^\varepsilon)^2 \lambda(S_m^\varepsilon, P_m^\varepsilon) k^\varepsilon \nabla P_m^\varepsilon \cdot \nabla P_m^\varepsilon \right) dx.$$

Finally we apply the assumptions (A.3) and (A.6) and obtain the estimate

$$Z_3 + Z_4 \geq C_1 \left(\int_{\Omega_f^\varepsilon} (|\nabla \theta_f^\varepsilon|^2 + |\nabla P_f^\varepsilon|^2) dx + \varepsilon^2 \int_{\Omega_m^\varepsilon} (|\nabla \theta_m^\varepsilon|^2 + |\nabla P_m^\varepsilon|^2) dx \right),$$

where $C_1 = \min(\frac{\rho_m^2 k_m}{\rho_M^2 \lambda_M}, \frac{\rho_m \lambda_m \omega_m^2 k_m}{\rho_M})$.

The integrals Z_5 and Z_6 are estimated by using the relations (6.37) and (6.38). Thereby it follows that

$$|Z_5 + Z_6| = |\int_{\Omega_f^\varepsilon} \lambda(S_f^\varepsilon, P_f^\varepsilon) \omega(S_f^\varepsilon, P_f^\varepsilon) \mathbb{K} \mathbf{g} \cdot \nabla P_f^\varepsilon dx + \int_{\Omega_m^\varepsilon} \lambda(S_m^\varepsilon, P_m^\varepsilon) \omega(S_m^\varepsilon, P_m^\varepsilon) \varepsilon^2 k^\varepsilon \frac{1}{\varepsilon} \mathbf{g} \cdot \nabla P_m^\varepsilon dx|.$$

Next, one concludes by using (A.1), (A.3) and (A.6) as well as Remark 7 and the Young inequality that for arbitrary $\alpha, \beta > 0$ it holds

$$|Z_5 + Z_6| \leq \frac{C_2}{2\alpha} + \varepsilon \frac{C_2}{2\beta} + \frac{C_2}{2} \alpha \int_{\Omega_f^\varepsilon} |\nabla P_f^\varepsilon|^2 dx + \varepsilon \frac{C_2}{2} \beta \int_{\Omega_m^\varepsilon} |\nabla P_m^\varepsilon|^2 dx,$$

with $C_2 = \lambda_M \rho_M k_M |\mathbf{g}| (|\Omega|)^{1/2}$.

Further, from relations (4.15), (4.16) and the nonnegativity of F_w in (A.7) it follows that the sum of the terms Z_7 and Z_8 can be written as

$$|Z_7 + Z_8| \leq |\int_\Omega \frac{1}{\rho_w} F_w P_w dx| + |\int_\Omega F_g \int_0^{P_g^\varepsilon} \frac{dp}{\rho_g(p)} dx| \quad \leq C_3 \int_\Omega (F_w + |F_g|)(|P^\varepsilon| + 1) dx,$$

with $C_3 = \frac{1}{\rho_m}$.

Finally, all obtained estimates for Z_k, $k = 1, \dots, 8$ are put together. Thus for a.e. $t \in]0, T[$ it follows that for any $\alpha, \beta > 0$:

$$\int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} \mathcal{G}(S^\varepsilon, P^\varepsilon) dx + C_1 \left(\int_{\Omega_f^\varepsilon} (|\nabla \theta_f^\varepsilon|^2 + |\nabla P_f^\varepsilon|^2) dx + \varepsilon^2 \int_{\Omega_m^\varepsilon} (|\nabla \theta_m^\varepsilon|^2 + |\nabla P_m^\varepsilon|^2) dx \right)$$
$$\leq \frac{C_2}{2\alpha} + \varepsilon \frac{C_2}{2\beta} + \frac{C_2}{2} \alpha \int_{\Omega_f^\varepsilon} |\nabla P_f^\varepsilon|^2 dx + \varepsilon \frac{C_2}{2} \beta \int_{\Omega_m^\varepsilon} |\nabla P_m^\varepsilon|^2 dx + C_3 \int_\Omega (F_w + |F_g|)(|P^\varepsilon| + 1) dx. \tag{6.44}$$

By choosing $\alpha = \frac{C_1}{C_2}$ and $\beta = \beta(\varepsilon) = \frac{\varepsilon C_1}{C_2}$ we obtain

$$\begin{aligned} &\int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} \mathcal{G}(S^\varepsilon, P^\varepsilon) dx + C_1 \int_{\Omega_f^\varepsilon} |\nabla \theta_f^\varepsilon|^2 dx + \frac{C_1}{2} \int_{\Omega_f^\varepsilon} |\nabla P_f^\varepsilon|^2 dx \\ &+ \varepsilon^2 C_1 \int_{\Omega_m^\varepsilon} |\nabla \theta_m^\varepsilon|^2 dx + \varepsilon^2 \frac{C_1}{2} \int_{\Omega_m^\varepsilon} |\nabla P_m^\varepsilon|^2 dx \\ \leq & C_4 + \frac{C_2^2}{2C_1} + C_3 \int_\Omega (F_w + |F_g|)(|P^\varepsilon| + 1) dx. \end{aligned} \tag{6.45}$$

In the next step the integrating of (6.45) over $]0, T[$ yields

$$\begin{aligned} &\int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} \mathcal{G}(S^\varepsilon, P^\varepsilon)(T) dx + C_1 \int_{\Omega_f^{\varepsilon,T}} |\nabla \theta_f^\varepsilon|^2 dx dt + \frac{C_1}{2} \int_{\Omega_f^{\varepsilon,T}} |\nabla P_f^\varepsilon|^2 dx dt \\ &+ \varepsilon^2 C_1 \int_{\Omega_m^{\varepsilon,T}} |\nabla \theta_m^\varepsilon|^2 dx dt + \varepsilon^2 \frac{C_1}{2} \int_{\Omega_m^{\varepsilon,T}} |\nabla P_m^\varepsilon|^2 dx dt \\ &\leq C_5 + C_3 \int_Q (F_w + |F_g|)(|P^\varepsilon| + 1) dx dt + \int_\Omega \Phi^\varepsilon \frac{\partial}{\partial t} \mathcal{G}(s_0, p_0) dx, \end{aligned}$$

with $C_5 = C_4 T + \frac{C_2^2}{2C_1} T$.

From (A.4) and the relations (4.15), (4.16) it follows that $0 \leq \mathcal{G}(S, P) \leq C(|P| + 1)$ and hence $\int_\Omega \Phi^\varepsilon \mathcal{G}(s_0, p_0) dx \leq C$. Moreover, by using (6.43) it follows

$$C_1 \int_{\Omega_f^{\varepsilon,T}} |\nabla \theta_f^\varepsilon|^2 dx dt + \frac{C_1}{2} \int_{\Omega_f^{\varepsilon,T}} |\nabla P_f^\varepsilon|^2 dx dt + \varepsilon^2 C_1 \int_{\Omega_m^{\varepsilon,T}} |\nabla \theta_m^\varepsilon|^2 dx dt + \varepsilon^2 \frac{C_1}{2} \int_{\Omega_m^{\varepsilon,T}} |\nabla P_m^\varepsilon|^2 dx dt$$
$$\leq C + C_3 (\|F_w\|_{L^2(Q)} + \|F_g\|_{L^2(Q)})(\|P^\varepsilon\|_{L^2(Q)} + 1),$$

and eventually after employing (A.7) we obtain

$$\begin{aligned} &C_1 \|\nabla \theta_f^\varepsilon\|^2_{L^2(\Omega_f^{\varepsilon,T})} + \frac{C_1}{2} \|\nabla P_f^\varepsilon\|^2_{L^2(\Omega_f^{\varepsilon,T})} + \varepsilon^2 C_1 \|\nabla \theta_m^\varepsilon\|^2_{L^2(\Omega_m^{\varepsilon,T})} + \varepsilon^2 \frac{C_1}{2} \|\nabla P_m^\varepsilon\|^2_{L^2(\Omega_m^{\varepsilon,T})} \\ &\leq C(\|P_f^\varepsilon\|_{L^2(\Omega_f^{\varepsilon,T})} + \|P_m^\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} + 1). \end{aligned} \tag{6.46}$$

The first term on the right-hand side of (6.46) can be estimated by the Poincaré inequality taking into account the boundary condition (6.6). Next we estimate the second term on the right-hand side of (6.46). Namely, by using the continuity of the global pressure across the interface Γ^ε_{fm} and the Poincaré inequality we get for any matrix block $\varepsilon(Y_m+k)$, $k \in \mathbb{Z}^d$:

$$\int_0^T \int_{\varepsilon(Y_m+k)} |P^\varepsilon_m - \widetilde{P}^\varepsilon_f|^2 dxdt \le C\varepsilon^2 \int_0^T \int_{\varepsilon(Y_m+k)} |\nabla(P^\varepsilon_m - \widetilde{P}^\varepsilon_f)|^2 dxdt,$$

so it also holds

$$\int_0^T \int_{\Omega^\varepsilon_m} |P^\varepsilon_m - \widetilde{P}^\varepsilon_f|^2 dxdt \le C\varepsilon^2 \int_0^T \int_{\Omega^\varepsilon_m} |\nabla(P^\varepsilon_m - \widetilde{P}^\varepsilon_f)|^2 dxdt. \tag{6.47}$$

Now we can estimate by using (6.47), (6.27) and (6.28) as follows.

$$\begin{aligned} \|P^\varepsilon_m\|_{L^2(\Omega^{\varepsilon,T}_m)} \le& \|P^\varepsilon_m - \widetilde{P}^\varepsilon_f\|_{L^2(\Omega^{\varepsilon,T}_m)} + \|\widetilde{P}^\varepsilon_f\|_{L^2(\Omega^{\varepsilon,T}_m)} \\ \le& C\varepsilon\|\nabla P^\varepsilon_m - \nabla\widetilde{P}^\varepsilon_f\|_{L^2(\Omega^{\varepsilon,T}_m)} + \|\widetilde{P}^\varepsilon_f\|_{L^2(Q)} \\ \le& C\varepsilon\|\nabla P^\varepsilon_m\|_{L^2(\Omega^{\varepsilon,T}_m)} + C\varepsilon\|\nabla P^\varepsilon_f\|_{L^2(\Omega^{\varepsilon,T}_f)} + C\|P^\varepsilon_f\|_{L^2(\Omega^{\varepsilon,T}_f)}. \end{aligned} \tag{6.48}$$

From the continuity of the saturation potential across Γ^ε_{fm} by using the same arguments we can obtain the estimate

$$\|\theta^\varepsilon_m\|_{L^2(\Omega^{\varepsilon,T}_m)} \le C\varepsilon\|\nabla\theta^\varepsilon_m\|_{L^2(\Omega^{\varepsilon,T}_m)} + C\varepsilon\|\nabla\theta^\varepsilon_f\|_{L^2(\Omega^{\varepsilon,T}_f)} + C\|\theta^\varepsilon_f\|_{L^2(\Omega^{\varepsilon,T}_f)}. \tag{6.49}$$

The last terms on the right-hand sides of (6.48) and (6.49) are bounded using the Poincaré inequality. Finally from (6.46) by using the Young inequality we get for any $\gamma, \delta > 0$:

$$\begin{aligned} &C_1\|\nabla\theta^\varepsilon_f\|^2_{L^2(\Omega^{\varepsilon,T}_f)} + \frac{C_1}{2}\|\nabla P^\varepsilon_f\|^2_{L^2(\Omega^{\varepsilon,T}_f)} + \varepsilon^2 C_1\|\nabla\theta^\varepsilon_m\|^2_{L^2(\Omega^{\varepsilon,T}_m)} + \varepsilon^2\frac{C_1}{2}\|\nabla P^\varepsilon_m\|^2_{L^2(\Omega^{\varepsilon,T}_m)} \\ &\le C(1 + \gamma\|\nabla P^\varepsilon_f\|^2_{L^2(\Omega^{\varepsilon,T}_f)} + \delta\varepsilon^2\|\nabla P^\varepsilon_m\|^2_{L^2(\Omega^{\varepsilon,T}_m)}), \end{aligned} \tag{6.50}$$

which shows the estimates (6.33) and (6.35). Moreover, now (6.33) and (6.35) along with (6.48) and (6.49) yield the uniform estimates for the matrix solutions claimed in (6.34).

In order to establish the uniform estimates for the time derivatives of the functions $\Phi^\varepsilon S^\varepsilon$ and $\Phi^\varepsilon\rho_g(S^\varepsilon, P^\varepsilon)S^\varepsilon$, we set an arbitrary $\varphi \in L^2(0,T;V)$ and $\psi \in L^2(0,T;V)$ in the variational equations (6.9) and (6.10), respectively. By employing the estimates (6.33)-(6.35), the desired estimate (6.36) easily follows. This completes the proof of Proposition 5.

$\square$

Remark 14 *As a consequence of Proposition 5 we have the following uniform estimates for the extended fracture functions which follow by the extension operator properties* (6.27) *and* (6.28)*:*

$$\|\widetilde{P}_f^\varepsilon\|_{L^2(0,T;V)} + \|\widetilde{\theta}_f^\varepsilon\|_{L^2(0,T;V)} \leq C. \tag{6.51}$$

6.5 Convergence results

Now we are ready to state the main result of this Chapter.

Theorem 10 *Suppose* (*A.1-d*) - (*A.2-d*), (*A.3*) - (*A.7*) *and* (*A.8-d*). *Let* $(P^\varepsilon, \theta^\varepsilon)$ *be a weak solution of the problem* (6.4), (6.5), (6.6), (6.7), (6.8). *Denote* $S^\varepsilon = \mathcal{S}^\varepsilon(\theta^\varepsilon)$ *and let* $\widetilde{P}_f^\varepsilon$, $\widetilde{\theta}_f^\varepsilon$, $\widetilde{S}_f^\varepsilon$ *and* $\widetilde{V}_f^\varepsilon$ *be the functions defined in Remark 13. Then, up to a subsequence, it holds*

$$\widetilde{P}_f^\varepsilon \rightharpoonup P \text{ weakly in } L^2(0,T;V) \text{ and } \widetilde{P}_f^\varepsilon \stackrel{2s}{\rightharpoonup} P, \tag{6.52}$$

$$\widetilde{S}_f^\varepsilon \to S \text{ strongly in } L^2(Q), \tag{6.53}$$

$$\widetilde{\theta}_f^\varepsilon \rightharpoonup \theta \text{ weakly in } L^2(0,T;V), \text{ strongly in } L^2(Q) \text{ and } \widetilde{\theta}_f^\varepsilon \stackrel{2s}{\rightharpoonup} \theta, \tag{6.54}$$

$$\theta = \beta(S),$$

$$\widetilde{V}_f^\varepsilon \to \rho_g(S,P)S \text{ strongly in } L^2(Q), \tag{6.55}$$

$$\nabla \widetilde{P}_f^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla P(x,t) + \nabla_y P_1(x,t,y), \tag{6.56}$$

$$\nabla \widetilde{\theta}_f^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla \theta(x,t) + \nabla_y \theta_1(x,t,y), \tag{6.57}$$

$$\chi_m^\varepsilon P_m^\varepsilon \stackrel{2s}{\rightharpoonup} p, \tag{6.58}$$

$$\chi_m^\varepsilon \theta_m^\varepsilon \stackrel{2s}{\rightharpoonup} \vartheta, \tag{6.59}$$

$$\chi_m^\varepsilon S_m^\varepsilon \stackrel{2s}{\rightharpoonup} s, \tag{6.60}$$

$$\chi_m^\varepsilon V_m^\varepsilon \stackrel{2s}{\rightharpoonup} v, \tag{6.61}$$

$$\varepsilon \chi_m^\varepsilon \nabla_x P_m^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla_y p(x,t,y), \tag{6.62}$$

$$\varepsilon \chi_m^\varepsilon \nabla_x \theta_m^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla_y \vartheta(x,t,y), \tag{6.63}$$

where

$$P_1(x,t,y) = \sum_{i=1}^{d} \left(\frac{\partial P}{\partial x_i}(x,t) - B(S,P) g_i \right) \chi_i(x,y), \tag{6.64}$$

$$\theta_1(x,t,y) = \sum_{i=1}^{d} \left(\frac{\partial \theta}{\partial x_i}(x,t) - E(S,P) g_i \right) \chi_i(x,y), \tag{6.65}$$

with $\chi_i(x,y)$ being a solution of the cell problem (6.17)*, while the functions B and E are given by*

$$B(S,P) = \frac{\lambda_w(S)\rho_w^2 + \lambda_g(S)\rho_g(S,P)^2}{\Lambda_w(S,P) + \Lambda_g(S,P)}, \tag{6.66}$$

$$E(S,P) = \frac{\lambda(S,P)\sqrt{\lambda_w(S)\lambda_g(S)}\omega(S,P)}{\Lambda_w(S,P) + \Lambda_g(S,P)} (\rho_g(S,P) - \rho_w). \tag{6.67}$$

The pair (P, θ) is a weak solution of the problem (6.11)-(6.20) *and $S = \mathcal{S}(\theta)$. Finally, let us assume that the problem* (4.5)-(4.6) *with the boundary and initial conditions* (4.7)-(4.10) *has a unique weak solution. The pair (p, ϑ) is a weak solution of the problem* (6.21)-(6.24)*, $s = \mathcal{S}(\vartheta)$ a.e. in $\Omega \times Y_m \times]0,T[$ and $v = \rho_g(s,p)s$ a.e. in $\Omega \times Y_m \times]0,T[$.*

Here $\stackrel{2s}{\rightharpoonup}$ denotes the two-scale convergence which is presented in Section 3.2.1. Theorem 10 is proven in Section 6.7.

