A fully homogenized model for incompressible two-phase flow in double porosity media

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Abstract

In this paper we discuss a model describing the global behavior of the two-phase incompressible flow in fractured porous media. The fractured medium is regarded as a porous medium consisting of two superimposed continua, a connected fracture system, which is assumed to be thin of order $\varepsilon\delta$, where δ being the relative fracture thickness, and an ε–periodic system of disjoint matrix blocks. We derive the global behavior of the fractured medium by passing to the limit as $\varepsilon \to 0$, taking into account that the permeability of the blocks is proportional to $(\varepsilon\delta)^2$, while the permeability of the fractures is of order one and obtain the corresponding global δ–model, i.e., the homogenized model with the coefficients depending on the small parameter δ. In the δ–model we linearize the cell problem in the matrix block and then by letting $\delta \to 0$ we obtain the macroscopic model which does not depend on ε and δ and is fully homogenized in the sense that all the coefficients are calculated in terms of given data and do not depend on the additional coupling or cell problems.

Keywords: homogenization; incompressible; two-phase flow; double porosity media; thin fractures

AMS Subject Classifications: 35B27; 35K65; 35Q35; 74Q15; 76M50; 76S05

1 Introduction

A naturally fractured reservoir is a reservoir that contains fracture planes distributed as a connected network throughout the reservoir. This type of porous medium is frequently encountered in hydrology and petroleum applications, for instance the sedimentary rock that composes a hydrocarbon reservoir. The fluid flow mechanism in such reservoirs has been known to be significantly different from that of an ordinary, unfractured reservoir. Specifically, the flow occurs as if the reservoir possessed two porous structures, one associated to the porous rock, and the

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other one to the system of fractures. Accordingly, a naturally fractured reservoir is considered as a porous medium consisting of two superimposed continua, a discontinuous system of periodically distributed matrix blocks surrounded by a connected system of thin fissures. Characteristic features of fractured rocks are that the volume occupied by the fractures is much smaller than the volume of the pores; the matrix keeps most of the fluid while the fractures are notably more permeable (see [1]). The fluid exchange between matrix blocks and fractures is a microscale process whose strong influence on the flow must be embedded in a large scale flow description. The macroscopic behavior of fluid flow in such porous media is described by the so-called double porosity model which was first derived experimentally as a physical notion and described by several authors in the engineering literature ([1, 2]). In the standard double porosity model one assumes that the width of the fractures is of the same order as the block size. However, the model of [1] assumes that the measure of the fracture set is small with respect to the measure of the pore blocks. One of the approaches in modeling such problems is therefore to consider the thickness of the fractures as an additional small parameter. In this work we consider a double porosity type model for incompressible two-phase fluid flow in a porous medium with thin fractures.

The first contribution on the derivation of the double porosity model for two-phase flow in a fractured medium is [3], where the effective equations of the double porosity model are established by formal technique of asymptotic expansion for the cases of completely miscible incompressible flow, and immiscible incompressible two-phase flow. The double porosity model for immiscible incompressible two-phase flow in a reduced pressure formulation is rigorously justified by periodic homogenization in [4]. Another result on the two-phase incompressible immiscible flow in fractured porous media is established in [5]. For the displacement of one compressible miscible fluid by another in a naturally fractured reservoir, the double porosity model was rigorously derived in [6]. Furthermore, [7] and [8] study the existence of weak solutions for the two models of the immiscible two-phase flow in fractured porous media.

The method involving only one small parameter ε in modeling of the thin structures, now known as method of mesoscopic energy characteristics, was proposed by E. Khruslov (see, e.g., [9]). The method of two small parameters in modeling of periodic thin structures has been widely used in the mathematical literature (see, e.g., [10, 11]) and applied to various linear elliptic problems. However, all these works study problems with the coefficients which are uniformly bounded and elliptic with respect to the small parameters. The first result on homogenization of a linear double porosity problem in the case of thin fissures was obtained in [12] where the thickness of the fissures as well as the order of the permeability in the matrix blocks were modeled by one small parameter ε. That result was recently generalized in [13] where several applications were studied. The method of two small parameters ε and δ for the linear double porosity model was proposed in [14] and then used in [15] for the homogenization of a degenerate triple porosity model with thin fissures and in [16] for the homogenization of a single phase flow through a porous medium in a thin layer. Most of these results were presented in the review paper [17]. The nonlinear elliptic double porosity type problem in domains with thin fissures was studied in [18]. The main feature of the double porosity models with thin fissures, compared to the standard double porosity models, is that such models do not contain any coupling between the meso- and macro-scale through the coefficients that depend on additional cell problems.

This paper contains a new homogenization result for the system modeling immiscible incompressible two-phase flow in a periodic fractured porous medium with thin fractures, modeled by two small parameters. The first one, ε, stands for the periodicity of the structure, and the second one, δ, describes the relative thickness of the fissure system. The scheme of the derivation of the resulting macroscopic model is as follows. In the first step we pass to the limit as $\varepsilon \to 0$ and obtain the homogenized system (9). This passage is a well known result which was rigorously justified in [4, 5]. Notice that the homogenization process is finished at this step. We obtain the global model but with the coefficients depending on a small parameter δ. In order to simplify the asymptotic analysis with respect to δ we introduce a second, "heuristic" step, concerning a linearization of the cell problem (17) in the global δ–model. The idea of this linearization comes from the paper [19] by T. Arbogast. However, the obtained simplified system is not equivalent to the δ–model (9). We underline that our goal is to study the linearized system and not the nonlinear one. Finally, in the third step we pass to the limit as $\delta \to 0$ in the simplified δ–model in order to obtain the desired global model with the coefficients that do not depend on ε and δ. In this study we make use of the known result from [17] (see Lemma 2 below). However, the passage to the limit in our case is much more difficult than in [17], where it was done for the linear elliptic equation. The main difficulty concerns the fact that we pass to the limit in a nonlinear degenerating integro-differential equation. This result is new in the homogenization theory.

The paper is organized in the following way. In Section 2 we set up the problem which describes the model on the mesoscale (the Darcy scale) with the coefficients depending on ε and δ. Then in Section 3 we present the global double porosity δ-model which has been derived earlier in [4], [5] from the mesoscopic problem and we present a derivation of the imbibition equation. Section 4 is devoted to decoupling the global δ-model from the system defined on a matrix cell: following [19], we linearize the imbibition equation and estimate its asymptotic behavior by using the Laplace transform. The passage to the limit as $\delta \to 0$ in the global double porosity δ-model is performed in Section 5. Namely, in Subsection 5.1 we obtain the *a priori* estimates for the weak solutions of the problem with respect to the space and time variables and establish a necessary compactness result. The main difficulty in derivation of uniform *a priori* estimates is in treatment of the convolution term. In this paper this term is estimated without an additional step of discretization of the time derivative. Finally, Subsection 5.2 exhibits the global fully homogenized model for immiscible incompressible two-phase flow in double porosity media with thin fractures. Namely, in the limit as $\delta \to 0$ we obtain the integro-differential system with constant effective coefficients which are defined in terms of the mesoscale parameters, and with an additional non-local in time source term of the convolution type describing the impact of the mesoscale matrix-fracture fluid exchange on the global flow behavior.

Up to our knowledge this is the first rigorous justification of fully homogenized double porosity model in the framework of the two-phase flow in a reservoir with thin fissures system.