Remark 15 *For completeness we quote here a standard result on a cell problem (see e.g. [2]). If $A(x,y)$ is a uniformly positively definite tensor which is bounded and Y-periodic, then it can be shown by using Lax-Milgram lemma that the following cell problem has a unique, up to a constant, solution ϖ:*

$$\begin{cases} - \quad \mathrm{div}_y \left(A(x,y)(\nabla_y \varpi(x,y) + \boldsymbol{e}_i) \right) = 0 \text{ in } Y_f, \\ A(x,y)(\nabla_y \varpi(x,y) + \boldsymbol{e}_i) \cdot \nu = 0 \text{ on } \Gamma_{fm}, \\ y \mapsto \varpi(x,y) \; Y - \text{ periodic.} \end{cases} \tag{6.68}$$

6.6 A compactness result for the fracture solutions

The aim of this Section is to prove the relative compactness and the corresponding convergence results for the sequences $(\widetilde{S}_f^\varepsilon)_\varepsilon$ and $(\widetilde{V}_f^\varepsilon)_\varepsilon$. Namely, the family $\widetilde{S}_f^\varepsilon$ is treated in Subsection 6.6.1 by using the result of [95] as the key idea; for the sequence $\widetilde{V}_f^\varepsilon$ a variant of the result of [5] is employed in Subsection 6.6.2. This Section is based on the ideas of [14] (see also [103]).

We begin with two auxiliary results which will be needed in the sequel. First, in the following Lemma a technical result is given which is easily proved by using the Fubini theorem.

Lemma 15 *For* $0 < h < \frac{T}{2}$ *and for integrable functions* $G_1(t)$, $G_2(t)$, *it holds*

$$\int_0^T G_1(t)\left(\int_{\max(t,h)}^{\min(t+h,T)} G_2(\tau)d\tau\right)dt = \int_h^T G_2(t)\left(\int_{t-h}^{t} G_1(\tau)d\tau\right)dt.$$

Further, our assumptions on the capillary pressure function imply the result of the next Lemma, which is going to be used several times in what follows.

Lemma 16 *There exist constants* $0 < \delta \leq 1$ *and* $C > 0$ *such that for any* $P \in \mathbb{R}$ *and any* $S_1, S_2 \in [0,1]$ *it holds:*

$$S_2|\int_{S_1}^{S_2} f_w(s,P)P_c'(s)ds| \leq C|S_1 - S_2|^\delta.$$

Proof. Note first that the assumption (4.12) in (A.4) implies that $\lambda_w(S)P_c'(S) \leq C$ for all $S \in [S^\#, 1\rangle$ and for some constant $C > 0$. Therefore, if $S_1, S_2 \geq S^\#$ one can write

$$S_2|\int_{S_1}^{S_2} f_w(s,P)P_c'(s)ds| \leq C\frac{\rho_M}{\rho_m\lambda_m}|S_1 - S_2|.$$

Next, using (4.13) in (A.4) one can obtain for some $C > 0$ and for $\gamma > 0$ given by (A.4) the following estimate valid for $S_1, S_2 \in]0, S^\#]$ [12]:

$$\min(S_1,S_2)|P_c(S_1) - P_c(S_2)| \leq C|S_1 - S_2|^\gamma. \tag{6.69}$$

Thus in the case $S_2 < S_1 < S^\#$ we can obtain by applying (6.69)

$$S_2|\int_{S_1}^{S_2} f_w(s,P)P_c'(s)ds| \leq \frac{\rho_M\lambda_M}{\rho_m\lambda_m}S_2|P_c(S_1) - P_c(S_2)| \leq C|S_1 - S_2|^\gamma.$$

If $S_1 < S_2 < S^\#$, applying (6.69) and the boundedness of the capillary pressure in (A.4) yields the estimate

$$\begin{aligned}
S_2\left|\int_{S_1}^{S_2} f_w(s,P)P_c'(s)ds\right| \leq & \frac{\rho_M\lambda_M}{\rho_m\lambda_m} S_2|P_c(S_1)-P_c(S_2)| \\
\leq & \frac{\rho_M\lambda_M}{\rho_m\lambda_m}\left(|S_2-S_1||P_c(S_1)-P_c(S_2)|+S_1|P_c(S_1)-P_c(S_2)|\right) \\
\leq & C\left(|S_2-S_1|+|S_1-S_2|^{\gamma}\right).
\end{aligned}$$

The remaining cases $S_1 < S^{\#} < S_2$ and $S_2 < S^{\#} < S_1$ can be treated by combining the previous ones. Finally, we put $\delta=\gamma$ if $\gamma<1$, and $\delta=1$ if $\gamma\geq 1$. Therefore Lemma 16 is proved. □

6.6.1 The compactness of $\widetilde{S}_f^{\varepsilon}$

In this Subsection we prove the result of the next Proposition.

Proposition 6 *Assume* (*A.1-d*) - (*A.2-d*), (*A.3*) - (*A.7*) *and* (*A.8-d*). *Let* $(\widetilde{S}_f^{\varepsilon})_{\varepsilon}$ *be the sequence defined by* (6.31). *There exists a function* S *such that, possibly along a subsequence,*

$$\widetilde{S}_f^{\varepsilon} \to S \text{ strongly in } L^2(Q) \text{ as } \varepsilon \to 0. \tag{6.70}$$

Moreover, $0 \leq S \leq 1$.

Proof. The proof consists of the following. First the modulus of continuity for the function θ_f^{ε} is obtained and then this result is extended to the function $\widetilde{\theta}_f^{\varepsilon}$, which will assure the convergence result for $\widetilde{\theta}_f^{\varepsilon}$ and finally, the desired convergence (6.70).

Step 1.

Let us consider the variational equation (6.9). Taking into account (6.1) we rewrite it as follows: for all $\varphi \in L^2(0,T;V)$

$$\begin{aligned}
&\rho_w \int_{\Omega_f^{\varepsilon,T}} \Phi \frac{\partial S_f^{\varepsilon}}{\partial t}\varphi dxdt + \rho_w \int_{\Omega_m^{\varepsilon,T}} \phi^{\varepsilon}\frac{\partial S_m^{\varepsilon}}{\partial t}\varphi dxdt \\
&\quad = \int_Q \Lambda_w(S^{\varepsilon},P^{\varepsilon})\mathbb{K}^{\varepsilon}\nabla P^{\varepsilon}\cdot\nabla\varphi dxdt - \int_Q A(S^{\varepsilon},P^{\varepsilon})\mathbb{K}^{\varepsilon}\nabla\theta^{\varepsilon}\cdot\nabla\varphi dxdt \\
&\quad - \int_Q \lambda_w(S^{\varepsilon})\rho_w^2\mathbb{K}^{\varepsilon}\mathbf{g}^{\varepsilon}\cdot\nabla\varphi dxdt - \int_Q F_w\varphi dxdt.
\end{aligned} \tag{6.71}$$

Denote the integral terms in the equality (6.71) by $I_1^\varepsilon(\varphi), \cdots, I_6^\varepsilon(\varphi)$, respectively. Recall the notation for the time difference operator,

$$\partial^{-h} v(t) = \frac{v(t) - v(t-h)}{h}.$$

Following the ideas of [14] and [103], for $\varepsilon > 0$ and for $0 < h < T/2$ we insert as a test function in (6.71) the following function:

$$\varphi^{\varepsilon,h}(x,t) = \int_{\max(t,h)}^{\min(t+h,T)} h\partial^{-h}\theta^\varepsilon(x,\tau)d\tau. \tag{6.72}$$

The properties of $\varphi^{\varepsilon,h}$ are described by the next result.

Lemma 17 *Let $\varepsilon > 0$ and let $h > 0$ small enough. There is a constant C which does not depend on ε or h such that for the sequence of functions defined by (6.72) it holds*

$$\|\varphi^{\varepsilon,h}\|_{L^2(Q)} \leq Ch, \tag{6.73}$$

$$\|\nabla\varphi^{\varepsilon,h}\|_{L^2(\Omega_f^{\varepsilon,T})} \leq Ch, \tag{6.74}$$

$$\varepsilon\|\nabla\varphi^{\varepsilon,h}\|_{L^2(\Omega_m^{\varepsilon,T})} \leq Ch. \tag{6.75}$$

Proof of Lemma 17.

For $r \in \{f, m\}$ we have

$$\|\nabla\varphi^{\varepsilon,h}\|^2_{L^2(\Omega_r^{\varepsilon,T})} \leq \int_{\Omega_r^{\varepsilon,T}} \left(\int_{\max(t,h)}^{\min(t+h,T)} |\nabla\theta_r^\varepsilon(x,\tau) - \nabla\theta_r^\varepsilon(x,\tau-h)|d\tau \right)^2 dxdt. \tag{6.76}$$

Since $\min(t+h,T) - \max(t,h) \leq h$, one gets for a.e. $(x,t) \in \Omega_r^{\varepsilon,T}$:

$$\int_{\max(t,h)}^{\min(t+h,T)} |\nabla\theta_r^\varepsilon(x,\tau) - \nabla\theta_r^\varepsilon(x,\tau-h)|d\tau \leq h^{1/2} \left(\int_{\max(t,h)}^{\min(t+h,T)} |\nabla\theta_r^\varepsilon(x,\tau) - \nabla\theta_r^\varepsilon(x,\tau-h)|^2 d\tau \right)^{1/2}.$$

Therefore we have from (6.76)

$$\|\nabla\varphi^{\varepsilon,h}\|^2_{L^2(\Omega_r^{\varepsilon,T})} \leq h \int_{\Omega_r^{\varepsilon,T}} \left(\int_{\max(t,h)}^{\min(t+h,T)} |\nabla\theta_r^\varepsilon(x,\tau) - \nabla\theta_r^\varepsilon(x,\tau-h)|^2 d\tau \right) dxdt. \tag{6.77}$$

Now we apply Lemma 15 with $G_1(t) = 1$, $G_2(\tau) = |\nabla\theta_r^\varepsilon(x,\tau) - \nabla\theta_r^\varepsilon(x,\tau-h)|^2$ to (6.77) to establish for a.e. $x \in \Omega_r^\varepsilon$:

$$\int_0^T \left(\int_{\max(t,h)}^{\min(t+h,T)} |\nabla\theta_r^\varepsilon(x,\tau) - \nabla\theta_r^\varepsilon(x,\tau-h)|^2 d\tau \right) dt$$
$$= \int_h^T |\nabla\theta_r^\varepsilon(x,t) - \nabla\theta_r^\varepsilon(x,t-h)|^2 \left(\int_{t-h}^t 1 d\tau \right) dt = h \int_h^T |\nabla\theta_r^\varepsilon(x,t) - \nabla\theta_r^\varepsilon(x,t-h)|^2 dt.$$

It then follows from (6.77)

$$\begin{aligned} \|\nabla\varphi^{\varepsilon,h}\|^2_{L^2(\Omega_r^{\varepsilon,T})} \leq & h^2 \int_{\Omega_r^\varepsilon} \int_h^T |\nabla\theta_r^\varepsilon(x,t) - \nabla\theta_r^\varepsilon(x,t-h)|^2 dt dx \\ \leq & 2h^2 \left(\int_{\Omega_r^\varepsilon} \int_h^T |\nabla\theta_r^\varepsilon(x,t)|^2 dt dx + \int_{\Omega_r^\varepsilon} \int_h^T |\nabla\theta_r^\varepsilon(x,t-h)|^2 dt dx \right) \\ \leq & 4h^2 \|\nabla\theta_r^\varepsilon\|^2_{L^2(\Omega_r^{\varepsilon,T})}. \end{aligned}$$

Finally, we conclude by applying the uniform estimates (6.33) for the fractures and (6.35) for the matrix that the estimates (6.74) and (6.75) hold true. In the same manner one can establish the estimate (6.73) by employing the uniform a priori bounds (6.33) and (6.34).

□

Proof of Proposition 6 continued.

Note that by Lemma 17, $\varphi^{\varepsilon,h}$ is an admissible test function for any $\varepsilon > 0$ and a sufficiently small $h > 0$, due to the boundary condition (6.6).

With the chosen test function (6.72) inserted in the equation (6.71) first we can write

$$\begin{aligned} I_1^\varepsilon(\varphi^{\varepsilon,h}) + I_2^\varepsilon(\varphi^{\varepsilon,h}) =& \rho_w \int_0^T \int_{\Omega_f^\varepsilon} \Phi \frac{\partial S_f^\varepsilon}{\partial t} \left(\int_{\max(t,h)}^{\min(t+h,T)} h\partial^{-h}\theta_f^\varepsilon(x,\tau) d\tau \right) dx dt \\ &+ \rho_w \int_0^T \int_{\Omega_m^\varepsilon} \phi^\varepsilon \frac{\partial S_m^\varepsilon}{\partial t} \left(\int_{\max(t,h)}^{\min(t+h,T)} h\partial^{-h}\theta_m^\varepsilon(x,\tau) d\tau \right) dx dt. \end{aligned}$$

Applying Lemma 15 it follows that

$$
\begin{aligned}
I_1^\varepsilon(\varphi^{\varepsilon,h}) + I_2^\varepsilon(\varphi^{\varepsilon,h}) =&\rho_w \int_h^T \int_{\Omega_f^\varepsilon} \Phi h \partial^{-h} \theta_f^\varepsilon(x,t) \left(\int_{t-h}^t \frac{\partial S_f^\varepsilon}{\partial \tau} d\tau \right) dxdt \\
&+\rho_w \int_h^T \int_{\Omega_m^\varepsilon} \phi^\varepsilon h \partial^{-h} \theta_m^\varepsilon(x,t) \left(\int_{t-h}^t \frac{\partial S_m^\varepsilon}{\partial \tau} d\tau \right) dxdt \\
=&\rho_w \int_h^T \int_{\Omega_f^\varepsilon} \Phi \left(\theta_f^\varepsilon(x,t) - \theta_f^\varepsilon(x,t-h) \right) \left(S_f^\varepsilon(x,t) - S_f^\varepsilon(x,t-h) \right) dxdt \\
&+\rho_w \int_h^T \int_{\Omega_m^\varepsilon} \phi^\varepsilon \left(\theta_m^\varepsilon(x,t) - \theta_m^\varepsilon(x,t-h) \right) \left(S_m^\varepsilon(x,t) - S_m^\varepsilon(x,t-h) \right) dxdt.
\end{aligned}
$$