2 Mesoscale model

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a periodic structure which is a union of disjoint cubes congruent to a reference cell $Y = (0, 1)^d$. The reference cell Y

consists of two subdomains, corresponding to the two types of rock - the matrix, and the fractures. Moreover, we suppose the relative fracture thickness to be of order δ, where $\delta > 0$ is a small parameter. In particular, we use the standard Warren-Root model which assumes that Y consists of an open cube Y_m^δ with edge length $1-\delta$, centered at the center of Y, completely surrounded by a connected fracture subdomain Y_f^δ, with a piecewise smooth internal boundary Γ^δ between the two media in Y. Therefore it is $Y = Y_m^\delta \cup \Gamma^\delta \cup Y_f^\delta$, where $|Y_f^\delta| = O(\delta)$ so that $|Y_f^\delta| \to 0$ as $\delta \to 0$. The outward unit normal vector to Y_m^δ is denoted by $\boldsymbol{\nu}^\delta$.

The periodic structure of a reservoir is depicted by a small parameter $\varepsilon > 0$ representing the characteristic size of the heterogeneities with respect to the size of Ω. Accordingly, for $\varepsilon > 0$ the domain Ω is assumed to be covered by a pavement of cells εY. For $\delta > 0$ let $\mathbf{1}_m^\delta(y)$ and $\mathbf{1}_f^\delta(y)$ be the characteristic functions of Y_m^δ and Y_f^δ, respectively, extended Y–periodically to the whole $\mathbb{R}^d$. The system of the matrix blocks in Ω, the fractured part of Ω and the matrix-fracture interface are denoted by $\Omega_m^{\varepsilon,\delta}$, $\Omega_f^{\varepsilon,\delta}$ and $\Gamma^{\varepsilon,\delta}$, respectively. For simplicity, we assume that $\Omega_m^{\varepsilon,\delta} \cap \partial\Omega = \emptyset$.

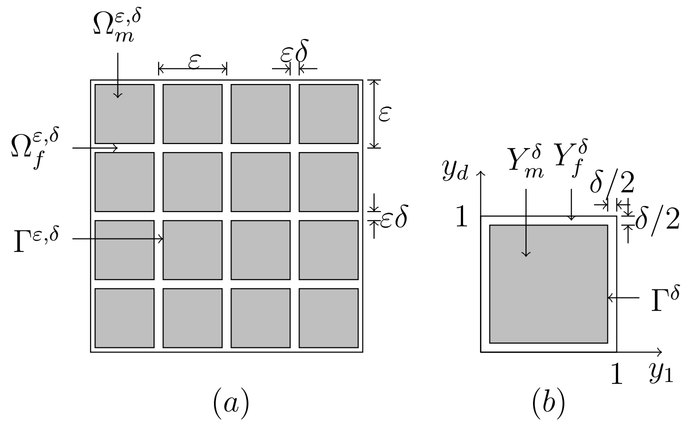

Figure 1: (a) The domain Ω with the mesostructure. (b) The reference cell Y.

We should point out here that in our starting mesoscopic model the fractures are represented as a porous medium with rock properties radically different from those of the matrix blocks. In particular, they are not represented as an empty space filled with the fluids. For an example of a numerical simulation over 3D matrix-fracture structure described here see e.g. [20].

The domain boundary $\partial\Omega$ consists of two parts, Γ_{inj} and Γ_{imp}, such that $\Gamma_{inj} \cap \Gamma_{imp} = \emptyset$, $\partial\Omega = \Gamma_{inj} \cup \Gamma_{imp}$. We will use the following notation: $\ell = f, m$ and $\Omega_T = \Omega \times (0,T)$, $\Omega_{\ell,T}^{\varepsilon,\delta} = \Omega_\ell^{\varepsilon,\delta} \times (0,T)$, $\Gamma_T^{\varepsilon,\delta} = \Gamma^{\varepsilon,\delta} \times (0,T)$, where $T > 0$ is fixed.

In this work we study the incompressible two-phase flow in the porous medium Ω over the time interval $(0,T)$. Let $S_\ell^{\varepsilon,\delta} \stackrel{\text{def}}{=} S_{w,\ell}^{\varepsilon,\delta}$, $S_{n,\ell}^{\varepsilon,\delta} = 1 - S_{w,\ell}^{\varepsilon,\delta}$ be the saturations of the wetting and the non-wetting phase in $\Omega_{\ell,T}^{\varepsilon,\delta}$, respectively; $\lambda_{w,\ell} = \lambda_{w,\ell}(S_\ell^{\varepsilon,\delta})$, $\lambda_{n,\ell} = \lambda_{n,\ell}(S_\ell^{\varepsilon,\delta})$ be the relative mobilities of the wetting and the non-wetting phase in $\Omega_{\ell,T}^{\varepsilon,\delta}$, respectively; let $P_{w,\ell}^{\varepsilon,\delta}$, $P_{n,\ell}^{\varepsilon,\delta}$ be the pressures of the wetting and the non-wetting phase in $\Omega_{\ell,T}^{\varepsilon,\delta}$, respectively. Finally, let $\Phi^{\varepsilon,\delta}(x)$ and

$\mathbb{K}^{\varepsilon,\delta}(x)$ be the porosity and the absolute permeability tensor of the porous medium Ω set by

$$\Phi^{\varepsilon,\delta}(x) \stackrel{\text{def}}{=} \begin{cases} \Phi_f & \text{in } \Omega^{\varepsilon,\delta}_{f,T} \\ \Phi_m & \text{in } \Omega^{\varepsilon,\delta}_{m,T} \end{cases} \quad \text{and} \quad \mathbb{K}^{\varepsilon,\delta}(x) \stackrel{\text{def}}{=} \begin{cases} k_f\, \mathbb{I} & \text{in } \Omega^{\varepsilon,\delta}_{f,T} \\ (\varepsilon\delta)^2\ k_m\, \mathbb{I} & \text{in } \Omega^{\varepsilon,\delta}_{m,T} \end{cases}, \tag{1}$$

where $\mathbb{I}$ is the unit tensor.

The mass conservation equations for the individual fluid phases in the subdomain $\Omega^{\varepsilon,\delta}_{\ell,T}$, $\ell = f, m$, are given by:

$$\Phi^{\varepsilon,\delta}\partial_t S^{\varepsilon,\delta}_\ell + \operatorname{div} \mathbf{q}^{\varepsilon,\delta}_{w,\ell} = 0, \quad -\Phi^{\varepsilon,\delta}\partial_t S^{\varepsilon,\delta}_\ell + \operatorname{div} \mathbf{q}^{\varepsilon,\delta}_{n,\ell} = 0, \tag{2}$$

with the velocities of the wetting and the non-wetting phases $\mathbf{q}^{\varepsilon,\delta}_{w,\ell}$, $\mathbf{q}^{\varepsilon,\delta}_{n,\ell}$ defined by the Darcy-Muskat's law (see, e.g., [21–23]):

$$\mathbf{q}^{\varepsilon,\delta}_{w,\ell} \stackrel{\text{def}}{=} -\mathbb{K}^{\varepsilon,\delta}(x)\lambda_{w,\ell}(S^{\varepsilon,\delta}_\ell)\nabla P^{\varepsilon,\delta}_{w,\ell}, \quad \mathbf{q}^{\varepsilon,\delta}_{n,\ell} \stackrel{\text{def}}{=} -\mathbb{K}^{\varepsilon,\delta}(x)\lambda_{n,\ell}(S^{\varepsilon,\delta}_\ell)\nabla P^{\varepsilon,\delta}_{n,\ell}, \tag{3}$$

where, for simplicity, the gravity effects are neglected.

The system (2)-(3) is closed by the capillary pressure law in each of the medium subdomains,

$$P_{c,\ell}(S^{\varepsilon,\delta}_\ell) = P^{\varepsilon,\delta}_{n,\ell} - P^{\varepsilon,\delta}_{w,\ell}, \quad \ell = f, m, \tag{4}$$

where $P_{c,\ell}$ is a given capillary pressure-saturation function.

Due to (1), (3), (4), the system (2) is now written in the subdomain $\Omega^{\varepsilon,\delta}_{f,T}$ as

$$\begin{aligned} &\Phi_f\partial_t S^{\varepsilon,\delta}_f - k_f \operatorname{div}\left(\lambda_{w,f}(S^{\varepsilon,\delta}_f)\nabla P^{\varepsilon,\delta}_{w,f}\right) = 0, \\ &-\Phi_f\partial_t S^{\varepsilon,\delta}_f - k_f \operatorname{div}\left(\lambda_{n,f}(S^{\varepsilon,\delta}_f)\nabla P^{\varepsilon,\delta}_{n,f}\right) = 0, \end{aligned} \tag{5}$$

with the capillary pressure law: $P_{c,f}(S^{\varepsilon,\delta}_f) = P^{\varepsilon,\delta}_{n,f} - P^{\varepsilon,\delta}_{w,f}$, and in the subdomain $\Omega^{\varepsilon,\delta}_{m,T}$ as

$$\begin{aligned} &\Phi_m\partial_t S^{\varepsilon,\delta}_m - (\varepsilon\delta)^2\ k_m \operatorname{div}\left(\lambda_{w,m}(S^{\varepsilon,\delta}_m)\nabla P^{\varepsilon,\delta}_{w,m}\right) = 0, \\ &-\Phi_m\partial_t S^{\varepsilon,\delta}_m - (\varepsilon\delta)^2\ k_m \operatorname{div}\left(\lambda_{n,m}(S^{\varepsilon,\delta}_m)\nabla P^{\varepsilon,\delta}_{n,m}\right) = 0, \end{aligned} \tag{6}$$

with the capillary pressure law: $P_{c,m}(S^{\varepsilon,\delta}_m) = P^{\varepsilon,\delta}_{n,m} - P^{\varepsilon,\delta}_{w,m}$.

On the matrix-fracture interface $\Gamma^{\varepsilon,\delta}$ the phase fluxes and pressures are required to be continuous. The boundary conditions for the system (5) are given by:

$$P^{\varepsilon,\delta}_{w,f} = P_{w,\Gamma} \text{ and } P^{\varepsilon,\delta}_{n,f} = P_{n,\Gamma} \text{ on } \Gamma_{inj} \times (0,T),\ \mathbf{q}^{\varepsilon,\delta}_{w,f} \cdot \boldsymbol{\nu} = \mathbf{q}^{\varepsilon,\delta}_{n,f} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_{imp} \times (0,T), \tag{7}$$

where $\boldsymbol{\nu}$ is the unit outward normal vector to $\partial\Omega$, and $P_{\alpha,\Gamma}$, $\alpha = w, n$, are given phase pressures. The initial conditions read:

$$S^{\varepsilon,\delta}_f(x,0) = S^0_f(x) \text{ in } \Omega^{\varepsilon,\delta}_f \quad \text{and} \quad S^{\varepsilon,\delta}_m(x,0) = S^0_m(x) \text{ in } \Omega^{\varepsilon,\delta}_m. \tag{8}$$

Let us now state the following assumptions on data.

(A.1) The porosity coefficients $0 < \Phi_f,\ \Phi_m < 1$ are constants independent of ε and δ.

(A.2) The absolute permeability coefficients $0 < k_f, k_m$ are constants independent of ε and δ.

(A.3) The capillary pressure functions satisfy for $\ell = f, m$: $P_{c,\ell} \in C^1((0,1];\mathbb{R}^+)$, $P'_{c,\ell}(s) < 0$ in $(0,1]$, $P_{c,m}(0^+) = P_{c,f}(0^+) \in (0,\infty]$, $P_{c,\ell}(1) = 0$. Furthermore, the initial data (8) are consistent in the sense that $P_{c,m}(S_m^0) = P_{c,f}(S_f^0)$ in Ω.

(A.4) The relative phase mobility functions satisfy $\lambda_{w,\ell}, \lambda_{n,\ell} \in C([0,1];\mathbb{R}^+)$, $\lambda_{w,\ell}(0) = \lambda_{n,\ell}(1) = 0$; $0 \leqslant \lambda_{w,\ell}, \lambda_{n,\ell} \leqslant 1$ in $[0,1]$; $\lambda_{w,\ell}$ is an increasing function in $[0,1]$ and $\lambda_{n,\ell}$ is a decreasing function in $[0,1]$. Moreover, there is a constant L_0 such that for all $s \in [0,1]$, $\lambda_\ell(s) \stackrel{\rm def}{=} \lambda_{w,\ell}(s) + \lambda_{n,\ell}(s) \geqslant L_0 > 0$.

The known theory (see, e.g., [24–26]) gives the existence of at least one weak solution to the problem (5)-(8) for fixed $\varepsilon > 0$, $\delta > 0$ under the conditions (A.1)–(A.4) and some supplementary regularity of saturation functions (see [25]). Notice that in [24] the existence of the solution is shown under the assumption that the total velocity field is given. In [25] the existence of fully coupled system is proved with homogeneous Neumman boundary conditions and the hypothesis on the capillary pressures that are more relaxed than (A.3), see [25], Assumption 2. Finally, the existence theorem in the case of compressible fluids is proven in [26] under additional assumption on Hölder continuity of the inverse of the function β_ℓ defined in (23).

3 Global double porosity δ-model

In the case when the typical size of the fractures is of the same order as the matrix block size, i.e. when $\delta = O(1)$, the homogenization process as $\varepsilon \to 0$ for the mesoscopic model (5)-(8) has been studied by formal homogenization techniques in [27–29], and rigorously in [4] and [5]. More precisely, in [5] the homogenization procedure for problem (5)-(8) with a fixed $\delta > 0$ as $\varepsilon \to 0$ was rigorously justified by using the notion of the two-scale convergence [30]. In this work various, rather strong assumptions were posed on the data which exclude appearance of one-phase zones and thus degeneracy of the system. On the other hand, the same type of result for the problem (5)-(8) in the global pressure formulation was established in [4] under an assumption of continuity of the saturations and the global pressure at the matrix-fracture boundary, but including possible one-phase zones.

We present now the global double porosity δ-model which was derived in [27], [4], [5] by keeping $\delta > 0$ fixed while passing to the limit as $\varepsilon \to 0$ in the mesoscopic problem (5)-(8). Namely, the global double porosity δ-model reads:

$$\begin{aligned} \Phi^\delta \partial_t S_f^\delta - \operatorname{div}\left(\mathbb{K}^{\star,\delta} \lambda_{w,f}(S_f^\delta) \nabla P_{w,f}^\delta\right) &= \mathfrak{Q}_w^\delta \quad \text{in } \Omega_T, \\ -\Phi^\delta \partial_t S_f^\delta - \operatorname{div}\left(\mathbb{K}^{\star,\delta} \lambda_{n,f}(S_f^\delta) \nabla P_{n,f}^\delta\right) &= \mathfrak{Q}_n^\delta \quad \text{in } \Omega_T, \end{aligned} \tag{9}$$

with the capillary pressure law: $P_{c,f}(S_f^\delta) = P_{n,f}^\delta - P_{w,f}^\delta$ in Ω_T. The boundary conditions for system (9) are given by:

$$\begin{aligned} &P^{\delta}_{w,f} = P_{w,\Gamma} \quad \text{and} \quad P^{\delta}_{n,f} = P_{n,\Gamma} \quad \text{on } \Gamma_{inj} \times (0,T), \\ &\mathbf{q}^{\delta}_{w,f} \cdot \boldsymbol{\nu} = \mathbf{q}^{\delta}_{n,f} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{imp} \times (0,T), \end{aligned} \tag{10}$$