Since β is a monotone function, we obtain by using (A.1-d)

$$
\begin{aligned}
I_1^\varepsilon(\varphi^{\varepsilon,h}) + I_2^\varepsilon(\varphi^{\varepsilon,h}) \geq \phi_m \rho_w [&\int_h^T \int_{\Omega_f^\varepsilon} \left(\theta_f^\varepsilon(x,t) - \theta_f^\varepsilon(x,t-h) \right) \left(S_f^\varepsilon(x,t) - S_f^\varepsilon(x,t-h) \right) dxdt \\
&+ \int_h^T \int_{\Omega_m^\varepsilon} \left(\theta_m^\varepsilon(x,t) - \theta_m^\varepsilon(x,t-h) \right) \left(S_m^\varepsilon(x,t) - S_m^\varepsilon(x,t-h) \right) dxdt] \\
= \phi_m \rho_w &\int_h^T \int_\Omega \left(\theta^\varepsilon(x,t) - \theta^\varepsilon(x,t-h) \right) \left(S^\varepsilon(x,t) - S^\varepsilon(x,t-h) \right) dxdt.
\end{aligned}
\tag{6.78}
$$

Next, we estimate the right-hand side of (6.71) with the test function $\varphi^{\varepsilon,h}$ and taking into account (6.2) as

$$
\begin{aligned}
&|I_3^\varepsilon(\varphi^{\varepsilon,h}) + I_4^\varepsilon(\varphi^{\varepsilon,h}) + I_5^\varepsilon(\varphi^{\varepsilon,h})| \\
\leq& \|\mathbb{K} \left(\Lambda_w(S_f^\varepsilon, P_f^\varepsilon) \nabla P_f^\varepsilon - A(S_f^\varepsilon, P_f^\varepsilon) \nabla \theta_f^\varepsilon - \lambda_w(S_f^\varepsilon) \rho_w^2 \mathbf{g} \right) \|_{L^2(\Omega_f^{\varepsilon,T})} \cdot \|\nabla \varphi^{\varepsilon,h}\|_{L^2(\Omega_f^{\varepsilon,T})} \\
+& \varepsilon \|k^\varepsilon (\Lambda_w(S_m^\varepsilon, P_m^\varepsilon) \nabla P_m^\varepsilon - A(S_m^\varepsilon, P_m^\varepsilon) \nabla \theta_m^\varepsilon - \lambda_w(S_m^\varepsilon) \rho_w^2 \frac{1}{\varepsilon} \mathbf{g}) \|_{L^2(\Omega_m^{\varepsilon,T})} \cdot \varepsilon \|\nabla \varphi^{\varepsilon,h}\|_{L^2(\Omega_m^{\varepsilon,T})}.
\end{aligned}
\tag{6.79}
$$

In a first step we use (A.1-d), (A.3) and (A.6), and apply the a priori estimates (6.33) and (6.35) to obtain the estimates

$$
\begin{aligned}
&\|\mathbb{K} \left(\Lambda_w(S_f^\varepsilon, P_f^\varepsilon) \nabla P_f^\varepsilon - A(S_f^\varepsilon, P_f^\varepsilon) \nabla \theta_f^\varepsilon - \lambda_w(S_f^\varepsilon) \rho_w^2 \mathbf{g} \right) \|_{L^2(\Omega_f^{\varepsilon,T})} \\
&\qquad \leq C_1 \|\nabla P_f^\varepsilon\|_{L^2(\Omega_f^{\varepsilon,T})} + C_2 \|\nabla \theta_f^\varepsilon\|_{L^2(\Omega_f^{\varepsilon,T})} + C_3 \leq C_4,
\end{aligned}
\tag{6.80}
$$

$$
\begin{aligned}
&\varepsilon \|k^\varepsilon (\Lambda_w(S_m^\varepsilon, P_m^\varepsilon) \nabla P_m^\varepsilon - A(S_m^\varepsilon, P_m^\varepsilon) \nabla \theta_m^\varepsilon - \lambda_w(S_m^\varepsilon) \rho_w^2 \frac{1}{\varepsilon} \mathbf{g}) \|_{L^2(\Omega_m^{\varepsilon,T})} \\
&\qquad \leq C_1 \varepsilon \|\nabla P_m^\varepsilon\|^2_{L^2(\Omega_m^{\varepsilon,T})} + C_2 \varepsilon \|\nabla \theta_m^\varepsilon\|^2_{L^2(\Omega_m^{\varepsilon,T})} + C_3 \leq C_5,
\end{aligned}
\tag{6.81}
$$

where $C_1 = k_M\lambda_M\rho_M$, $C_2 = \frac{k_M\lambda_M\rho_M^2}{\lambda_m\rho_m}$ and $C_3 = k_M\lambda_M\rho_M^2|\mathbf{g}|T^{1/2}|\Omega|^{1/2}$. From (6.79) by using (6.80), (6.81), (6.74) and (6.75) we have

$$|I_3^\varepsilon(\varphi^{\varepsilon,h}) + I_4^\varepsilon(\varphi^{\varepsilon,h}) + I_5^\varepsilon(\varphi^{\varepsilon,h})| \leq Ch. \tag{6.82}$$

Next, we can estimate by using (A.7) and the estimate (6.73):

$$|I_6^\varepsilon(\varphi^{\varepsilon,h})| \leq \|F_w\|_{L^2(Q)} \cdot \|\varphi^{\varepsilon,h}\|_{L^2(Q)} \leq Ch. \tag{6.83}$$

Finally, from (6.71), by taking into account (6.78), (6.82) and (6.83), and by using the monotonicity of $S \mapsto \beta(S)$ we obtain for sufficiently small $h > 0$ the following estimate:

$$\int_h^T \int_\Omega \big(\theta^\varepsilon(x,t) - \theta^\varepsilon(x,t-h)\big)\big(S^\varepsilon(x,t) - S^\varepsilon(x,t-h)\big)dxdt \leq Ch. \tag{6.84}$$

Remark 16 *The estimate* (6.84) *can obviously be rewritten into the following estimates for the individual subdomains of* Ω*:*

$$\int_h^T \int_{\Omega_f^\varepsilon} \big(\theta_f^\varepsilon(x,t) - \theta_f^\varepsilon(x,t-h)\big)\big(S_f^\varepsilon(x,t) - S_f^\varepsilon(x,t-h)\big)dxdt \leq Ch, \tag{6.85}$$

$$\int_h^T \int_{\Omega_m^\varepsilon} \big(\theta_m^\varepsilon(x,t) - \theta_m^\varepsilon(x,t-h)\big)\big(S_m^\varepsilon(x,t) - S_m^\varepsilon(x,t-h)\big)dxdt \leq Ch. \tag{6.86}$$

Step 2.
From the definition of the function β it follows for a.e. $(x,t) \in \Omega_f^\varepsilon \times]h,T[$ by using (4.12) in (A.4)

$$|\theta_f^\varepsilon(x,t) - \theta_f^\varepsilon(x,t-h)| = |\int_{S_f^\varepsilon(x,t-h)}^{S_f^\varepsilon(x,t)} \sqrt{\lambda_w(s)\lambda_g(s)}P_c'(s)ds \leq M|S_f^\varepsilon(x,t) - S_f^\varepsilon(x,t-h)|,$$

which along with (6.85) yields the estimate

$$\int_h^T \int_{\Omega_f^\varepsilon} |\theta_f^\varepsilon(x,t) - \theta_f^\varepsilon(x,t-h)|^2 dxdt \leq Ch. \tag{6.87}$$

Remark 17 *It is easy to see that the following estimate can be established from* (6.86) *analogously:*

$$\int_h^T \int_{\Omega_m^\varepsilon} |\theta_m^\varepsilon(x,t) - \theta_m^\varepsilon(x,t-h)|^2 dxdt \leq Ch, \tag{6.88}$$

so we have also on the whole Ω,

$$\int_h^T \int_\Omega |\theta^\varepsilon(x,t) - \theta^\varepsilon(x,t-h)|^2 dxdt \leq Ch. \tag{6.89}$$

Step 3.
Using the fact that the extension operator Π^ε from the fracture domain to the whole domain is defined by reflection (see [1]), the consequences of (6.84) and (6.87) are the estimates

$$\int_h^T \int_\Omega \left(\widetilde{\theta}_f^\varepsilon(x,t) - \widetilde{\theta}_f^\varepsilon(x,t-h)\right) \left(\widetilde{S}_f^\varepsilon(x,t) - \widetilde{S}_f^\varepsilon(x,t-h)\right) dxdt \leq Ch \tag{6.90}$$

and

$$\int_h^T \int_\Omega |\widetilde{\theta}_f^\varepsilon(x,t) - \widetilde{\theta}_f^\varepsilon(x,t-h)|^2 dxdt \leq Ch, \tag{6.91}$$

respectively.

Step 4.
Now we focus our attention to the sequence $(\widetilde{\theta}_f^\varepsilon)_\varepsilon$. Its space derivatives are bounded by (6.51). On the other hand, regarding the time variable we have obtained the modulus of continuity (6.91). Now we apply the result of [95, Theorem 3] to conclude that $\{\widetilde{\theta}_f^\varepsilon : \varepsilon > 0\}$ is a relatively compact set in $L^2(Q)$. It follows that there exists $\overline{\theta} \in L^2(Q)$ such that, possibly along a subsequence,

$$\widetilde{\theta}_f^\varepsilon \to \overline{\theta} \text{ strongly in } L^2(Q) \text{ and a.e. in } Q. \tag{6.92}$$

Now we define

$$S := \mathcal{S}(\overline{\theta}). \tag{6.93}$$

Due to the Hölder continuity of $\mathcal{S}$ given in (4.18) and the definition (6.93) we have

$$\|\widetilde{S}_f^\varepsilon - S\|_{L^2(Q)} \leq C\|\widetilde{\theta}_f^\varepsilon - \overline{\theta}\|_{L^2(Q)}^\tau$$

and therefore with (6.92) we obtain, up to a subsequence,

$$\widetilde{S}_f^\varepsilon \to S \text{ strongly in } L^2(Q) \text{ and a.e. in } Q, \tag{6.94}$$

which is the assertion of Proposition 6.

□

At this point we establish the additional two estimates that will be needed in the subsequent Subsection.

Corollary 2 *For $\varepsilon > 0$ and for a sufficiently small $h > 0$, it holds*

$$\int_h^T \int_\Omega |\widetilde{S}_f^\varepsilon(x,t) - \widetilde{S}_f^\varepsilon(x,t-h)|^2 dxdt \le Ch^\tau, \tag{6.95}$$

$$\int_h^T \int_\Omega |S^\varepsilon(x,t) - S^\varepsilon(x,t-h)|^2 dxdt \le Ch^\tau. \tag{6.96}$$

The constant C is independent of ε and h, and τ is given by (A.5).

Proof. Concerning (6.96), we use the Hölder continuity of $\mathcal{S} = \beta^{-1}$ in (4.18) as well as the Hölder inequality to get

$$|S^\varepsilon(x,t) - S^\varepsilon(x,t-h)| \le C|\theta^\varepsilon(x,t) - \theta^\varepsilon(x,t-h)|^\tau,$$

and hence from (6.89) we conclude that

$$\begin{aligned}\int_h^T \int_\Omega |S^\varepsilon(x,t) - S^\varepsilon(x,t-h)|^2 dxdt \le & C\int_h^T \int_\Omega |\theta^\varepsilon(x,t) - \theta^\varepsilon(x,t-h)|^{2\tau} dxdt \\ \le & C\left(\int_h^T \int_\Omega |\theta^\varepsilon(x,t) - \theta^\varepsilon(x,t-h)|^2 dxdt\right)^\tau \\ \le & Ch^\tau. \end{aligned} \tag{6.97}$$

The same argument can be applied by using (6.91) instead of (6.89) to establish (6.95). Note that (6.95) and (6.96) represent the modula of continuity for the functions $\widetilde{S}_f^\varepsilon$ and S^ε, respectively.

□

6.6.2 The compactness of $\widetilde{V}_f^\varepsilon$

The present Subsection is devoted to the proof of the following Proposition.

Proposition 7 *Suppose (A.1-d) - (A.2-d), (A.3) - (A.7) and (A.8-d). Let the sequence $(\widetilde{V}_f^\varepsilon)_\varepsilon$ be defined by* (6.32)*, and let the function S be given in* (6.93)*. There exists a function*

$P \in L^2(0,T;V)$ *such that, possibly along a subsequence,*

$$\widetilde{V}_f^\varepsilon \to \rho_g(S,P)S \text{ strongly in } L^2(Q) \text{ as } \varepsilon \to 0. \tag{6.98}$$

Proof.

Following [14], we decompose the function $V_f^\varepsilon = V_f^\varepsilon(S_f^\varepsilon, P_f^\varepsilon) = \rho_g(S_f^\varepsilon, P_f^\varepsilon)S_f^\varepsilon$ as

$$V_f^\varepsilon = W(P_f^\varepsilon) + U_1(S_f^\varepsilon, P_f^\varepsilon) + U_2(S_f^\varepsilon, P_f^\varepsilon), \tag{6.99}$$

where it is denoted

$$\begin{aligned} W(P_f^\varepsilon) &= \rho_g(S, P_f^\varepsilon)S, \\ U_1(S_f^\varepsilon, P_f^\varepsilon) &= (S_f^\varepsilon - S)\rho_g(S_f^\varepsilon, P_f^\varepsilon), \\ U_2(S_f^\varepsilon, P_f^\varepsilon) &= S\left(\rho_g(S_f^\varepsilon, P_f^\varepsilon) - \rho_g(S, P_f^\varepsilon)\right), \end{aligned}$$

with $S \in L^2(Q)$ given by (6.93) and (6.94). Then we can also write

$$\widetilde{V}_f^\varepsilon = W(\widetilde{P}_f^\varepsilon) + U_1(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) + U_2(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon). \tag{6.100}$$

The convergence result (6.98) will be established in four steps. First we show that the second and the third summand in the representation (6.100) tend strongly to 0. Thereby we are left with the task of proving the strong convergence of $W(\widetilde{P}_f^\varepsilon)$ to the desired limit $\rho_g(S,P)S$. This will be accomplished in further three steps. Namely, we begin by obtaining the modulus of continuity for the sequence $W(P_f^\varepsilon)$, and then we utilize the properties of the extension operator in order to pass to the analogous conclusion for the family $W(\widetilde{P}_f^\varepsilon)$. Finally, a result from [5] is used to show the required compactness for the sequence $W(\widetilde{P}_f^\varepsilon)$.

Step 0.

By using the boundedness of the gas density in (A.6) we obtain

$$\|U_1(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)\|_{L^2(Q)} \leq \rho_M \|\widetilde{S}_f^\varepsilon - S\|_{L^2(Q)}$$

and hence, due to (6.94),

$$U_1(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) \to 0 \text{ strongly in } L^2(Q). \tag{6.101}$$

Next, $(\mathsf{A.6})$ and the relations (2.23) and (2.24) give the estimate

$$|U_2(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)| \le \max_{p\in\mathbb{R}} \rho_g'(p) S |P_g(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) - P_g(S, \widetilde{P}_f^\varepsilon)| \le \rho_M S |\int_S^{\widetilde{S}_f^\varepsilon} f_w(s, \widetilde{P}_f^\varepsilon) P_c'(s) ds|.$$

Now we employ Lemma 16 to conclude that

$$|U_2(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)| \le C|\widetilde{S}_f^\varepsilon - S|^\delta \text{ a.e. in } Q,$$

which means that, using (6.94),

$$U_2(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) \to 0 \text{ strongly in } L^2(Q). \tag{6.102}$$

Step 1.
As announced, our next step is to prove the following.

Lemma 18 *(Modulus of continuity for $W(P_f^\varepsilon)$)*
There is a constant C independent of ε and h such that for $\varepsilon > 0$ and for a sufficiently small $h > 0$, it holds

$$\int_h^T \int_{\Omega_f^\varepsilon} \big(W(P_f^\varepsilon(x,t)) - W(P_f^\varepsilon(x,t-h))\big)\big(P_f^\varepsilon(x,t) - P_f^\varepsilon(x,t-h)\big)\, dxdt \le Ch^{\frac{\delta\tau}{2}}, \tag{6.103}$$

where τ and $0 < \delta \le 1$ are given by ($\mathsf{A.5}$) and Lemma 16, respectively.

Proof of Lemma 18.