where

$$\mathbf{q}^{\delta}_{w,f} = -\mathbb{K}^{\star,\delta}\lambda_{w,f}(S^{\delta}_f)\nabla P^{\delta}_{w,f} \quad \text{and} \quad \mathbf{q}^{\delta}_{n,f} = -\mathbb{K}^{\star,\delta}\lambda_{n,f}(S^{\delta}_f)\nabla P^{\delta}_{n,f}, \tag{11}$$

and the initial condition reads:

$$S^{\delta}_f(x,0) = S^0_f(x) \text{ in } \Omega. \tag{12}$$

The effective porosity Φ^{δ} is given as:

$$\Phi^{\delta} \stackrel{\text{def}}{=} \Phi_f \frac{|Y^{\delta}_f|}{|Y^{\delta}_m|} = \delta\, d\, \Phi_f + O(\delta^2), \tag{13}$$

where $|Y^{\delta}_m|$ and $|Y^{\delta}_f|$ denote the measure of the set Y^{δ}_m and Y^{δ}_f, respectively. $\mathbb{K}^{\star,\delta} = (\mathbb{K}^{\star,\delta}_{ij})$ is the effective permeability tensor given for $i, j = 1, \ldots, d$ by:

$$\mathbb{K}^{\star,\delta}_{ij} \stackrel{\text{def}}{=} \frac{k_f}{|Y^{\delta}_m|} \int_{Y^{\delta}_f} \left[\nabla_y \xi^{\delta}_i + \mathbf{e}_i\right] \left[\nabla_y \xi^{\delta}_j + \mathbf{e}_j\right] dy, \tag{14}$$

with $\mathbf{e}_j$ being the j–th coordinate vector. The function ξ^{δ}_j, $j = 1, \ldots, d$, is a Y-periodic solution of the cell problem:

$$-\Delta_y \xi^{\delta}_j = 0 \quad \text{in } Y^{\delta}_f, \quad (\nabla_y \xi^{\delta}_j + \mathbf{e}_j) \cdot \boldsymbol{\nu}^{\delta} = 0 \quad \text{on } \Gamma^{\delta}. \tag{15}$$

The matrix-fracture source terms $\mathcal{Q}^{\delta}_w$ and $\mathcal{Q}^{\delta}_n$ are given by:

$$\mathcal{Q}^{\delta}_w(x,t) \stackrel{\text{def}}{=} -\frac{\Phi_m}{|Y^{\delta}_m|} \int_{Y^{\delta}_m} \partial_t S^{\delta}_m(x,y,t)\, dy = -\mathcal{Q}^{\delta}_n(x,t), \tag{16}$$

where the function S^{δ}_m is the matrix block saturation defined below.

To each point $x \in \Omega$ there is an associated matrix block congruent to Y^{δ}_m. For any $x \in \Omega$ the flow equations in a matrix block $Y^{\delta}_m \times (0,T)$ are:

$$\begin{aligned} \Phi_m \partial_t S^{\delta}_m - \delta^2 k_m \operatorname{div}_y \left(\lambda_{w,m}(S^{\delta}_m)\nabla_y P^{\delta}_{w,m}\right) = 0, \\ -\Phi_m \partial_t S^{\delta}_m - \delta^2 k_m \operatorname{div}_y \left(\lambda_{n,m}(S^{\delta}_m)\nabla_y P^{\delta}_{n,m}\right) = 0, \end{aligned} \tag{17}$$

completed by the capillary pressure law $P_{c,m}(S^{\delta}_m) = P^{\delta}_{n,m} - P^{\delta}_{w,m}$. On the interface Γ^{δ} in Y we have the continuity conditions for the phase pressures and fluxes. The boundary conditions are

$$P^{\delta}_{w,m}(x,y,t) = P^{\delta}_{w,f}(x,t) \quad \text{and} \quad P^{\delta}_{n,m}(x,y,t) = P^{\delta}_{n,f}(x,t) \quad \text{on } \Omega \times \Gamma^{\delta} \times (0,T). \tag{18}$$

Finally, the initial condition is

$$S^{\delta}_m(x,y,0) = S^0_m(x) \text{ in } \Omega \times Y^{\delta}_m. \tag{19}$$

The existence of weak solutions of the global δ-problem (9)-(19) is a consequence of the homogenization result in [4, 5] and it has also been studied in [8].

It can be seen, as in [11], that there exist positive constants $\hat{k}_m^1$, $\hat{k}_m^2$ such that the effective permeability tensor $\mathbb{K}^{\star,\delta}$ satisfies for any $\boldsymbol{\xi} \in \mathbb{R}^d$:

$$\hat{k}_m^1 \, |\boldsymbol{\xi}|^2 \leq \delta^{-1} \, \mathbb{K}^{\star,\delta} \boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq \hat{k}_m^2 \, |\boldsymbol{\xi}|^2. \tag{20}$$

Following [11], Chapter 2, the asymptotic behavior of the homogenized permeability tensor $\mathbb{K}^{\star,\delta}$ with respect to δ is given by

$$\frac{\mathbb{K}_{ij}^{\star,\delta}}{|Y_f^\delta|} = \mathbb{K}_{ij}^\star + \bar{\mathbb{K}}_{ij}^\delta, \tag{21}$$

where $\bar{\mathbb{K}}_{ij}^\delta \to 0$ and $|Y_f^\delta| = d\,\delta + O(\delta^2)$. The tensor $\mathbb{K}^\star$ is calculated as (see [11])

$$\mathbb{K}^\star = k^* \, \mathbb{I} \quad \text{with } k^* = \frac{d-1}{d} \, k_f, \quad d = 2, 3. \tag{22}$$

The problem (17)-(19) can be simplified due to the constant in y boundary conditions by eliminating the matrix phase pressures as follows. Let us introduce the functions

$$\beta_\ell(s) \stackrel{\text{def}}{=} \int_0^s \alpha_\ell(\xi) \, d\xi, \quad \text{where } \alpha_\ell(s) \stackrel{\text{def}}{=} \frac{\lambda_{w,\ell}(s) \, \lambda_{n,\ell}(s)}{\lambda_\ell(s)} |P'_{c,\ell}(s)|, \quad \text{for } \ell = f, m. \tag{23}$$

Lemma 1. *Let* $S_m^\delta(x,y,t)$ *be the solution of the cell problem* (17)-(19)*. It holds:*

$$\begin{aligned} &\Phi_m \partial_t S_m^\delta - \delta^2 \, k_m \, \Delta_y \beta_m(S_m^\delta) = 0 \quad \text{in } \Omega_T \times Y_m^\delta, \\ &S_m^\delta(x,y,t) = \mathcal{P}(S_f^\delta(x,t)) \quad \text{on } \Omega_T \times \Gamma^\delta, \qquad S_m^\delta(x,y,0) = S_m^0(x) \quad \text{in } \Omega \times Y_m^\delta, \end{aligned} \tag{24}$$

where the function $\mathcal{P}(s) \stackrel{\text{def}}{=} (P_{c,m}^{-1} \circ P_{c,f})(s)$.

Equation $(24)_1$ is known as the *imbibition equation*.

Proof. Let us first introduce the global pressure P_m^δ in the matrix block (see [22, 31]) by

$$P_{w,m}^\delta = \mathsf{P}_m^\delta - \int_{S_m^\delta}^1 \frac{\lambda_{n,m}(\xi)}{\lambda_m(\xi)} \, P'_{c,m}(\xi) \, d\xi, \quad P_{n,m}^\delta = \mathsf{P}_m^\delta + \int_{S_m^\delta}^1 \frac{\lambda_{w,m}(\xi)}{\lambda_m(\xi)} \, P'_{c,m}(\xi) \, d\xi, \tag{25}$$

where the total mobility function λ_m is defined in (A.4). From the boundary conditions (18) at the interface Γ^δ we immediately get $(24)_2$. Since the function S_m^δ does not depend on y on $\Omega_T \times \Gamma^\delta$, it follows that the global pressure P_m^δ does not depend on y on $\Omega_T \times \Gamma^\delta$. Therefore,

$$\mathsf{P}_m^\delta(x,y,t) = P_{m,\Gamma}^\delta(x,t) \quad \text{on } \Omega_T \times \Gamma^\delta. \tag{26}$$

By summing the two equations in (17) and by applying the definition of P_m^δ we get ([22, 31])

$$-\delta^2 \, k_m \, \mathrm{div}\left(\lambda_m(S_m^\delta) \nabla \mathsf{P}_m^\delta\right) = 0 \quad \text{in } \Omega_T \times Y_m^\delta, \tag{27}$$

and by multiplying the equation (27) by $\mathsf{P}_m^\delta - P_{m,\Gamma}^\delta$ and integrating over $\Omega_T \times Y_m^\delta$, using (26) and (A.4) we obtain:

$$0 = \delta^2\, k_m \int_{\Omega_T \times Y_m^\delta} \lambda_m(S_m^\delta) |\nabla_y \mathsf{P}_m^\delta|^2\, dx\, dy\, dt \geqslant \delta^2\, k_m\; L_0 \int_{\Omega_T \times Y_m^\delta} |\nabla_y \mathsf{P}_m^\delta|^2\, dx\, dy\, dt,$$

which gives

$$\nabla_y \mathsf{P}_m^\delta = 0 \quad \text{a.e. in } \Omega_T \times Y_m^\delta. \tag{28}$$

This result allows us to reduce the two equations in the problem (17) to only one, as announced. Namely, by taking into account (28) and the identity

$$\lambda_{w,m}(S_m^\delta) \nabla_y P_{w,m}^\delta = \lambda_{w,m}(S_m^\delta) \nabla_y \mathsf{P}_m^\delta - \nabla_y \beta_m(S_m^\delta),$$

from $(17)_1$ we establish $(24)_1$. This completes the proof of Lemma 1.

□

Let us point out that the matrix-fracture source terms $\mathcal{Q}_w^\delta, \mathcal{Q}_n^\delta$ of the system (9), given in an implicit form by (16), involve the function S_m^δ which is a solution of the local boundary value problem (24), which is coupled with the global problem (9)-(12) through its boundary condition. This feature of the system (9)-(16), (24) is captured by the concept introduced in [28]: the homogenized system of equations is said to be **fully homogenized** if it does not involve the unknown functions which are defined as the solutions of the coupled local problems. The global δ-problem (9)-(16), (24) is not fully homogenized in the said sense. The purpose of the succeeding sections is to express the source terms $\mathcal{Q}_w^\delta, \mathcal{Q}_n^\delta$ in an explicit form by decoupling the global system (9)-(16) from the local problem (24). This will be done by passing to the limit as $\delta \to 0$ in the system (9)-(16), (24) and thereby establishing the fully homogenized model. Following the idea of [19] we will first linearize the imbibition equation (24) and perform the asymptotic analysis of the linearized imbibition equation.

4 Linearized imbibition equation

Our next step is to simplify the matrix cell problem (24) by introducing a linearized version of that problem. This is a "heuristic" step, as explained in the Introduction. As suggested by Arbogast in [19], we consider a function $\psi_m(x)$ such that

$$\psi_m \approx \alpha_m(S_m^\delta). \tag{29}$$

Moreover, we assume that there are constants $\psi_m^{min}, \psi_m^{max}$ such that for any $x \in \Omega$ it holds

$$0 < \psi_m^{min} \leq \psi_m(x) \leq \psi_m^{max}. \tag{30}$$

Thus we replace the imbibition equation (24) by its linearized version

$$\begin{aligned} &\Phi_m \partial_t S_m^\delta - \delta^2 k_m\, \psi_m(x) \Delta_y S_m^\delta = 0 \quad \text{in } \Omega_T \times Y_m^\delta, \\ &S_m^\delta(x,y,t) = \mathcal{P}(S_f^\delta(x,t)) \quad \text{on } \Omega_T \times \Gamma^\delta, \qquad S_m^\delta(x,y,0) = S_m^0(x) \quad \text{in } \Omega \times Y_m^\delta. \end{aligned} \tag{31}$$

The particular choice of function ψ_m was proposed and validated in [19]. The numerical simulations were performed for exact and linearized models and the computational results show that the linearized model is computationally less complex while essentially without significant loss in accuracy compared to the exact model. We point out that in some settings, as in water-oil systems relevant for the petroleum industry, where the matrix is not fully saturated by water, hypothesis (4.1) is a reasonable approximation in the case of a special relation between the capillary pressure function and the phase mobilities. For instance, by using the well-known Brooks-Corey capillary pressure model $P_{c,m}(S) = P_d\, S^{-1/\lambda}$ and Corey-type phase mobilities models $\lambda_{w,m}(S) = S^\alpha$, $\lambda_{n,m}(S) = (1-S)^\beta$, where P_d is the entry pressure, λ is the pore size distribution index and α, $\beta > 1$, with $\alpha = -1/\lambda - 1$ and for λ small enough, one obtains the function α_m which has values near to a constant for all values of S of interest. Small values of λ correspond to the heterogeneous soils with a wide range of pore sizes. An existence result for the model (9)-(16), (31) is proved in [7].

In order to analyze the behavior of S_m^δ as $\delta \to 0$ we replace the parabolic problem (31) by an elliptic problem by use of the Laplace transform $\mathcal{L}$. Let $S_m^\delta(x, y, t)$ be the solution of the linearized problem (31). We denote for $\lambda > 0$: $s_m^\delta \stackrel{\text{def}}{=} \mathcal{L}(S_m^\delta)$.

By using the basic properties of the Laplace transformation, it follows easily that the function $s_m^\delta(x, y, \lambda)$ satisfies the following problem:

$$\begin{aligned} &\lambda \Phi_m\, s_m^\delta(x,y,\lambda) - \delta^2 k_m \psi_m(x) \Delta_y s_m^\delta(x,y,\lambda) = \Phi_m S_m^0(x) \quad \text{in } \Omega \times Y_m^\delta, \\ &s_m^\delta(x,y,\lambda) = \mathcal{L}\big(\mathcal{P}(S_f^\delta)\big)(x,\lambda) \quad \text{on } \Omega \times \Gamma^\delta. \end{aligned} \tag{32}$$

Introducing the associated auxiliary problem with constant boundary data:

$$\begin{aligned} &\lambda \Phi_m\, \mathsf{u}^\delta(x,y,\lambda) - \delta^2 k_m \psi_m(x) \Delta_y \mathsf{u}^\delta(x,y,\lambda) = 0 \quad \text{in } \Omega \times Y_m^\delta, \\ &\mathsf{u}^\delta(x,y,\lambda) = 1 \quad \text{on } \Omega \times \Gamma^\delta, \end{aligned} \tag{33}$$

it is easy to see that the solution s_m^δ of (32) is given by

$$s_m^\delta(x,y,\lambda) = \frac{1}{\lambda}\, S_m^0(x) + \mathsf{u}^\delta(x,y,\lambda)\ \mathcal{L}\big(\mathcal{P}(S_f^\delta(x,t)) - S_m^0(x)\big). \tag{34}$$

The matrix-fracture source terms $\mathcal{Q}_w^\delta, \mathcal{Q}_n^\delta$ are given by (16) in which S_m^δ is the solution of nonlinear imbibition equation (24). We define simplified matrix-fracture source terms $\hat{\mathcal{Q}}_w^\delta, \hat{\mathcal{Q}}_n^\delta$ by introducing solution S_m^δ of linearized imbibition equation (31) into (16).