We consider the variational equation (6.10). Taking into account (6.1) and the decomposition (6.99) we rewrite (6.10) as follows: for all $\psi \in L^2(0,T;V)$

$$\begin{aligned}
&\int_{\Omega_f^{\varepsilon,T}} \Phi \frac{\partial W(P_f^\varepsilon)}{\partial t} \psi dxdt + \int_{\Omega_f^{\varepsilon,T}} \Phi \frac{\partial\big(U_1(S_f^\varepsilon, P_f^\varepsilon) + U_2(S_f^\varepsilon, P_f^\varepsilon)\big)}{\partial t} \psi dxdt + \int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon \frac{\partial V_m^\varepsilon}{\partial t} \psi dxdt \\
&\quad = -\int_Q \Lambda_g(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla P^\varepsilon \cdot \nabla\psi dxdt - \int_Q A(S^\varepsilon, P^\varepsilon)\mathbb{K}^\varepsilon \nabla\theta^\varepsilon \cdot \nabla\psi dxdt \\
&\qquad + \int_Q \lambda_g(S^\varepsilon)\rho_g(S^\varepsilon, P^\varepsilon)^2 \mathbb{K}^\varepsilon \mathbf{g}^\varepsilon \cdot \nabla\psi dxdt + \int_Q F_g \psi dxdt.
\end{aligned} \tag{6.104}$$

Denote the integral terms in the equality (6.71) by $J_1^\varepsilon(\psi), \cdots, J_7^\varepsilon(\psi)$, respectively. Similarly as in Subsection 6.6.1, for $\varepsilon > 0$ and sufficiently small $h > 0$ we insert as a test function in (6.104) the following function [14, 103]:

$$\psi^{\varepsilon,h}(x,t) = \int_{\max(t,h)}^{\min(t+h,T)} h\partial^{-h} P^{\varepsilon}(x,\tau)d\tau. \tag{6.105}$$

Remark 18 *By repeating the arguments of the proof of Lemma 17, for $\varepsilon > 0$ and for $h > 0$ small enough one can obtain the uniform estimate*

$$\|\psi^{\varepsilon,h}\|_{L^2(Q)} + \|\nabla\psi^{\varepsilon,h}\|_{L^2(\Omega_f^{\varepsilon,T})} + \varepsilon\|\nabla\psi^{\varepsilon,h}\|_{L^2(\Omega_m^{\varepsilon,T})} \leq Ch, \tag{6.106}$$

where C is a constant independent of ε and h. In particular, $\psi^{\varepsilon,h}$ is a valid test function for such ε and h by considering the boundary condition (6.6).

With the chosen test function we first have by applying Lemma 15

$$\begin{aligned} J_1^{\varepsilon}(\psi^{\varepsilon,h}) &= \int_0^T \int_{\Omega_f^{\varepsilon}} \Phi \frac{\partial W(P_f^{\varepsilon})}{\partial t} \left(\int_{\max(t,h)}^{\min(t+h,T)} h\partial^{-h} P_f^{\varepsilon}(x,\tau)d\tau \right) dxdt \\ &= \int_h^T \int_{\Omega_f^{\varepsilon}} \Phi h\partial^{-h} P_f^{\varepsilon}(x,t) \left(\int_{t-h}^{t} \frac{\partial (W(P_f^{\varepsilon}))}{\partial \tau} d\tau \right) dxdt \\ &= \int_h^T \int_{\Omega_f^{\varepsilon}} \Phi \left[P_f^{\varepsilon}(x,t) - P_f^{\varepsilon}(x,t-h) \right] \left[(W(P_f^{\varepsilon}))(x,t) - (W(P_f^{\varepsilon}))(x,t-h) \right] dxdt. \end{aligned}$$

Further, the following decomposition is easily seen to hold a.e. in $\Omega_f^{\varepsilon} \times]h,T[$ (the x variable is omitted in writing in this paragraph):

$$\begin{aligned} (W(P_f^{\varepsilon}))(t) - (W(P_f^{\varepsilon}))(t-h) =& S(t) \left[\rho_g(S(t), P_f^{\varepsilon}(t)) - \rho_g(S(t), P_f^{\varepsilon}(t-h)) \right] \\ &+ S(t) \left[\rho_g(S(t), P_f^{\varepsilon}(t-h)) - \rho_g(S(t-h), P_f^{\varepsilon}(t-h)) \right] \\ &+ \left[S(t) - S(t-h) \right] \rho_g(S(t-h), P_f^{\varepsilon}(t-h)). \end{aligned}$$

We denote the summands in this decomposition by $a_1^{\varepsilon,h}(x,t)$, $a_2^{\varepsilon,h}(x,t)$ and $a_3^{\varepsilon,h}(x,t)$, respectively. It follows that

$$J_1^{\varepsilon}(\psi^{\varepsilon,h}) = A_1^{\varepsilon,h} + A_2^{\varepsilon,h} + A_3^{\varepsilon,h}, \tag{6.107}$$

where it is denoted for $i \in \{1,2,3\}$

$$A_i^{\varepsilon,h} = \int_h^T \int_{\Omega_f^{\varepsilon}} \Phi \left[P_f^{\varepsilon}(x,t) - P_f^{\varepsilon}(x,t-h) \right] a_i^{\varepsilon,h}(x,t) dxdt.$$

We note that the first term in (6.107) appears, up to the function Φ, as the integral term in (6.103):

$$A_1^{\varepsilon,h} = \int_h^T \int_{\Omega_f^\varepsilon} \Phi \left[P_f^\varepsilon(x,t) - P_f^\varepsilon(x,t-h)\right] \left[W(P_f^\varepsilon(x,t)) - W(P_f^\varepsilon(x,t-h))\right] dxdt. \quad (6.108)$$

Regarding the second term in (6.107), the following bound is valid a.e. in $\Omega_f^\varepsilon \times]h,T[$:

$$|a_2^{\varepsilon,h}(x,t)| \le \rho_M |S(x,t)| |\int_{S(x,t-h)}^{S(x,t)} f_w(s, P_f^\varepsilon(x,t-h)) P_c'(s) ds|.$$

From Lemma 16 we obtain

$$|a_2^{\varepsilon,h}(x,t)| \le C|S(x,t) - S(x,t-h)|^\delta$$

and this can be further estimated as

$$|a_2^{\varepsilon,h}(x,t)| \le C\left(|S(x,t) - \widetilde{S}_f^\varepsilon(x,t)|^\delta + |\widetilde{S}_f^\varepsilon(x,t) - \widetilde{S}_f^\varepsilon(x,t-h)|^\delta + |\widetilde{S}_f^\varepsilon(x,t-h) - S(x,t-h)|^\delta\right).$$

This way we have

$$\begin{aligned} \|a_2^{\varepsilon,h}(x,t)\|^2_{L^2(h,T;L^2(\Omega_f^\varepsilon))} \le & C\big(\|S(x,t) - \widetilde{S}_f^\varepsilon(x,t)\|^{2\delta}_{L^2(h,T;L^2(\Omega_f^\varepsilon))} \\ + \|\widetilde{S}_f^\varepsilon(x,t) - \widetilde{S}_f^\varepsilon(x,t-h)\|^{2\delta}_{L^2(h,T;L^2(\Omega_f^\varepsilon))} + & \|\widetilde{S}_f^\varepsilon(x,t-h) - S(x,t-h)\|^{2\delta}_{L^2(h,T;L^2(\Omega_f^\varepsilon))}\big). \end{aligned} \quad (6.109)$$

The first and the third term in (6.109) tend to zero as $\varepsilon \to 0$ (uniformly in h) because of (6.94). For the second term in (6.109) we utilize the estimate (6.95), which is uniform in ε. Hence we have for sufficiently small h and for $\varepsilon < \varepsilon(h)$ (where $\varepsilon(h)$ tends to 0 as h tends to 0):

$$\|a_2^{\varepsilon,h}(x,t)\|_{L^2(h,T;L^2(\Omega_f^\varepsilon))} \le Ch^{\frac{\delta\tau}{2}}.$$

Now it holds for sufficiently small h and for $\varepsilon < \varepsilon(h)$, taking into account (6.33)

$$\begin{aligned} |A_2^{\varepsilon,h}| \le & \phi_M \|P_f^\varepsilon(x,t) - P_f^\varepsilon(x,t-h)\|_{L^2(h,T;L^2(\Omega_f^\varepsilon))} \cdot \|a_2^{\varepsilon,h}(x,t)\|_{L^2(h,T;L^2(\Omega_f^\varepsilon))} \\ \le & 2\phi_M \|P_f^\varepsilon\|_{L^2(\Omega_f^{\varepsilon,T})} Ch^{\frac{\delta\tau}{2}} \le Ch^{\frac{\delta\tau}{2}}, \end{aligned} \quad (6.110)$$

where C does not depend on ε neither on h.

The term $A_3^{\varepsilon,h}$ is treated in a similar way. In fact, we have a.e. in $\Omega_f^\varepsilon \times]h,T[$:

$$|a_3^{\varepsilon,h}(x,t)| \le \rho_M |S(x,t) - S(x,t-h)|^\delta,$$

so it is

$$
\begin{aligned}
|A_3^{\varepsilon,h}| \leq & \phi_M \|P_f^\varepsilon(x,t) - P_f^\varepsilon(x,t-h)\|_{L^2(h,T;L^2(\Omega_f^\varepsilon))} \cdot \|a_3^{\varepsilon,h}(x,t)\|_{L^2(h,T;L^2(\Omega_f^\varepsilon))} \\
\leq & 2\phi_M \|P_f^\varepsilon\|_{L^2(\Omega_f^{\varepsilon,T})} \|S(x,t) - S(x,t-h)\|^\delta_{L^2(h,T;L^2(\Omega_f^\varepsilon))}.
\end{aligned}
$$

With the uniform estimates (6.33) and (6.96) we conclude that

$$
|A_3^{\varepsilon,h}| \leq C h^{\frac{\delta\tau}{2}}, \tag{6.111}
$$

for any $\varepsilon > 0$ and for h small enough. Now we take into account (6.110) and (6.111) to finally obtain from (6.107) for h small enough, and for $\varepsilon < \varepsilon(h)$:

$$
J_1^\varepsilon(\psi^{\varepsilon,h}) = A_1^{\varepsilon,h} + O(h^{\frac{\delta\tau}{2}}), \text{ as } h \to 0. \tag{6.112}
$$

Next we consider the term $J_2^\varepsilon(\psi^{\varepsilon,h})$. To this aim we apply the following decomposition which is easy to verify:

$$
U_1(S_f^\varepsilon, P_f^\varepsilon) + U_2(S_f^\varepsilon, P_f^\varepsilon) = S_f^\varepsilon \left(\rho_g(S_f^\varepsilon, P_f^\varepsilon) - \rho_g(S, P_f^\varepsilon)\right) + \rho_g(S, P_f^\varepsilon)\left(S_f^\varepsilon - S\right).
$$

Let us denote the terms on the right-hand side of this equality by $B_1^\varepsilon(S_f^\varepsilon, P_f^\varepsilon)$ and $B_2^\varepsilon(S_f^\varepsilon, P_f^\varepsilon)$, respectively. By using Lemma 15 we can write

$$
\begin{aligned}
J_2^\varepsilon(\psi^{\varepsilon,h}) = & \int_h^T \int_{\Omega_f^\varepsilon} \Phi \left[P_f^\varepsilon(x,t) - P_f^\varepsilon(x,t-h)\right] \cdot \left[B_1^\varepsilon(S_f^\varepsilon, P_f^\varepsilon)(x,t) - B_1^\varepsilon(S_f^\varepsilon, P_f^\varepsilon)(x,t-h)\right] dxdt \\
& + \int_h^T \int_{\Omega_f^\varepsilon} \Phi \left[P_f^\varepsilon(x,t) - P_f^\varepsilon(x,t-h)\right] \cdot \left[B_2^\varepsilon(S_f^\varepsilon, P_f^\varepsilon)(x,t) - B_2^\varepsilon(S_f^\varepsilon, P_f^\varepsilon)(x,t-h)\right] dxdt.
\end{aligned}
$$

In order to make use of the convergence result (6.94), we consider first the term

$$
\begin{aligned}
\widetilde{J}_2^\varepsilon(\psi^{\varepsilon,h}) := & \int_h^T \int_\Omega \Phi \left[\widetilde{P}_f^\varepsilon(x,t) - \widetilde{P}_f^\varepsilon(x,t-h)\right] \cdot \left[B_1^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)(x,t) - B_1^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)(x,t-h)\right] dxdt \\
& + \int_h^T \int_\Omega \Phi \left[\widetilde{P}_f^\varepsilon(x,t) - \widetilde{P}_f^\varepsilon(x,t-h)\right] \cdot \left[B_2^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)(x,t) - B_2^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)(x,t-h)\right] dxdt
\end{aligned}
$$

which can be estimated as

$$
|\widetilde{J}_2^\varepsilon(\psi^{\varepsilon,h})| \leq 4\phi_M \|\widetilde{P}_f^\varepsilon\|_{L^2(Q)} \left(\|B_1^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)\|_{L^2(Q)} + \|B_2^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon)\|_{L^2(Q)}\right).
$$

Now we notice that

$$B_1^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) \to 0 \text{ strongly in } L^2(Q),$$
$$B_2^\varepsilon(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) \to 0 \text{ strongly in } L^2(Q),$$

due to (A.6), (6.94) and Lemma 16 (as in (6.102)). Taking into account (6.51), this means that, uniformly in h,

$$\widetilde{J}_2^\varepsilon(\psi^{\varepsilon,h}) \to 0 \text{ as } \varepsilon \to 0.$$

Since the extension is by reflection, we obtain

$$J_2^\varepsilon(\psi^{\varepsilon,h}) \to 0 \text{ as } \varepsilon \to 0, \text{ uniformly in } h. \tag{6.113}$$

Next we look at the term $J_3^\varepsilon(\psi^{\varepsilon,h})$. It can be handled in much the same way as $J_1^\varepsilon(\psi^{\varepsilon,h})$. Namely, upon applying Lemma 15, we can write

$$J_3^\varepsilon(\psi^{\varepsilon,h}) = \int_h^T \int_{\Omega_m^\varepsilon} \phi^\varepsilon \left[P_m^\varepsilon(x,t) - P_m^\varepsilon(x,t-h)\right] \left[V_m^\varepsilon(x,t) - V_m^\varepsilon(x,t-h)\right] dxdt.$$

As before, now we decompose the function V_m^ε a.e. in $\Omega_m^\varepsilon \times]h,T[$ as

$$\begin{aligned} V_m^\varepsilon(t) - V_m^\varepsilon(t-h) =& S_m^\varepsilon(t) \left[\rho_g(S_m^\varepsilon(t), P_m^\varepsilon(t)) - \rho_g(S_m^\varepsilon(t), P_m^\varepsilon(t-h))\right] \\ &+ S_m^\varepsilon(t) \left[\rho_g(S_m^\varepsilon(t), P_m^\varepsilon(t-h)) - \rho_g(S_m^\varepsilon(t-h), P_m^\varepsilon(t-h))\right] \\ &+ \left[S_m^\varepsilon(t) - S_m^\varepsilon(t-h)\right] \rho_g(S_m^\varepsilon(t-h), P_m^\varepsilon(t-h)). \end{aligned}$$

With the summands in this decomposition denoted by $e_1^{\varepsilon,h}(x,t)$, $e_2^{\varepsilon,h}(x,t)$ and $e_3^{\varepsilon,h}(x,t)$, respectively, it holds

$$J_3^\varepsilon(\psi^{\varepsilon,h}) = E_1^{\varepsilon,h} + E_2^{\varepsilon,h} + E_3^{\varepsilon,h}, \tag{6.114}$$

where we set for $i \in \{1,2,3\}$

$$E_i^{\varepsilon,h} = \int_h^T \int_{\Omega_m^\varepsilon} \phi^\varepsilon \left[P_m^\varepsilon(x,t) - P_m^\varepsilon(x,t-h)\right] e_i^{\varepsilon,h}(x,t) dxdt.$$

Note that $P \mapsto \rho_g(S_m^\varepsilon, P)$ is an increasing function by (A.6), the definition of the function ω and (4.17). This yields

$$E_1^{\varepsilon,h} \geq 0. \tag{6.115}$$

Next, in the same manner as with the term $A_2^{\varepsilon,h}$, we have a.e. $\Omega_m^\varepsilon\times]h,T[$:

$$|e_2^{\varepsilon,h}(x,t)| \le \rho_M |S_m^\varepsilon(x,t)| \cdot |\int_{S_m^\varepsilon(x,t-h)}^{S_m^\varepsilon(x,t)} f_w(s, P_m^\varepsilon(x,t-h)) P_c'(s) ds|,$$

and then by using Lemma 16 one obtains

$$|e_2^{\varepsilon,h}(x,t)| \le C|S_m^\varepsilon(x,t) - S_m^\varepsilon(x,t-h)|^\delta.$$

Therefore we have for sufficiently small h:

$$\begin{aligned} |E_2^{\varepsilon,h}| \le& \phi_M \|P_m^\varepsilon(x,t) - P_m^\varepsilon(x,t-h)\|_{L^2(h,T;L^2(\Omega_m^\varepsilon))} \cdot \|e_2^{\varepsilon,h}(x,t)\|_{L^2(h,T;L^2(\Omega_m^\varepsilon))} \\ \le& 2\phi_M \|P_m^\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} \cdot \|S_m^\varepsilon(x,t) - S_m^\varepsilon(x,t-h)\|^\delta_{L^2(h,T;L^2(\Omega_m^\varepsilon))} \qquad (6.116) \\ \le& Ch^{\frac{\delta\tau}{2}}, \qquad (6.117) \end{aligned}$$

where the estimates (6.34) and (6.96) have been employed, and C is independent of ε and h.

Lastly, let us consider the term $E_3^{\varepsilon,h}$ which can be treated by the same arguments as $A_3^{\varepsilon,h}$. Namely, we have a.e. in $\Omega_m^\varepsilon\times]h,T[$:

$$|e_3^{\varepsilon,h}(x,t)| \le \rho_M |S_m^\varepsilon(x,t) - S_m^\varepsilon(x,t-h)|^\delta,$$

so it is

$$\begin{aligned} |A_3^{\varepsilon,h}| \le& \phi_M \|P_m^\varepsilon(x,t) - P_m^\varepsilon(x,t-h)\|_{L^2(h,T;L^2(\Omega_m^\varepsilon))} \cdot \|e_3^{\varepsilon,h}(x,t)\|_{L^2(h,T;L^2(\Omega_m^\varepsilon))} \\ \le& 2\phi_M \|P_m^\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} \cdot \|S_m^\varepsilon(x,t) - S_m^\varepsilon(x,t-h)\|^\delta_{L^2(h,T;L^2(\Omega_m^\varepsilon))}. \end{aligned}$$

With the uniform estimates (6.34) and (6.96) we conclude that for h small enough,

$$|E_3^{\varepsilon,h}| \le Ch^{\frac{\delta\tau}{2}}. \qquad (6.118)$$

Now we take into account (6.115), (6.117) and (6.118) to finally obtain from (6.114) for h small enough:

$$J_3^\varepsilon(\psi^{\varepsilon,h}) = E_1^{\varepsilon,h} + O(h^{\frac{\delta\tau}{2}}), \text{ as } h \to 0. \qquad (6.119)$$

Lastly we consider the terms on the right-hand side of the equation (6.104). By arguing as in the derivation of the estimates (6.82) and (6.83), we use (A.1-d), (A.3), (A.6) and (A.7), the a priori estimates (6.33) and (6.35), as well as Remark 18 to establish the following estimates:

$$
\begin{aligned}
&|J_4^\varepsilon(\psi^{\varepsilon,h})+J_5^\varepsilon(\psi^{\varepsilon,h})+J_6^\varepsilon(\psi^{\varepsilon,h})|\\
\leq&|\int_Q \mathbb{K}^\varepsilon\left(\Lambda_g(S^\varepsilon,P^\varepsilon)\nabla P^\varepsilon\cdot\nabla+A(S^\varepsilon,P^\varepsilon)\nabla\theta^\varepsilon-\lambda_g(S^\varepsilon)\rho_g(S^\varepsilon,P^\varepsilon)^2\mathbf{g}^\varepsilon\right)\cdot\nabla\psi^{\varepsilon,h}dxdt|
\end{aligned}
$$

$$
\begin{aligned}
\leq&\|\mathbb{K}\left(\Lambda_g(S_f^\varepsilon,P_f^\varepsilon)\nabla P_f^\varepsilon+A(S_f^\varepsilon,P_f^\varepsilon)\nabla\theta_f^\varepsilon-\lambda_g(S_f^\varepsilon)\rho_g(S^\varepsilon,P^\varepsilon)^2\mathbf{g}\right)\|_{L^2(\Omega_f^{\varepsilon,T})}\cdot\|\nabla\psi^{\varepsilon,h}\|_{L^2(\Omega_f^{\varepsilon,T})}\\
+\varepsilon&\|k^\varepsilon(\Lambda_g(S_m^\varepsilon,P_m^\varepsilon)\nabla P_m^\varepsilon+A(S_m^\varepsilon,P_m^\varepsilon)\nabla\theta_m^\varepsilon-\lambda_g(S_m^\varepsilon)\rho_g(S^\varepsilon,P^\varepsilon)^2\frac{1}{\varepsilon}\mathbf{g})\|_{L^2(\Omega_m^{\varepsilon,T})}\cdot\varepsilon\|\nabla\psi^{\varepsilon,h}\|_{L^2(\Omega_m^{\varepsilon,T})}\\
\leq&Ch, \qquad (6.120)
\end{aligned}
$$

and

$$
|J_7^\varepsilon(\psi^{\varepsilon,h})|=|\int_Q F_g\psi^{\varepsilon,h}dxdt|\leq\|F_g\|_{L^2(Q)}\cdot\|\psi^{\varepsilon,h}\|_{L^2(Q)}\leq Ch. \qquad (6.121)
$$

Now we collect the estimates for the integral terms in the equation (6.104), with the test function (6.105). Namely, by taking into account (6.107), (6.114), (6.108), (6.110), (6.111), (6.115), (6.117), (6.118), (6.120) and (6.121) we obtain for any sufficiently small $h>0$ and for $\varepsilon<\varepsilon(h)$

$$
\int_h^T\int_{\Omega_f^\varepsilon}\Phi\left[P_f^\varepsilon(x,t)-P_f^\varepsilon(x,t-h)\right]\left[W(P_f^\varepsilon(x,t))-W(P_f^\varepsilon(x,t-h))\right]dxdt\leq Ch^{\frac{\delta\tau}{2}}.
$$

Finally we note that the function $P\mapsto W(P)=\rho_g(S,P)S$ is strictly increasing in $\{(x,t)\in Q; S(x,t)>0\}$ so we have by using (A.1-d) the estimate

$$
\int_h^T\int_{\Omega_f^\varepsilon}\left[P_f^\varepsilon(x,t)-P_f^\varepsilon(x,t-h)\right]\left[W(P_f^\varepsilon(x,t))-W(P_f^\varepsilon(x,t-h))\right]dxdt\leq Ch^{\frac{\delta\tau}{2}},
$$

as claimed. This completes the proof of Lemma 18.

□

Proof of Proposition 7 continued.
Step 2.

Remark 19 *(Modulus of continuity for $W(\widetilde{P}_f^\varepsilon)$)*
Since the extension is by reflection [1], from (6.103) *we conclude that there is a constant C independent of ε and h, and $0<\tau<1$, $0<\delta\leq 1$ such that for sufficiently small $\varepsilon>0$*

and $h > 0$, it holds

$$\int_h^T \int_\Omega \Big(W(\widetilde{P}_f^\varepsilon(x,t)) - W(\widetilde{P}_f^\varepsilon(x,t-h)) \Big) \Big(\widetilde{P}_f^\varepsilon(x,t) - \widetilde{P}_f^\varepsilon(x,t-h) \Big)\, dxdt \le C h^{\frac{\delta\tau}{2}}. \quad (6.122)$$

Step 3. In order to prove the compactness for the sequence $W(\widetilde{P}_f^\varepsilon)$ we rely on the following modification of [5, Lemma 1.9] (see also [14]).

Lemma 19 *Suppose that the sequence $(u_\varepsilon)_\varepsilon$ converges weakly to u in $L^2(0,T;H^1(\Omega))$. Let $F = F(z,x,t)$ be a function defined by*

$$F(z,x,t) = F_1(z + b(z,x,t))F_2(x,t),$$

where F_1 is a continuous, monotone and bounded function in $\mathbb{R}$, $b \in L^\infty(\mathbb{R}\times Q)$, and $F_2 \in L^\infty(Q)$, $F_2 \ge 0$ a.e. in Q. Assume that

$$\int_h^T \int_\Omega \big(F(u_\varepsilon(x,t),x,t) - F(u_\varepsilon(x,t-h),x,t))\big)\big(u_\varepsilon(x,t) - u_\varepsilon(x,t-h)\big)dxdt \le Ch^\alpha$$

for some $\alpha > 0$ and with a constant C independent of h and ε. Then $F(u_\varepsilon,x,t)$ converges to $F(u,x,t)$ strongly in $L^2(Q)$.

Proof.

One can follow similar arguments as in the proof of [5, Lemma 1.9] (see [5, Remark 1.10]), to conclude that $F(u_\varepsilon,x,t)$ converges to $F(u,x,t)$ strongly in $L^1(Q)$. Since in addition the sequence $F(u_\varepsilon,x,t)$ is uniformly bounded in $L^\infty(Q)$, by the interpolation inequality for L^p-spaces we conclude that convergence is valid in $L^2(Q)$. □

From the uniform estimate for the global pressure in (6.51) we conclude that there exists a function $P \in L^2(0,T;V)$ such that, along a subsequence,

$$\widetilde{P}_f^\varepsilon \rightharpoonup P \text{ weakly in } L^2(0,T;V).$$

Now we apply Lemma 19 to $\widetilde{P}_f^\varepsilon$ and $W(\widetilde{P}_f^\varepsilon)$. To that aim, the conditions on the functions $F_1(p) = \rho_g(p)$ and $b(z,x,t) = P_c(0) + \int_0^{S(x,t)} f_w(s,z)P_c'(s)ds$ are verified by using our assumptions on the coefficients. Eventually this yields

$$W(\widetilde{P}_f^\varepsilon) \to \rho_g(S,P)S \text{ strongly in } L^2(Q). \quad (6.123)$$

Taking into account (6.100) and combining (6.123) with the convergence results (6.101) and (6.102) completes the proof of Proposition 7.

□

6.7 Proof of the homogenization result

6.7.1 Passage to the limit

Now we formulate the convergence results for the sequences of solutions of the microscopic problem parametrized by ε. This will enable us to pass to the limit as $\varepsilon \to 0$ in the weak equations (6.9)-(6.10). More precisely, similarly as in Chapter 5, the results on the two-scale convergence cited in Subsection 3.2.1 are now going to be applied to the uniform a priori estimates for the functions P^ε, θ^ε, S^ε, V^ε and for the corresponding extensions of the solutions in the fractures, which were established in Section 6.4. Furthermore, we employ the compactness results from the Section 6.6.

Accordingly, the a priori estimate (6.51) and Theorem 2 imply that there exist $P,\ \theta \in L^2(0,T;V)$ and $P_1(x,t,y),\ \theta_1(x,t,y) \in L^2(Q;H)$ such that as $\varepsilon \to 0$, up to a subsequence,

$$\widetilde{P}_f^\varepsilon \rightharpoonup P \text{ weakly in } L^2(0,T;V) \text{ and } \widetilde{P}_f^\varepsilon \overset{2s}{\rightharpoonup} P, \tag{6.124}$$

$$\widetilde{\theta}_f^\varepsilon \rightharpoonup \theta \text{ weakly in } L^2(0,T;V) \text{ and } \widetilde{\theta}_f^\varepsilon \overset{2s}{\rightharpoonup} \theta, \tag{6.125}$$

$$\nabla \widetilde{P}_f^\varepsilon \overset{2s}{\rightharpoonup} \nabla P + \nabla_y P_1(x,t,y), \tag{6.126}$$

$$\nabla \widetilde{\theta}_f^\varepsilon \overset{2s}{\rightharpoonup} \nabla \theta + \nabla_y \theta_1(x,t,y). \tag{6.127}$$

The limit θ is identified as $\theta = \beta(S)$ by using (6.92) and (6.93).

Regarding the microscopic solutions in the matrix, from the a priori estimates (6.34) and the boundedness of the saturation and ρ_g by using Theorem 1 we obtain

$$\chi_m^\varepsilon P_m^\varepsilon \overset{2s}{\rightharpoonup} p(x,t,y), \tag{6.128}$$

$$\chi_m^\varepsilon \theta_m^\varepsilon \overset{2s}{\rightharpoonup} \vartheta(x,t,y), \tag{6.129}$$

$$\chi_m^\varepsilon S_m^\varepsilon \overset{2s}{\rightharpoonup} s(x,t,y), \tag{6.130}$$

$$\chi_m^\varepsilon V_m^\varepsilon \overset{2s}{\rightharpoonup} v(x,t,y), \tag{6.131}$$

for some $p, \vartheta, s, v \in L^2(Q; L^2_p(Y_m))$. Moreover, the uniform estimates (6.34) and (6.35) along with Theorem 2 imply that $p, \vartheta \in L^2(Q; H^1_p(Y_m))$ and

$$\varepsilon \chi_m^\varepsilon \nabla P_m^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla_y p(x,t,y), \tag{6.132}$$

$$\varepsilon \chi_m^\varepsilon \nabla \theta_m^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla_y \vartheta(x,t,y). \tag{6.133}$$

Remark 20 *It is easy to see that*

$$\chi_r^\varepsilon(x) = \chi_r(\frac{x}{\varepsilon}) \stackrel{2s}{\rightharpoonup} \chi_r(y), \; r \in \{f, m\}.$$

Our next step is to pass to the limit in the variational formulation (6.9)-(6.10). First the equation (6.9) is treated. We choose suitable test function following [2]. Namely, the test function in the equation (6.9) is taken in the form

$$\varphi_w(x, \frac{x}{\varepsilon}, t) = \varphi(x,t) + \varepsilon \zeta(x, \frac{x}{\varepsilon}, t) = \varphi(x,t) + \varepsilon \zeta_1(x,t) \zeta_2(\frac{x}{\varepsilon}),$$

for some $\varphi \in \mathcal{D}(Q)$, $\zeta_1 \in \mathcal{D}(Q)$ and $\zeta_2 \in C_p^\infty(Y)$. In this way one obtains

$$\begin{aligned}
&\rho_w \int_Q \Phi \chi_f^\varepsilon \widetilde{S}_f^\varepsilon \left[\partial_t \varphi(x,t) + \varepsilon \partial_t \zeta(x, \frac{x}{\varepsilon}, t)\right] dxdt \\
&+\rho_w \int_Q \phi(\frac{x}{\varepsilon}) \chi_m^\varepsilon S_m^\varepsilon \left[\partial_t \varphi(x,t) + \varepsilon \partial_t \zeta(x, \frac{x}{\varepsilon}, t)\right] dxdt \\
&+ \int_Q \chi_f^\varepsilon \mathbb{K} \left[\Lambda_w(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) \nabla \widetilde{P}_f^\varepsilon - A(\widetilde{S}_f^\varepsilon, \widetilde{P}_f^\varepsilon) \nabla \widetilde{\theta}_f^\varepsilon - \lambda_w(\widetilde{S}_f^\varepsilon) \rho_w^2 \mathbf{g}\right] \cdot [\nabla \varphi + \varepsilon \nabla_x \zeta + \nabla_y \zeta] \, dxdt \\
&+\varepsilon \int_{\Omega_m^{\varepsilon,T}} k(\frac{x}{\varepsilon}) \left[\Lambda_w(S_m^\varepsilon, P_m^\varepsilon) k(\frac{x}{\varepsilon}) \varepsilon \nabla P_m^\varepsilon - A(S_m^\varepsilon, P_m^\varepsilon) \varepsilon \nabla \theta_m^\varepsilon - \lambda_w(S_m^\varepsilon) \rho_w^2 \varepsilon \frac{1}{\varepsilon} \mathbf{g}\right] \\
&\quad \cdot [\nabla \varphi + \varepsilon \nabla_x \zeta + \nabla_y \zeta] \, dxdt \\
&= \int_Q F_w \left[\varphi(x,t) + \varepsilon \zeta(x, \frac{x}{\varepsilon}, t)\right] dxdt.
\end{aligned} \tag{6.134}$$

We denote the integral terms in the expression (6.134) by $I_1^w, \cdots, I_5^w$, respectively. The passage to the limit as $\varepsilon \to 0$ in I_1^w and I_2^w is done using the strong convergence in Proposition 6, the boundedness of the porosity and the saturation, Remark 20 and the two-scale convergence result (6.130). On the other hand, the terms I_3^w, I_4^w and I_5^w are treated by employing the a priori estimates (6.51) and (6.35), the two-scale convergence results (6.126) and (6.127), and the Lebesgue theorem. Regarding the latter result, let us remark that the almost everywhere convergence of the global pressure function $\widetilde{P}_f^\varepsilon$ can be established only on a subset of Q where $S > 0$, as already noted in Chapter 4. Nevertheless, due to the continuity and the boundedness of the coefficients, the strong convergence result from Proposition 6 and the weak convergence in (6.124), we can apply Lemma 7 which assures the pointwise convergence of all nonlinear functions of $\widetilde{S}_f^\varepsilon$ and $\widetilde{P}_f^\varepsilon$ and accordingly the Lebesgue theorem can be applied.