From (16), using (34) we obtain

$$\hat{\mathcal{Q}}_w^\delta(x,t) = -\frac{\Phi_m}{|Y_m^\delta|}\,\mathcal{L}^{-1}\bigg(\lambda\,\mathcal{L}\big(\mathcal{P}(S_f^\delta(x,t)) - S_m^0(x)\big)\int_{Y_m^\delta} \mathsf{u}^\delta(x,y,\lambda)dy\bigg). \tag{35}$$

In order to estimate the asymptotic behavior of $\hat{\mathcal{Q}}_w^\delta$ as δ tends to 0, we need to estimate asymptotically in δ the integral term $\int_{Y_m^\delta} \mathsf{u}^\delta(x,y,\lambda)dy$ in (35). Slightly modifying the proof of Lemma 7.2 from [12], we have:

Lemma 2. *For any $x \in \Omega$, let $\mathsf{u}^\delta(x,y,\lambda)$ be the solution of the problem* (33) *with parameter x. Then it holds as $\delta \to 0$, uniformly in x,*

$$\int_{Y_m^\delta} \mathsf{u}^\delta(x,y,\lambda)dy = \frac{2d\sqrt{k_m\psi_m(x)}}{\sqrt{\Phi_m}\sqrt{\lambda}}\,\delta\,(1+o(1)). \tag{36}$$

Finally, from (35) and (36), by applying the basic properties of the Laplace transformation, we obtain the following result.

Corollary 1. *The simplified matrix-fracture source terms $\hat{\mathfrak{Q}}_w^\delta, \hat{\mathfrak{Q}}_n^\delta$ satisfy*

$$\hat{\mathfrak{Q}}_w^\delta(x,t) = -\partial_t\left[\left(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)\right) * \omega^\delta\right](x,t) = -\hat{\mathfrak{Q}}_n^\delta(x,t), \tag{37}$$

where we denote

$$\omega^\delta(x,t) \stackrel{\text{def}}{=} D^\delta(x)t^{-\frac{1}{2}},\ D^\delta(x) \stackrel{\text{def}}{=} \delta\left[C_m(x)/|Y_m^\delta| + o(1)\right],\ C_m(x) \stackrel{\text{def}}{=} 2d\sqrt{\Phi_m k_m \psi_m(x)}/\sqrt{\pi}, \tag{38}$$

and $*$ denotes convolution with respect to time.

Note that for all $x \in \Omega$ and sufficiently small δ it holds

$$D^\delta(x) \le 2\,\delta\,C_m^{max}, \quad C_m^{max} = 2d\sqrt{\Phi_m k_m \psi_m^{max}}/\sqrt{\pi}. \tag{39}$$

5 Passage to the limit as $\delta \to 0$

In order to derive the fully homogenized model we need to pass to the limit as $\delta \to 0$ in the problem (9) with corresponding boundary and initial conditions. However, we will not make the asymptotic analysis of the full system (9), but only of the system in which the nonlinear matrix-fracture source terms $\mathfrak{Q}_w^\delta, \mathfrak{Q}_n^\delta$ are replaced by simplified matrix-fracture source terms $\hat{\mathfrak{Q}}_w^\delta, \hat{\mathfrak{Q}}_n^\delta$, for which the asymptotic behavior is given by Corollary 1. This passage is performed rigorously by establishing uniform in δ estimates and the compactness criterion for the sequence $\{S_f^\delta\}_{(\delta>0)}$. We start by transforming the system (9) by employing new variables: the global pressure P_f^δ and a "complementary pressure" θ_f^δ. First the global pressure P_f^δ in the fractures is inducted analogously to (25) by

$$P_{w,f}^\delta = \mathsf{P}_f^\delta - \int_{S_f^\delta}^1 \frac{\lambda_{n,f}(\xi)}{\lambda_f(\xi)}\,P'_{c,f}(\xi)\,d\xi, \quad P_{n,f}^\delta = \mathsf{P}_f^\delta + \int_{S_f^\delta}^1 \frac{\lambda_{w,f}(\xi)}{\lambda_f(\xi)}\,P'_{c,f}(\xi)\,d\xi. \tag{40}$$

A "complementary pressure" θ_f^δ is defined (see [7]) by

$$\theta_f^\delta \stackrel{\text{def}}{=} \beta_f(S_f^\delta), \tag{41}$$

where β_f is defined in (23). We denote $\theta_f^\star = \beta_f(1)$ and the inverse function

$$S_f^\delta = \mathcal{B}_f(\theta_f^\delta) \stackrel{\text{def}}{=} \beta_f^{-1}(\theta_f^\delta) \quad \text{for } 0 \le \theta_f^\delta \le \theta_f^\star. \tag{42}$$

Note that $\mathcal{B}_f : [0, \theta_f^*] \to [0, 1]$ is a continuous and monotone increasing function.

Finally, the system (9), with simplified matrix-fracture source terms, in terms of the global pressure and the complementary pressure reads:

$$\begin{aligned} &\operatorname{div}\big(\lambda_f(S_f^\delta)\mathbb{K}^{\star,\delta}\nabla \mathsf{P}_f^\delta\big) = 0 \quad \text{in } \Omega_T, \\ &\Phi^\delta \partial_t S_f^\delta - \operatorname{div}\big(\mathbb{K}^{\star,\delta}\,\big(\lambda_f(S_f^\delta)\nabla\theta_f^\delta + \lambda_{w,f}(S_f^\delta)\nabla \mathsf{P}_f^\delta\big)\,\big) = \hat{\mathbb{Q}}_w^\delta \quad \text{in } \Omega_T, \\ &S_f^\delta = \mathcal{B}_f(\theta_f^\delta) \quad \text{in } \Omega_T. \end{aligned} \tag{43}$$

The boundary conditions for the system (43) are given by:

$$\begin{aligned} &\mathsf{P}_f^\delta = P_\Gamma \quad \text{and} \quad \theta_f^\delta = \theta_\Gamma \ \text{ on } \Gamma_{inj} \times (0,T), \\ &\mathbb{K}^{\star,\delta}\lambda_f(S_f^\delta)\nabla \mathsf{P}_f^\delta \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_{imp} \times (0,T), \\ &\mathbb{K}^{\star,\delta}\,\big(\lambda_f(S_f^\delta)\nabla\theta_f^\delta + \lambda_{w,f}(S_f^\delta)\nabla \mathsf{P}_f^\delta\big)\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma_{imp} \times (0,T). \end{aligned} \tag{44}$$

The initial condition reads:

$$\theta_f^\delta(x,0) = \theta_f^0(x) \text{ in } \Omega. \tag{45}$$

The boundary and initial data P_Γ, θ_Γ and θ_f^0 in (44) and (45) are given by the corresponding transformations of the functions $P_{w,\Gamma}$, $P_{n,\Gamma}$, $S_{f,\Gamma} \stackrel{\text{def}}{=} P_{c,f}^{-1}(P_{n,\Gamma} - P_{w,\Gamma})$ and S_f^0.

Now we state the rest of the assumptions on the data which will assure the existence for weak solutions of the problem (43)-(45).

(A.5) The boundary and initial data satisfy: $P_\Gamma \in L^2(0,T;H^1(\Omega))$, $\theta_\Gamma \in L^2(0,T;H^1(\Omega))$, $\partial_t\theta_\Gamma \in L^1(\Omega_T)$, $\theta_f^0 \in L^2(\Omega)$, $0 \le \theta_f^0, \theta_\Gamma \le \theta_f^*$ a.e. in Ω.