After taking the two-scale limit in the equation (6.134) one gets

$$\begin{aligned}
&\rho_w |Y_f| \int_Q \Phi(x) S \partial_t \varphi(x,t) dxdt + \rho_w \int_Q \int_{Y_m} \phi(y) s(x,y,t) \partial_t \varphi(x,t) dydxdt \\
&+ \int_Q \int_{Y_f} \Lambda_w(S,P) \mathbb{K}(x) (\nabla P + \nabla_y P_1(x,y,t)) \cdot [\nabla \varphi(x,t) + \nabla_y \zeta(x,y,t)] dydxdt \\
&- \int_Q \int_{Y_f} A(S,P) \mathbb{K}(x) (\nabla \theta + \nabla_y \theta_1(x,y,t)) \cdot [\nabla \varphi(x,t) + \nabla_y \zeta(x,y,t)] dydxdt \\
&- \int_Q \int_{Y_f} \lambda_w(S) \rho_w^2 \mathbb{K}(x) \mathbf{g} \cdot [\nabla \varphi + \nabla_y \zeta] dydxdt = \int_Q F_w \varphi(x,t) dxdt.
\end{aligned} \tag{6.135}$$

Now we handle the equation (6.10) in much the same way, with corresponding application of the Proposition 7 and the two-scale convergence result (6.131). Accordingly, upon inserting the appropriate test function and passing to the limit when $\varepsilon \to 0$ in (6.10), we obtain

$$\begin{aligned}
&- |Y_f| \int_Q \Phi(x) S \rho_g(S,P) \partial_t \varphi(x,t) dxdt - \int_Q \int_{Y_m} \phi(y) v(x,y,t) \partial_t \varphi(x,t) dydxdt \\
&+ \int_Q \int_{Y_f} \Lambda_g(S,P) \mathbb{K}(x) (\nabla P + \nabla_y P_1(x,y,t)) \cdot [\nabla \varphi(x,t) + \nabla_y \zeta(x,y,t)] dydxdt \\
&+ \int_Q \int_{Y_f} A(S,P) \mathbb{K}(x) (\nabla \theta + \nabla_y \theta_1(x,y,t)) \cdot [\nabla \varphi(x,t) + \nabla_y \zeta(x,y,t)] dydxdt \\
&- \int_Q \int_{Y_f} \lambda_g(S) \rho_g(S,P)^2 \mathbb{K}(x) \mathbf{g} \cdot [\nabla \varphi + \nabla_y \zeta] dydxdt = \int_Q F_g \varphi(x,t) dxdt.
\end{aligned} \tag{6.136}$$

In our next step we aim to identify the functions P_1 and θ_1. To this end, we set $\varphi \equiv 0$ in the equations (6.135) and (6.136) and sum the two equations. This yields

$$\int_{Y_f} \mathbb{K}(x)(\nabla P(x,t) - B(S,P)\mathbf{g}) \cdot \nabla_y \zeta_2(y) dy = - \int_{Y_f} \mathbb{K}(x) \nabla_y P_1(x,y,t) \cdot \nabla_y \zeta_2(y) dy, \tag{6.137}$$

where the coefficient B is given by (6.66). From this equation we obtain P_1 in a form given by (6.64).

Finally, by setting $\varphi \equiv 0$ in (6.135) and by taking into account (6.66) we get

$$\begin{aligned}&\left(\Lambda_w(S,P)B(S,P) - \lambda_w(S)\rho_w^2\right)\int_{Y_f} \mathbb{K}(x)\mathbf{g}\cdot\nabla_y\zeta_2(y)dy\\&= A(S,P)\left(\int_{Y_f}\mathbb{K}(x)\nabla\theta(x,t)\cdot\nabla_y\zeta_2(y)dy + \int_{Y_f}\mathbb{K}(x)\nabla_y\theta_1(x,y,t)\cdot\nabla_y\zeta_2(y)dy\right).\end{aligned}$$

Denoting $\Lambda(S,P) = \Lambda_w(S,P) + \Lambda_g(S,P)$ and using the fact that

$$\Lambda_w(S,P)B(S,P) - \lambda_w(S)\rho_w^2 = \frac{\Lambda_w(S,P)\Lambda_g(S,P)}{\omega(S,P)\Lambda(S,P)}(\rho_g(S,P) - \rho_w)$$

and the expression (6.67), we can write

$$\int_{Y_f}\mathbb{K}(x)(\nabla\theta(x,t) - E(S,P)\mathbf{g})\cdot\nabla_y\zeta_2(y)dy = -\int_{Y_f}\mathbb{K}(x)\nabla_y\theta_1(x,y,t)\cdot\nabla_y\zeta_2(y)dy.$$

This equation leads to the form of the function θ_1 given by the formula (6.65).

In our final step towards establishing the homogenized fracture flow equations, we choose $\zeta_2 \equiv 0$ in (6.135) and (6.136), take into account the representations (6.64) and (6.65) and use the definition of the homogenized tensor $\mathbb{K}^H$ given by (6.16) and (6.17). In this way, for all $\varphi \in \mathcal{D}(Q)$ we obtain

$$\begin{aligned}&-\rho_w\int_Q \Phi^H(x)\partial_t S\varphi(x,t)dxdt + \int_Q \Lambda_w(S,P)\mathbb{K}^H\nabla P(x,t)\cdot\nabla\varphi(x,t)dxdt\\&-\int_Q A(S,P)\mathbb{K}^H\nabla\theta(x,t)\cdot\nabla\varphi(x,t)dxdt - \int_Q \lambda_w(S)\rho_w^2\mathbb{K}^H\mathbf{g}\cdot\nabla\varphi(x,t)dxdt\\&= \rho_w\int_Q\int_{Y_m}\phi(y)\partial_t s(x,y,t)\varphi(x,t)dydxdt + \int_Q F_w\varphi(x,t)dxdt\end{aligned} \tag{6.138}$$

and

$$\begin{aligned}&\int_Q \Phi^H(x)\partial_t(\rho_g(S,P)S)\varphi(x,t)dxdt + \int_Q \Lambda_g(S,P)\mathbb{K}^H\nabla P(x,t)\cdot\nabla\varphi(x,t)dxdt\\&+\int_Q A(S,P)\mathbb{K}^H\nabla\theta(x,t)\cdot\nabla\varphi(x,t)dxdt - \int_Q \lambda_g(S)\rho_g(S,P)^2\mathbb{K}^H\mathbf{g}\cdot\nabla\varphi(x,t)dxdt\\&= -\int_Q\int_{Y_m}\phi(y)\partial_t v(x,y,t)\varphi(x,t)dydxdt + \int_Q F_g\varphi(x,t)dxdt.\end{aligned} \tag{6.139}$$

As explained in Section 3.3, in a double porosity model the transport of fluids occurs primarily through the fractures. The effective flow in fractures for our model is described by the system (6.138)-(6.139), with the corresponding boundary and initial conditions (6.18)-

(6.20). This system is coupled to the matrix through locally defined macroscopic matrix source terms, appearing as the first terms on right-hand sides of the equations (6.138) and (6.139), that represent matrix-to-fracture flow; at this point, the corresponding term in (6.139) contains a non-identified two-scale limit v, defined by (6.131). Furthermore, as announced in Section 6.5, at the macroscopic level a set of equations arises to describe the flow within each matrix block.

What is left in the proof of Theorem 10 is to derive the effective matrix system (6.21) - (6.24) which is satisfied by the two-scale limits of matrix solutions p, ϑ, s and v, given by (6.128)-(6.133); in addition, one needs to identify the functions ϑ and v. Both are accomplished in the following Subsection. Then the proof of Theorem 10 will be completed.

6.7.2 The identification of the limit term

Due to the nonlinearities and to the strong coupling of the system, the notion of the two-scale convergence does not provide an explicit form for the source-like term generated by the flow in the matrix which models the influence of the matrix flow on the flow in the fractures at the macroscopic level. To overcome this difficulty, we make use of the appropriate dilation operator presented in Subsection 3.3.1, and the corresponding definitions and results cited therein. Namely, as in [14, 25, 39, 54, 103], the weak matrix solutions of the microscopic system are transformed using the dilation operator and the asymptotic behavior of the dilated solutions is deduced. Finally, after passing to the limit as $\varepsilon \to 0$ in the system for the dilated functions, the effective matrix equations are established.

The outline of the process is the following. First we find the equations satisfied by the dilated functions p_m^ε, ϑ_m^ε, s_m^ε, v_m^ε in $\Omega \times Y_m \times]0, T[$. The uniform a priori estimates ensure the weak convergence of the dilations as $\varepsilon \to 0$. By applying Lemma 2, these weak limits are recognized as the two-scale limits of matrix solutions: p, ϑ, s, v, respectively. On the other hand, we note that for any ε the restrictions of the dilated functions to a fixed translation of εY are independent of x. Thus, for $\varepsilon > 0$ and for each fixed $x_0 \in \Omega$, we define the corresponding functions p_{m,x_0}^ε, $\vartheta_{m,x_0}^\varepsilon$, s_{m,x_0}^ε, v_{m,x_0}^ε of variables y and t. For each fixed $x_0 \in \Omega$, we establish enough compactness results to find the equations satisfied by their limits. Lastly, the equations for the limits p, ϑ, s, v are deduced and the limits ϑ and v are identified.

Accordingly, the dilated functions (see Definition 2) will be denoted by

$$p_m^\varepsilon := D^\varepsilon P_m^\varepsilon, \ \vartheta_m^\varepsilon := D^\varepsilon \theta_m^\varepsilon, \ s_m^\varepsilon := D^\varepsilon S_m^\varepsilon, \ v_m^\varepsilon := D^\varepsilon V_m^\varepsilon. \tag{6.140}$$

Obviously it holds $\vartheta_m^\varepsilon = \beta(s_m^\varepsilon)$ and $v_m^\varepsilon = \rho_g(s_m^\varepsilon, p_m^\varepsilon) s_m^\varepsilon$.

Remark 21 *In the proofs of the identification of the source-like terms we will use several technical assumptions that are not given explicitly in the theorem statement. We believe that these supplementary assumptions could be weakened by refinement of our arguments which will be a subject of further research. Therefore, we give these assumptions in places where they are needed.*

Lemma 20 *Let $\varepsilon > 0$. The dilated functions defined by (6.140) satisfy the problem*

$$-\rho_w \phi(y)\frac{\partial s_m^\varepsilon}{\partial t} - div_y\big(k(y)\big[\Lambda_w(s_m^\varepsilon, p_m^\varepsilon)\nabla p_m^\varepsilon - A(s_m^\varepsilon, p_m^\varepsilon)\nabla \vartheta_m^\varepsilon - \lambda_w(s_m^\varepsilon)\rho_w^2 \boldsymbol{g}\big]\big) = D^\varepsilon F_w, \tag{6.141}$$

$$\phi(y)\frac{\partial v_m^\varepsilon}{\partial t} - div_y\big(k(y)\big[\Lambda_g(s_m^\varepsilon, p_m^\varepsilon)\nabla p_m^\varepsilon + A(s_m^\varepsilon, p_m^\varepsilon)\nabla \vartheta_m^\varepsilon - \lambda_g(s_m^\varepsilon)\rho_g(s_m^\varepsilon, p_m^\varepsilon)^2 \boldsymbol{g}\big]\big) = D^\varepsilon F_g \tag{6.142}$$

in $L^2(0,T;H^{-1}(Y_m))$ for a.e. $x \in \Omega$.

Proof. Let us fix $\hat{x} \in \Omega$ and set $c^\varepsilon(\hat{x}) = \varepsilon j$, $j \in \mathbb{Z}^d$. For $x \in \mathbb{R}^d$ and $t \in]0,T[$ we define

$$\zeta_1^\varepsilon(\hat{x}, x, t) = \begin{cases} \zeta(\frac{x - c^\varepsilon(\hat{x})}{\varepsilon}, t) & \text{for } x \in \varepsilon Y_m + c^\varepsilon(\hat{x}), \\ 0 & \text{otherwise }, \end{cases}$$

where $\zeta \in H^1(0,T;L^2(Y_m)) \cap L^2(0,T;C_c^\infty(Y_m))$. Now we consider the weak equation (6.9) with a test function in the form

$$\varphi^\varepsilon(x,t) = \zeta_1^\varepsilon(\hat{x}, x, t).$$

By using the definitions (6.1) and (6.2), the equation (6.9) hence becomes

$$\begin{aligned} &-\rho_w \int_0^T \int_{\varepsilon(Y_m+j)} \phi^\varepsilon(x)\partial_t S_m^\varepsilon \varphi^\varepsilon(x,t)dxdt \\ &+\varepsilon^2 \int_0^T \int_{\varepsilon(Y_m+j)} k^\varepsilon(x)\left[\Lambda_w(S_m^\varepsilon, P_m^\varepsilon)\nabla P_m^\varepsilon - A(S_m^\varepsilon, P_m^\varepsilon)\nabla\theta_m^\varepsilon - \lambda_w(S_m^\varepsilon)\rho_w^2\frac{1}{\varepsilon}\mathbf{g}\right]\cdot\nabla\varphi^\varepsilon(x,t)dxdt \\ &= \int_0^T \int_{\varepsilon(Y_m+j)} F_w \varphi^\varepsilon(x,t)dxdt. \end{aligned} \tag{6.143}$$

Upon performing a change of variables $x = \varepsilon y + c^\varepsilon(\hat{x}) = \varepsilon(y + j)$, from (6.143) we obtain

$$
\begin{aligned}
&-\rho_w \int_0^T \int_{Y_m} \phi(y)\partial_t s_m^\varepsilon \zeta(y,t)dydt \\
&\quad + \int_0^T \int_{Y_m} k(y)\left[\Lambda_w(s_m^\varepsilon, p_m^\varepsilon)\nabla p_m^\varepsilon - A(s_m^\varepsilon, p_m^\varepsilon)\nabla \vartheta_m^\varepsilon - \lambda_w(s_m^\varepsilon)\rho_w^2 \mathbf{g}\right]\cdot \nabla_y \zeta(y,t)dydt \\
&\quad = \int_0^T \int_{Y_m} D^\varepsilon F_w(\hat{x}, y, t)\zeta(y,t)dydt,
\end{aligned}
$$

where the periodicity of the functions ϕ^ε and k^ε has been used and Lemma 1 has been applied. Therefore, we have established (6.141). The equation (6.142) can be derived analogously, which completes the proof.

□

Remark 22 *The boundary and initial conditions for the system* (6.141)-(6.142) *are as follows [14, 39, 54, 103]:*

$$
\begin{aligned}
p_m^\varepsilon(x,y,t) =& D^\varepsilon \widetilde{P}_f^\varepsilon(x,y,t), \quad \vartheta_m^\varepsilon(x,y,t) = D^\varepsilon \widetilde{\theta}_f^\varepsilon(x,y,t) \\
& \text{in } H^{1/2}(\Gamma_{fm}) \text{ for } (x,t) \in Q,
\end{aligned} \tag{6.144}
$$

$$
p_m^\varepsilon(x,y,0) = D^\varepsilon p_0(x,y), \quad \vartheta_m^\varepsilon(x,y,0) = D^\varepsilon \theta_0(x,y) \quad \text{in } \Omega \times Y_m. \tag{6.145}
$$

It will cause no confusion that in (6.145) *we use again the notation D^ε, this time for the dilation operator defined for $L^2(\Omega)$-functions which maps them to the functions in $L^2(\Omega \times Y_m)$ by the formula*

$$
(D^\varepsilon \varphi)(x,y) = \varphi(\varepsilon y + c^\varepsilon(x)).
$$

We note that the problem (6.141)-(6.142) *with the non-homogenous Dirichlet boundary conditions* (6.144)-(6.145) *has at least one weak solution due to Theorem 5. The system* (6.141)-(6.142), (6.144)-(6.145) *corresponds to a family of problems in $Y_m \times]0,T[$, parametrized by x and depending on x through the source terms on the right-hand side of the equations* (6.141) *and* (6.142) *as well as through the boundary data in* (6.144) *and the initial data in* (6.145).

The following Lemma contains the uniform a priori estimates for the dilated solutions.

Lemma 21 *Let $(p_m^\varepsilon, \vartheta_m^\varepsilon)_\varepsilon$ be the sequence of solutions to* (6.141)-(6.142), (6.144)-(6.145).

There exists a constant C independent of ε such that

$$\|p_m^\varepsilon\|_{L^2(Q;H_p^1(Y_m))} + \|\vartheta_m^\varepsilon\|_{L^2(Q;H_p^1(Y_m))} \le C, \tag{6.146}$$

$$\|\partial_t(\phi s_m^\varepsilon)\|_{L^2(Q;H_p^{-1}(Y_m))} + \|\partial_t(\phi v_m^\varepsilon)\|_{L^2(Q;H_p^{-1}(Y_m))} \le C. \tag{6.147}$$

Moreover, $0 \le s_m^\varepsilon \le 1$ *a.e. in* $Q \times Y_m$.

Proof. The estimate (6.146) follows immediately from the estimates (6.34) and (6.35) by applying Lemma 1. On the other hand, by choosing a test function $\varphi \in L^2(0,T;H_0^1(\Omega_m^\varepsilon))$ extended by zero outside the matrix part in the weak formulation (6.9), the estimates (6.34) and (6.35) yield the uniform bound

$$\|\partial_t(\phi^\varepsilon S_m^\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega_m^\varepsilon))} + \|\partial_t(\phi^\varepsilon V_m^\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega_m^\varepsilon))} \le C$$

by arguing as in the proof of Proposition 5. Now (6.147) can be obtained by employing Lemma 1. This finishes the proof of Lemma 21.

□

By combining the uniform estimate (6.146) with the two-scale convergence results (6.128) and (6.129) through Lemma 2 for the functions p_m^ε and ϑ_m^ε, and analogously by using the boundedness of the saturation and of the gas density along with the convergence results (6.130) and (6.131) and Lemma 2 for s_m^ε and v_m^ε, the following results on the convergence of the dilated solutions can be established.

Corollary 3 *Let $(p_m^\varepsilon, \vartheta_m^\varepsilon)_\varepsilon$ be the sequence of solutions to* (6.141)-(6.142), (6.144)-(6.145) *and let the functions p, ϑ, s, v be defined by* (6.128)-(6.131). *Then it holds as $\varepsilon \to 0$, up to a subsequence,*

$$p_m^\varepsilon \rightharpoonup p \text{ weakly in } L^2(Q;H^1(Y_m)), \tag{6.148}$$

$$\vartheta_m^\varepsilon \rightharpoonup \vartheta \text{ weakly in } L^2(Q;H^1(Y_m)), \tag{6.149}$$

$$s_m^\varepsilon \rightharpoonup s \text{ weakly in } L^2(Q;L^2(Y_m)), \tag{6.150}$$

$$v_m^\varepsilon \rightharpoonup v \text{ weakly in } L^2(Q;L^2(Y_m)). \tag{6.151}$$

Hence the limit behavior of the dilated functions is known by Corollary 3. Still, the established weak convergence does not suffice to pass to the limit in the equations for the dilated functions (6.141)-(6.142) neither to identify the limits ϑ and v.

In order to resolve this holdback, we modify the idea of [14, 54] (see also [39]) which is based on the observation that for any ε the dilated functions are constant in x on a fixed block of an ε-reservoir. Namely, for a fixed $x_0 \in \Omega$ and for $\varepsilon > 0$ we write $\mathcal{C}^\varepsilon(x_0)$ for an ε-cell containing x_0 and we set $\mathcal{C}^\varepsilon_m(x_0) = \mathcal{C}^\varepsilon(x_0) \cap \Omega^\varepsilon_m$. The construction of ε-reservoir assures that $\mathcal{C}^\varepsilon(x_0)$ is well defined a.e. in Ω. We denote by $k(x_0, \varepsilon) \in \mathbb{Z}^d$ such that $\mathcal{C}^\varepsilon(x_0) = \varepsilon(Y + k(x_0, \varepsilon))$. Now for fixed $x_0 \in \Omega$ and $\varepsilon > 0$, we consider the restrictions of the dilated functions to the ε-cell $\mathcal{C}^\varepsilon(x_0)$. More precisely, we define for $y \in Y_m,\ t \in]0, T[$:

$$f^\varepsilon_{m,x_0}(y, t) = f^\varepsilon_m(x, y, t), \text{ for } x \in \mathcal{C}^\varepsilon(x_0), \tag{6.152}$$

where f stands for p, ϑ, s and v. Since the dilated functions are constant in x in $\mathcal{C}^\varepsilon(x_0)$, as noted before, the definition (6.152) does not depend on the choice of x. Moreover, we have $f^\varepsilon_{m,x_0}(y, t) = f^\varepsilon_m(x_0, y, t)$ for all $\varepsilon > 0$. The key feature of the newly introduced functions is that they possess more compactness than the dilated functions introduced by (6.140), as will be established in what follows.

First, from the definition of the functions $p^\varepsilon_{m,x_0}, \vartheta^\varepsilon_{m,x_0}$ it is clear that they solve the problem (6.141)-(6.142), (6.144)-(6.145) in the space $L^2(0, T; H^{-1}(Y_m))$.

Remark 23 *For a fixed $x_0 \in \Omega$, the boundary and initial conditions* (6.144)-(6.145) *applied to the system satisfied by $p^\varepsilon_{m,x_0}, \vartheta^\varepsilon_{m,x_0}$ become (see Remark 22)*

$$\begin{gathered} p^\varepsilon_{m,x_0}(y, t) = D^\varepsilon \widetilde{P}^\varepsilon_f(x_0, y, t), \quad \vartheta^\varepsilon_{m,x_0}(y, t) = D^\varepsilon \widetilde{\theta}^\varepsilon_f(x_0, y, t) \\ \text{in } H^{1/2}(\Gamma_{fm}) \text{ for } t \in]0, T[, \end{gathered} \tag{6.153}$$

$$p^\varepsilon_{m,x_0}(y, 0) = D^\varepsilon p_0(x_0, y), \quad \vartheta^\varepsilon_{m,x_0}(y, 0) = D^\varepsilon \theta_0(x_0, y) \quad \text{in } Y_m, \tag{6.154}$$

with D^ε in (6.154) *denoting the dilation operator acting on the functions defined in Q.*

Consequently, one can obtain the following uniform estimates.

Lemma 22 *For $x_0 \in \Omega$ and $\varepsilon > 0$ there is a constant $C(x_0)$ which does not depend on ε such that*

$$\|p^\varepsilon_{m,x_0}\|_{L^2(0,T;H^1_p(Y_m))} + \|\vartheta^\varepsilon_{m,x_0}\|_{L^2(0,T;H^1_p(Y_m))} \le C(x_0), \tag{6.155}$$

$$\|\partial_t(\phi s^\varepsilon_{m,x_0})\|_{L^2(0,T;H^{-1}_p(Y_m))} + \|\partial_t(\phi v^\varepsilon_{m,x_0})\|_{L^2(0,T;H^{-1}_p(Y_m))} \le C(x_0). \tag{6.156}$$

In addition, $0 \le s^\varepsilon_{m,x_0} \le 1$ a.e. in $Y_m \times]0, T[$.

Proof. We are dealing with the matrix problem (6.141)-(6.142), (6.144)-(6.145) with the non-homogenous Dirichlet boundary data and hence we use the same technique as in Proposition 2. This yields an estimate analogous to (4.65), involving for a.e. $x \in \Omega$ the corresponding norms of the dilatations of the extended fracture solutions, in variables y

and t. First we note that (A.7) and the a priori estimate (6.51), by applying Lemma 1, guarantee the following bounds for a.e. $x \in \Omega$, uniform in ε:

$$\|D^\varepsilon F_w(x,\cdot,\cdot)\|_{L^2(Y_m\times]0,T[)} + \|D^\varepsilon F_g(x,\cdot,\cdot)\|_{L^2(Y_m\times]0,T[)} \le C(x);$$
$$\|D^\varepsilon \widetilde{P}^\varepsilon_f(x,\cdot,\cdot)\|_{L^2(0,T;H^1(Y_m))} \le C(x).$$

At this point we need to pose the additional uniform regularity conditions on the fracture solutions as follows: we assume that there exists a constant C which does not depend on ε such that

$$\|P_c(\widetilde{S}^\varepsilon_f)\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{6.157}$$
$$\|\partial_t \widetilde{P}^\varepsilon_f\|_{L^2(Q)} + \|\partial_t(P_c(\widetilde{S}^\varepsilon_f))\|_{L^2(Q)} \le C, \tag{6.158}$$
$$\|\widetilde{P}^\varepsilon_f\|_{L^\infty(0,T;L^2(Q))} + \|P_c(\widetilde{S}^\varepsilon_f)\|_{L^\infty(0,T;L^2(Q))} \le C. \tag{6.159}$$

The assumptions (6.157)-(6.159) yield the following uniform estimates for a.e. x:

$$\|D^\varepsilon \widetilde{P}^\varepsilon_{gf}(x,\cdot,\cdot)\|_{L^2(0,T;H^1(Y_m))} + \|D^\varepsilon \widetilde{P}^\varepsilon_{wf}(x,\cdot,\cdot)\|_{L^2(0,T;H^1(Y_m))} \le C(x),$$
$$\|\partial_t D^\varepsilon \widetilde{P}^\varepsilon_{gf}(x,\cdot,\cdot)\|_{L^1(Y_m\times]0,T[)} + \|\partial_t D^\varepsilon \widetilde{P}^\varepsilon_{wf}(x,\cdot,\cdot)\|_{L^1(Y_m\times]0,T[)} \le C(x),$$
$$\|D^\varepsilon \widetilde{P}^\varepsilon_{gf}(x,\cdot,\cdot)\|_{L^\infty(0,T;L^1(Y_m))} + \|D^\varepsilon \widetilde{P}^\varepsilon_{wf}(x,\cdot,\cdot)\|_{L^\infty(0,T;L^1(Y_m))} \le C(x).$$

As argued in Remarks 5 and 8, by taking into account the inequality of the type (4.65), the last estimates enable us to establish the uniform bounds (6.155) and (6.156). □

The a priori estimates (6.155) and (6.156) yield the corresponding weak convergence results. Besides, the following compactness result is established by virtue of Lemma 14, that is, by [6, Lemma 4.2].

Lemma 23 *Under the assumptions (A.1-d) - (A.2-d), (A.3) - (A.7) and (A.8-d), for a.e. $x_0 \in \Omega$ the sets $\{s_{m,x_0} : \varepsilon > 0\}$ and $\{v_{m,x_0} : \varepsilon > 0\}$ are compact in $L^2(Y_m\times]0,T[)$.*

Proof. We follow the lines of the proof of Lemma 13. Namely, by arguing as in the proof of Lemma 12, one first gets the Hölder continuity of s_{m,x_0} and v_{m,x_0} in terms of p_{m,x_0} and ϑ_{m,x_0}. Then the uniform a priori estimate (6.155) is taken into account to validate the condition 2. of Lemma 14. Finally, the estimate (6.156) shows directly the third condition in Lemma 14 so this Lemma can be applied to the sequences s_{m,x_0} and v_{m,x_0}, which gives the desired conclusion. □

Finally, in next Lemma we compile the convergence results for the functions $p^{\varepsilon}_{m,x_0}, \vartheta^{\varepsilon}_{m,x_0}, s^{\varepsilon}_{m,x_0}$ and v^{ε}_{m,x_0} which follow from Lemmata 22 and 23.

Lemma 24 *Let $x_0 \in \Omega$. The following convergence results hold true as $\varepsilon \to 0$, up to a subsequence (which depends on x_0):*

$$p^{\varepsilon}_{m,x_0} \rightharpoonup p_{x_0} \text{ weakly in } L^2(0,T; H^1_p(Y_m)), \tag{6.160}$$
$$\vartheta^{\varepsilon}_{m,x_0} \rightharpoonup \vartheta_{x_0} \text{ weakly in } L^2(0,T; H^1_p(Y_m)) \text{ and strongly in } L^2(Y_m \times]0,T[), \tag{6.161}$$
$$s^{\varepsilon}_{m,x_0} \to s_{x_0} \text{ strongly in } L^2(Y_m \times]0,T[), \tag{6.162}$$
$$v^{\varepsilon}_{m,x_0} \to v_{x_0} \text{ strongly in } L^2(Y_m \times]0,T[), \tag{6.163}$$
$$\partial_t(\phi s^{\varepsilon}_{m,x_0}) \rightharpoonup \partial_t(\phi s_{x_0}) \text{ weakly in } L^2(0,T; H^{-1}(Y_m)), \tag{6.164}$$
$$\partial_t(\phi v^{\varepsilon}_{m,x_0}) \rightharpoonup \partial_t(\phi v_{x_0}) \text{ weakly in } L^2(0,T; H^{-1}(Y_m)), \tag{6.165}$$

where $\vartheta_{x_0} = \beta(s_{x_0})$ and $v_{x_0} = \rho_g(s_{x_0}, p_{x_0}) s_{x_0}$.

Let us note that the identification of v_{x_0} in Lemma 24 is achieved by employing the convergences (6.160) and (6.162) and the boundedness and continuity of the coefficients, and by applying Lemma 5.

The convergence results of Lemma 24 allow for a fixed $x_0 \in \Omega$ the passage to the limit as $\varepsilon \to 0$ in the equations (6.141)-(6.142). As in Chapter 4, the terms which contain the nonlinear functions of s_{m,x_0}, p_{m,x_0} are treated by applying Lemma 7. One passes to the limit in the source term by employing the result (3.5). The passage to the limit in the boundary conditions (6.153) is performed by using the following result.

Lemma 25 *Up to a subsequence, for a.e. $x \in \Omega$ it holds*

$$D^{\varepsilon}\widetilde{\theta}^{\varepsilon}_f(x,y,t) \to \theta(x,t) \text{ in } L^2(0,T; H^1(Y_m)), \tag{6.166}$$
$$D^{\varepsilon}\widetilde{P}^{\varepsilon}_f(x,y,t) \rightharpoonup P(x,t) \text{ in } L^2(0,T; H^1(Y_m)), \tag{6.167}$$

where the functions P and θ are defined by (6.124) *and* (6.125).