A weak solution of this problem is defined as follows. Let $V \stackrel{\text{def}}{=} \{u \in H^1(\Omega), u|_{\Gamma_{inj}} = 0\}$.

Definition 1. *A weak solution to the system* (43)-(45) *is a pair* $(\mathsf{P}_f^\delta, \theta_f^\delta)$ *such that*

$$\mathsf{P}_f^\delta - P_\Gamma \in L^2(0,T;V),\ \ \theta_f^\delta - \theta_\Gamma \in L^2(0,T;V),\ \ 0 \le \theta_f^\delta \le \theta_f^* \quad \textit{a.e. in } \Omega_T,\ \ S_f^\delta = \mathcal{B}_f(\theta_f^\delta),$$
$$\partial_t\left(\Phi^\delta S_f^\delta + \big(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)\big) * \omega^\delta\right) \in L^2(0,T;V'),$$

for any $\zeta, \varphi \in L^2(0,T;V)$

$$\int_{\Omega_T} \lambda_f(S_f^\delta)\mathbb{K}^{\star,\delta}\nabla \mathsf{P}_f^\delta \cdot \nabla\zeta \,dx\,dt = 0, \tag{46}$$

$$\begin{aligned} &\int_0^T \langle \partial_t\big(\Phi^\delta S_f^\delta + \big(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)\big) * \omega^\delta\big), \varphi\rangle dt \\ &\qquad + \int_{\Omega_T} \mathbb{K}^{\star,\delta}\,\big(\lambda_f(S_f^\delta)\nabla\theta_f^\delta + \lambda_{w,f}(S_f^\delta)\nabla \mathsf{P}_f^\delta\big)\cdot\nabla\varphi\,dx\,dt = 0. \end{aligned} \tag{47}$$

Furthermore, the initial condition is satisfied in the following sense:

for any $\varphi \in L^2(0,T;V) \cap W^{1,1}(0,T;L^1(\Omega))$ *such that* $\varphi(\cdot,T) = 0$ *in* Ω*,*

$$\begin{aligned} &\int_0^T \langle \partial_t\big(\Phi^\delta S_f^\delta + \big(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)\big) * \omega^\delta\big), \varphi\rangle dt \\ &\qquad + \int_{\Omega_T} \big(\Phi^\delta(S_f^\delta - S_f^0) + \big(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)\big) * \omega^\delta\big)\,\partial_t\varphi\,dx\,dt = 0. \end{aligned} \tag{48}$$

The existence of a weak solution from Definition 1 under conditions (A.1)–(A.5) follows from the result of [7], Theorem 1. Our goal is to pass to the limit as $\delta \to 0$ in the system (43)-(45).

5.1 Uniform a priori estimates

First we establish the following uniform estimates.

Proposition 1. *Let $\delta > 0$. Let $(\mathsf{P}_f^\delta, \theta_f^\delta)$ be a weak solution of the problem* (43)-(45). *The following estimates, uniform with respect to δ, hold:*

$$\|\mathsf{P}_f^\delta\|_{L^2(0,T;H^1(\Omega))} + \|\theta_f^\delta\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{49}$$

$$\|\frac{1}{\delta}\,\partial_t \left(\Phi^\delta S_f^\delta + \left(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)\right) * \omega^\delta\right)\|_{L^2(0,T;V')} \le C. \tag{50}$$

Proof. We first insert $\zeta = \mathsf{P}_f^\delta - P_\Gamma$ into the equation (46). This yields

$$\int_{\Omega_T} \lambda_f(S_f^\delta)\mathbb{K}^{\star,\delta}|\nabla \mathsf{P}_f^\delta|^2\,dx\,dt = \int_{\Omega_T} \lambda_f(S_f^\delta)\mathbb{K}^{\star,\delta}\nabla \mathsf{P}_f^\delta \cdot \nabla P_\Gamma\,dx\,dt. \tag{51}$$

Taking into account the representation (21) of the tensor $\mathbb{K}^{\star,\delta}$, we get

$$\int_{\Omega_T} \lambda_f(S_f^\delta)\mathbb{K}^{\star}|\nabla \mathsf{P}_f^\delta|^2\,dx\,dt = \frac{1}{|Y_f^\delta|}\int_{\Omega_T} \lambda_f(S_f^\delta)\mathbb{K}^{\star,\delta}\nabla \mathsf{P}_f^\delta \cdot \nabla P_\Gamma\,dx\,dt - \int_{\Omega_T} \lambda_f(S_f^\delta)\bar{\mathbb{K}}^\delta|\nabla \mathsf{P}_f^\delta|^2 dxdt,$$

and by applying (22) and (A.4), we finally obtain $\|\nabla \mathsf{P}_f^\delta\|_{L^2(\Omega_T)} \le C$, with a constant C which is independent of δ.

Now we choose $\varphi = \theta_f^\delta - \theta_\Gamma$ in (47). This yields

$$\begin{aligned} &\int_0^T \langle \partial_t(\Phi^\delta S_f^\delta + [(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)) * \omega^\delta]), \theta_f^\delta - \theta_\Gamma\rangle dt \\ &+ \int_{\Omega_T} \mathbb{K}^{\star,\delta}\lambda_f(S_f^\delta)\nabla\theta_f^\delta \cdot \nabla\theta_f^\delta\,dxdt + \int_{\Omega_T} \mathbb{K}^{\star,\delta}\lambda_{w,f}(S_f^\delta)\nabla \mathsf{P}_f^\delta \cdot \nabla\theta_f^\delta\,dxdt \\ &= \int_{\Omega_T} \mathbb{K}^{\star,\delta}\lambda_f(S_f^\delta)\nabla\theta_f^\delta \cdot \nabla\theta_\Gamma\,dxdt + \int_{\Omega_T} \mathbb{K}^{\star,\delta}\lambda_{w,f}(S_f^\delta)\nabla \mathsf{P}_f^\delta \cdot \nabla\theta_\Gamma\,dxdt. \end{aligned} \tag{52}$$

The integral terms in the equality (52) are denoted by $X_1, X_2, \ldots, X_5$, respectively. Assume for the moment that the function S_f^δ is sufficiently regular in time. Then we can write $X_1 = Y_1 + Y_2$. For Y_1, by (A.5), we have:

$$\begin{aligned} Y_1 \stackrel{\text{def}}{=} &\int_\Omega \Phi^\delta(H(\theta_f^\delta(T)) - S_f^\delta(T)\theta_\Gamma(T))\,dx - \int_\Omega \Phi^\delta(H(\theta_f^\delta(0)) - S_f^\delta(0)\theta_\Gamma(0))\,dx \\ &+ \int_0^T\int_\Omega \Phi^\delta S_f^\delta \partial_t\theta_\Gamma\,dxdt \ge -\Phi^\delta\left[4\,\theta_f^*\,|\Omega| + \|\partial_t\theta_\Gamma\|_{L^1(\Omega_T)}\right], \end{aligned} \tag{53}$$

where the function H is defined by $H(\theta) \stackrel{\text{def}}{=} \int_0^\theta \mathcal{B}_f'(r) r \, dr$. Moreover, it is easy to see,

$$|H(\theta_f^\delta)| = |\mathcal{B}_f(\theta_f^\delta)\theta_f^\delta - \int_0^{\theta_f^\delta} \mathcal{B}_f(r)\, dr| \leq 2\,\theta_f^*.$$