Proof. Let us write

$$D^{\varepsilon}\widetilde{\theta}^{\varepsilon}_f(x,y,t) - \theta(x,t) = \big(D^{\varepsilon}\widetilde{\theta}^{\varepsilon}_f(x,y,t) - D^{\varepsilon}\theta(x,y,t)\big) + \big(D^{\varepsilon}\theta(x,y,t) - \theta(x,t)\big), \tag{6.168}$$
$$D^{\varepsilon}\widetilde{P}^{\varepsilon}_f(x,y,t) - P(x,t) = \big(D^{\varepsilon}\widetilde{P}^{\varepsilon}_f(x,y,t) - D^{\varepsilon}P(x,y,t)\big) + \big(D^{\varepsilon}P(x,y,t) - P(x,t)\big). \tag{6.169}$$

By using Lemma 1 it holds

$$\|D^\varepsilon \widetilde{\theta}_f^\varepsilon(x,y,t) - D^\varepsilon \theta(x,y,t)\|_{L^2(\Omega\times Y_m\times]0,T[)} = \|\widetilde{\theta}_f^\varepsilon(x,t) - \theta(x,t)\|_{L^2(\Omega_m^{\varepsilon,T})},$$

so due to the convergence result (6.54), the first summand in (6.168) converges strongly to 0 in $L^2(\Omega \times Y_m\times]0,T[)$. The second term in (6.168) tends to zero in $L^2(\Omega\times Y_m\times]0,T[)$ by (3.5). Further, we have from Lemma 1

$$\|\nabla_y D^\varepsilon \widetilde{\theta}_f^\varepsilon(x,y,t)\|_{L^2(\Omega\times Y_m\times]0,T[)} = \varepsilon\|\nabla_x \widetilde{\theta}_f^\varepsilon(x,t)\|_{L^2(\Omega_m^{\varepsilon,T})} \le \varepsilon\|\nabla_x \widetilde{\theta}_f^\varepsilon(x,t)\|_{L^2(Q)}$$

and hence, due to the uniform a priori estimates, it follows that $\nabla_y D^\varepsilon \widetilde{\theta}_f^\varepsilon(x,y,t) \to 0 = \nabla_y \theta(x,t)$ in $L^2(\Omega \times Y_m\times]0,T[)$. Together, we have obtained $D^\varepsilon \widetilde{\theta}_f^\varepsilon(x,y,t) \to \theta(x,t)$ in $L^2(Q; H^1(Y_m))$. Then we conclude that, along a subsequence, $D^\varepsilon \widetilde{\theta}_f^\varepsilon(x,y,t) \to \theta(x,t)$ for a.e. $x\in\Omega$, strongly in $L^2(0,T;H^1(Y_m))$. Hence (6.166) is established.

Now we consider the global pressure term. Assuming that the limit S of the fracture saturations (which is defined by (6.93) and (6.92)) is strictly positive almost everywhere in Q, we can obtain the a.e. in Q convergence of $\widetilde{P}_f^\varepsilon$ to P by using the convergence results (6.94), (6.98) and (6.124) (as in the proof of Lemma 5). Therefore the same arguments can be applied to $\widetilde{P}_f^\varepsilon$ as we have done with $\widetilde{\theta}_f^\varepsilon$. In this way we prove (6.167) and finish the proof of Lemma 25.

□

After passing to the limit as $\varepsilon \to 0$ in the matrix problem (6.141)-(6.142), (6.153)-(6.154) for fixed $x_0\in\Omega$, the following system is obtained which is satisfied in $L^2(0,T;H^{-1}(Y_m))$

$$-\rho_w\phi(y)\partial_t s_{x_0} - \operatorname{div}_y\left(k(y)\left[\Lambda_w(s_{x_0},p_{x_0})\nabla_y p_{x_0} - A(s_{x_0},p_{x_0})\nabla_y\vartheta_{x_0} - \lambda_w(s_{x_0})\rho_w^2\mathbf{g}\right]\right) = F_w, \tag{6.170}$$

$$\phi(y)\partial_t v_{x_0} - \operatorname{div}_y\left(k(y)\left[\Lambda_g(s_{x_0},p_{x_0})\nabla_y p_{x_0} + A(s_{x_0},p_{x_0})\nabla_y\vartheta_{x_0} - \lambda_g(s_{x_0})\rho_g(s_{x_0},p_{x_0})^2\mathbf{g}\right]\right) = F_g, \tag{6.171}$$

which is complemented by the boundary and initial conditions

$$p_{x_0}(y,t) = P(x_0,t), \quad \vartheta_{x_0}(y,t) = \theta(x_0,t) \text{ in } H^{1/2}(\Gamma_{fm}) \text{ for } t\in]0,T[,$$
$$p_{x_0}(y,0) = p_0(x_0), \quad \vartheta_{x_0}(y,0) = \theta_0(x_0) \quad \text{in } Y_m.$$

Moreover, one has the identification $\vartheta_{x_0} = \beta(s_{x_0})$ and $v_{x_0} = \rho_g(s_{x_0},p_{x_0})s_{x_0}$ (Lemma 24).

The only point remaining is to relate the functions $p_{x_0}, \vartheta_{x_0}, s_{x_0}, v_{x_0}$ to the corresponding weak limits p, ϑ, s, v given by (6.148)-(6.151). To that aim, at this point we set the assumption that the problem (4.5)-(4.6) with the boundary and initial conditions (4.7)-(4.10) has a unique weak solution. In this case the convergence results (6.160)-(6.163) hold for the whole corresponding sequences, as $\varepsilon \to 0$. By taking into account the definition (6.152) and by considering the test functions $\varphi(x,y,t) \in L^2(\Omega \times Y_m \times]0,T[)$ in a separated form $\varphi(x,y,t) = \varphi_1(x,t)\varphi_2(y)$ with $\varphi_1 \in L^2(Q)$, $\varphi_2 \in L^2(Y_m)$, we can conclude that this weak convergence is in fact in $L^2(\Omega \times Y_m \times]0,T[)$. Due to the convergence results (6.148)-(6.151), we can identify the weak limits. Thereby from the system (6.170)-(6.171) we conclude that the following equations are satisfied for a.e. $x \in \Omega$ in $L^2(0,T;H^{-1}(Y_m))$:

$$-\rho_w \phi(y)\partial_t s - \operatorname{div}_y \left(k(y) \left[\Lambda_w(s,p)\nabla_y p - A(s,p)\nabla_y \vartheta - \lambda_w(s)\rho_w^2 \mathbf{g} \right] \right) = F_w, \tag{6.172}$$

$$\phi(y)\partial_t v - \operatorname{div}_y \left(k(y) \left[\Lambda_g(s,p)\nabla_y p + A(s,p)\nabla_y \vartheta - \lambda_g(s)\rho_g(s,p)^2 \mathbf{g} \right] \right) = F_g, \tag{6.173}$$

with the boundary and initial conditions (6.23)-(6.24), where the functions p, ϑ, s and v are defined by (6.148)-(6.151); moreover, $\vartheta = \beta(s)$ and $v = \rho_g(s,p)s$ a.e. in $\Omega \times Y_m \times]0,T[$ (see Lemma 24). Therefore Theorem 10 is proved.

□

Chapter 7

Conclusion and perspectives

The purpose of this dissertation is to investigate three standard problems for the new formulation of the immiscible compressible two-phase flow, fully equivalent to the original one, which is derived by using the notion of the global pressure in [8, 11]. First, we extend the existence results for two compressible fluids of [12] to the case of water-gas flow. In this work we permit the realistic case of an unbounded capillary pressure function and its derivatives at both ends $S = 0, 1$. Compared to the corresponding assumptions on the data made in [12], in our case the restrictions on the capillary pressure are less strong. More precisely, they are set only at $S = 0$, which is a consequence of incompressibility of the wetting phase. Our requirements on the sign of the boundary and source data, $F_w \geq 0$ and $G_w \leq 0$, are needed only if the capillary pressure curve is unbounded at $S = S_g = 1$. In such case the said restrictions correspond to not allowing extraction of the wetting phase from the domain, which is expected since otherwise we can not control the decay of the wetting phase pressure to $-\infty$. While we follow the strategy used in [12], the incompressibility of one phase causes additional difficulties in obtaining a priori estimates and passage to the limit, and makes the proof essentially more involved. Namely, in contrast to the case of strictly increasing mass densities, the characteristic change of variables $(S, P) \mapsto (\rho_w(S, P)(1 - S), \rho_g(S, P)S)$ is not a diffeomorfism in our situation. By taking advantage of the global pressure we are able to derive the uniform a priori estimates for the global pressure, which is not the case with the phase pressures in a usual phase form of the flow equations due to the vanishing of the phase relative permeabilities in the zones without one of the fluids. However, due to that vanishing property of the relative permeabilities one could still not obtain the uniform L^2 bounds for the gradient of the saturation, which motivates the induction of the saturation potential $\theta = \beta(S)$, as in [12]. Despite both transformations, the degeneracy of the equations is still present through the diffusion term $A(S, P)$ which is equal to zero for the saturation values $S = 0$ and $S = 1$.

The integrability problems caused by the singularity of the capillary pressure as well as the diffusion degeneracy are overcame by an appropriate regularization, as in [12]. Another difficulty is vanishing of the pressure time derivative term, which is treated with the aid of an adjusted compactness result of [12]. Let us note that the assumption (A.8) in Chapter 4 on the boundary data is the sufficient condition to obtain the uniform a priori estimates, which arises from the particular choice of the test functions (suggested by [68]).

Furthermore, in this work the homogenization result for the single rock-type heterogenous porous medium has been established for the case of two compressible phases, but the extension of all the results to an incompressible wetting and a compressible non-wetting phases is possible quite straightforward. This can be seen by bringing together the a priori estimates established as in Subsection 5.4, and the almost everywhere convergence for the saturation and the global pressure which may be established by the arguments of Chapter 4. Moreover, one can use the existence result of Chapter 4 to consider in this manner the homogenization of a water-gas flow subject to the non-homogenous conditions on the boundary.

In the double porosity model for the flow of gas and water presented in this thesis with the matrix permeability being scaled by ε^2, the gravitational term is additionally compensated by ε^{-1} since otherwise the macroscopic model would contain no gravitational terms in the matrix equations, which can be observed from the limiting process as well as from the formal asymptotic expansion which is not included in this work. The derivation of the compactness result for the extensions of the fracture solutions S_f^ε and $\rho_g(S_f^\varepsilon, P_f^\varepsilon)S_f^\varepsilon$ to the whole domain is complicated by the fact that one has no uniform estimates for their time derivatives in a fractured medium. The suitable test functions are hence employed to establish the uniform bounds for the time translations which enables the use of standard compactness results of [5] and [95]. The essential part of the fluid flow in a fractured reservoir is the microscopic flow from matrix to the fissures which is captured at the global level by the source-like terms that contain the local scale. Their identification is the most involved part of the homogenization process and it relies on the dilation operator of [26].

The results of the research presented in this work can be continued in several directions. It would be interesting to extend the existence results presented in this thesis to the porous media with several rock types. In such setting the capillary pressure and relative permeability functions are different in each type of porous media and the continuity of the physical quantities at the separating interfaces between different media gives rise to the nonlinear transmission conditions. Our future study will include also the extension of the homogenization results obtained in this thesis to the case of multiple rock type porous media, where the upscaling process becomes more involved. A suggestion for further work concerns also the extension of the results obtained for the double porosity model studied in Chapter 6 under weak assumptions and the derivation of a double porosity model for the fractured reservoir in the case of the two strictly compressible phases. To our knowledge, there have been no results for this problem yet. The proof of the compactness for the microscopic solutions may be more demanding in this situation. Another direction of suggested future research is the study of the existence and the homogenization for the partially miscible compressible two-phase flow, such as hydrogen dissolving in water in a nuclear waste management context.

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- Workshop "Young Women in PDEs", Bonn, Germany, May 2012

Talks and Posters

- *Homogenization of immiscible compressible two phase flow in porous media by the concept of global pressure*, Conference on applied mathematics and scientific computing ApplMath11, Trogir, June 2009 (talk)
- *An existence result for a system modeling a water-gas flow in porous media in a fully equivalent global pressure formulation*, Workshop on "Fluid Dynamics in Porous Media", Coimbra, Portugal, September 2011 (talk)
- *On existence for a system modeling water-gas flow in porous media in a fully equivalent global pressure formulation*, Workshop "Young Women in PDEs", Bonn, Germany, May 2012 (poster presentation)

Short term visits

- Université de Pau et des Pays de l'Adour, Laboratoire de Mathématiques et de leurs Applications, Pau, France, March 17 - June 14, 2011; October 31 - November 24, 2011; January 19 - February 17, 2012; May 29 - June 25, 2012

Životopis

Osobni podaci

- rodjena 11. 11. 1980. u Zagrebu

Obrazovanje

- **studeni 2006. –** doktorski studij, Prirodoslovno-matematički fakultet, Matematički odsjek, Zagreb
- **rujan 1999. – listopad 2005.** dodiplomski studij, Prirodoslovno-matematički fakultet, Matematički odsjek, Zagreb, diplomski rad *Stokesov paradoks* pod vodstvom prof. dr. sc. Eduarda Marušića Paloke
- **rujan 1995. – lipanj 1999.** V. gimnazija, Zagreb
- **rujan 1987. – lipanj 1995.** Osnovna škola Ljudevita Gaja, Zaprešić

Profesionalno iskustvo

- **2006 –** asistent i znanstveni novak, Prirodoslovno-matematički fakultet, Matematički odsjek, Zagreb
- **2005–2006** profesor matematike i informatike, Srednja škola bana Josipa Jelačića, Zaprešić

Znanstveni radovi

1. B. Amaziane, M. Jurak, A. Vrbaški. *Homogenization results for a coupled system modeling immiscible compressible two-phase flow in porous media by the concept of global pressure*, prihvaćeno za objavljivanje u Applicable Analysis, DOI:10.1080/00036811.2012.682059.

2. B. Amaziane, M. Jurak, A. Vrbaški. *Existence for a global pressure formulation of water-gas flow in porous media*, Electronic Journal of Differential Equations, 2012 (102): 1–22, 2012.

Projekti

- **2007 –** : Ministarstvo znanosti, obrazovanja i športa Republike Hrvatske, projekt br. 037-0372787-2798 *Numeričko modeliranje strujanja fluida kroz poroznu sredinu*
- **2002 – 2006**: Ministarstvo znanosti i tehnologije Republike Hrvatske, projekt br. 0037111 *Efektivni parametri u matematičkim modelima ležišta*

Konferencije, radionice i ljetne škole

- 5th Conference on Applied Mathematics and Scientific Computing, ApplMath 07, Brijuni, srpanj 2007.
- Advanced School on Numerical Solutions of Partial Differential Equations, Bellatera, Španjolska, studeni 2007.
- International Summer School on Numerical Linear Algebra (SIAM), Castro Urdiales, Španjolska, srpanj 2008.
- Conference on Scaling Up for Modeling Transport and Flow in Porous Media, Dubrovnik, listopad 2008.
- 6th Conference on Applied Mathematics and Scientific Computing, ApplMath 09, Zadar, rujan 2009.
- Conference "Multiscale problems in science and technology - challenges to mathematical analysis and perspectives III", Dubrovnik, Croatia, lipanj 2010.
- Workshop "Homogenisation, multi-scale methods and applications" (DAAD), Dubrovnik, lipanj 2010.
- 7th Conference on Applied Mathematics and Scientific Computing, ApplMath 11, Trogir, lipanj 2011.
- Workshop on "Fluid Dynamics in Porous Media", Coimbra, Portugal, rujan 2011.
- Workshop "Young Women in PDEs", Bonn, Njemačka, svibanj 2012.

Javna izlaganja i posteri

- *Homogenization of immiscible compressible two phase flow in porous media by the concept of global pressure*, Conference on applied mathematics and scientific computing ApplMath11, Trogir, June 2009. (izlaganje)
- *An existence result for a system modeling a water-gas flow in porous media in a fully equivalent global pressure formulation*, Workshop on "Fluid Dynamics in Porous Media", Coimbra, Portugal, September 2011. (izlaganje)
- *On existence for a system modeling water-gas flow in porous media in a fully equivalent global pressure formulation*, Workshop "Young Women in PDEs", Bonn, Germany, May 2012. (poster)

Studijski boravci

- Université de Pau et des Pays de l'Adour, Laboratoire de Mathématiques et de leurs Applications, Pau, Francuska, 17.3.-14.6.2011., 31.10.-24.11.2011., 19.1.-17.2.2012., 29.5.-25.6.2012.