For Y_2 part of the X_1 we obtain, using integration by parts,

$$Y_2 \stackrel{\text{def}}{=} \int_0^T \int_\Omega \partial_t \left((\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)) * \omega^\delta \right) (\theta_f^\delta - \theta_\Gamma)\, dx dt = Y_2^1 + Y_2^2 + Y_2^3,$$

with

$$|Y_2^1| \leq 2\,\delta\, C_m^{max}, \quad |Y_2^3| \leq 4\,\delta\,\sqrt{T}\, C_m^{max}\, \|\partial_t \theta_\Gamma\|_{L^1(\Omega_T)}$$

and

$$Y_2^2 \stackrel{\text{def}}{=} -\int_0^T \int_\Omega \int_0^t \left(\mathcal{P}(S_f^\delta(t-\tau)) - \mathcal{P}(S_f^0) \right)\, \omega^\delta(\tau)\, d\tau\, \partial_t \theta_f^\delta(t)\, dt\, dx.$$

By changing the order of the time integration and integrating by parts in the term Y_2^2, we can write $Y_2^2 = Y_2^{2,1} + Y_2^{2,2}$, where

$$|Y_2^{2,1}| \leq 8\,\delta\, C_m^{max}\, \theta_f^*\, |\Omega|\, \sqrt{T} \text{ and } Y_2^{2,2} \stackrel{\text{def}}{=} \int_0^T \int_\Omega \int_\tau^T \partial_t \mathcal{P}(S_f^\delta(t-\tau))\, \theta_f^\delta(t)\, dt\, \omega^\delta(\tau)\, dx\, d\tau.$$

Finally, the term $Y_2^{2,2}$ can be written as $Y_2^{2,2} = A + B$, where

$$|A| \leq 4\,\delta\, C_m^{max}\, \theta_f^*\, |\Omega|\, \sqrt{T} \;\text{ and }\; B \stackrel{\text{def}}{=} -\int_0^T \int_\Omega \partial_\tau \left(\int_\tau^T \mathcal{P}(S_f^\delta(t-\tau))\, \theta_f^\delta(t)\, dt \right) \omega^\delta(\tau)\, dx\, d\tau.$$

It can be proved, as in [7], that for any $\tau \in [0, T]$ it holds

$$h^\delta(\tau) \stackrel{\text{def}}{=} \int_\tau^T \mathcal{P}(S_f^\delta(t-\tau))\, \theta_f^\delta(t)\, dt \leq h(0) \tag{54}$$

and from (54) it follows that $\partial_\tau h(\tau) \leq 0$ in $[0, T]$. Then $B \geq 0$ which gives $Y_2^{2,2} \geq A \geq -4\,\delta\, C_m^{max}\, \theta_f^*\, |\Omega|\, \sqrt{T}$. Summing all the obtained inequalities, we have for the first term in (52) the estimate: $X_1 \geq -C\left(\Phi^\delta + \delta\right)$, where the constant C depends on C_m^{max}, $|\Omega|$, T, θ_f^*, $\|\partial_t \theta_\Gamma\|_{L^1(\Omega_T)}$. These calculations are applicable for regularized in time S_f^δ but they remain true for the desired S_f^δ by a passage to the limit as the regularization parameter tends to 0.

We treat the terms X_2,…,X_5 in a standard way using the Cauchy-Schwartz inequality and the already obtained estimate for the global pressure in (49). Finally, we have

$$L_0\, \delta\, \hat{k}_m^1\, \|\nabla \theta_f^\delta\|_{L^2(\Omega_T)}^2 \leq C\,\delta + C\,\delta\, \hat{k}_m^2\, (1 + \|\nabla \theta_f^\delta\|_{L^2(\Omega_T)}) \tag{55}$$

and, therefore, $\|\nabla \theta_f^\delta\|_{L^2(\Omega_T)} \leq C$, with a constant C which is independent of δ. The estimate (50) follows in the standard way from (49). This completes the proof of Proposition 1.

□

Lemma 3. *There exists a constant C which is independent of δ and h such that, as $h \to 0$,*

$$\int_h^T \int_\Omega \big(S_f^\delta(x,t) - S_f^\delta(x,t-h)\big)\big(\theta_f^\delta(x,t) - \theta_f^\delta(x,t-h)\big)\, dx\, dt \le C\sqrt{h}.$$

Proof. Let us first note that for integrable functions G_1, G_2 and for $0 < h < T/2$ it holds

$$\int_0^T G_1(t) \int_{\max(t,h)}^{\min(t+h,T)} G_2(\tau) d\tau dt = \int_h^T G_2(t) \int_{t-h}^t G_1(\tau) d\tau dt. \tag{56}$$

We define the test function in (47) by

$$\varphi = \varphi^{\delta,h}(x,t) = \int_{\max(t,h)}^{\min(t+h,T)} \big((\theta_f^\delta(x,\tau) - \theta_\Gamma(x,\tau)) - (\theta_f^\delta(x,\tau-h) - \theta_\Gamma(x,\tau-h))\big)\; d\tau.$$

Then $\varphi \in L^2(0,T;V)$. Plugging it in (47) we have:

$$\begin{aligned}\int_0^T \langle \partial_t(\Phi^\delta S_f^\delta + [\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)] * \omega^\delta), \varphi^{\delta,h}\rangle dt = \\ -\int_{\Omega_T} \mathbb{K}^{*,\delta}(\lambda_f(S_f^\delta)\nabla\theta_f^\delta + \lambda_{w,f}(S_f^\delta)\nabla \mathsf{P}_f^\delta)\cdot\nabla\varphi^{\delta,h}\, dxdt.\end{aligned} \tag{57}$$

By using (56) the left-hand side term can be written as

$$\begin{aligned}&\int_0^T \langle \partial_t(\Phi^\delta S_f^\delta + [\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0)] * \omega^\delta), \varphi^{\delta,h}\rangle dt \\ &= \int_h^T \int_\Omega \Phi^\delta (S_f^\delta(x,t) - S_f^\delta(x,t-h))(\theta_f^\delta(x,t) - \theta_f^\delta(x,t-h))\, dxdt \\ &- \int_h^T \int_\Omega \Phi^\delta (S_f^\delta(x,t) - S_f^\delta(x,t-h))(\theta_\Gamma(x,t) - \theta_\Gamma(x,t-h))\, dxdt \\ &+ \int_h^T \int_\Omega [(\theta_f^\delta(x,t) - \theta_f^\delta(x,t-h)) - (\theta_\Gamma(x,t) - \theta_\Gamma(x,t-h))] X_h^\delta(x,t)\, dxdt,\end{aligned} \tag{58}$$

where

$$\begin{aligned}X_h^\delta(x,t) = &\int_h^t [\mathcal{P}(S_f^\delta(x,t-\tau) - \mathcal{P}(S_f^0(x))][\omega^\delta(\tau) - \omega^\delta(\tau-h)]\, d\tau \\ &+ \int_0^h [\mathcal{P}(S_f^\delta(x,t-\tau) - \mathcal{P}(S_f^0(x))]\omega^\delta(\tau)\, d\tau.\end{aligned} \tag{59}$$

Let us denote the integral terms at the right-hand side of the equality (58) by Z_1, Z_2, Z_3, respectively. For Z_2 we have by using (A.5):

$$|Z_2| \le \Phi^\delta \int_h^T \int_\Omega |\theta_\Gamma(x,t) - \theta_\Gamma(x,t-h)|\, dxdt \le C\, h\, \delta\, \|\partial_t\theta_\Gamma\|_{L^1(\Omega_T)}, \tag{60}$$

since $\Phi^\delta/\delta \le C$, uniformly with respect to δ. Next, due to $0 \le \mathcal{P} \le 1$ and $\omega^\delta > 0$ we have

$$|X_h^\delta(x,t)| \le 2\int_0^h \omega^\delta(\tau)\,d\tau = 4\,\delta\, C_m^{max}\sqrt{h}$$

and therefore

$$|Z_3| \le 8|\Omega_T|\,\theta_f^*\, C_m^{max}\,\delta\,\sqrt{h}. \tag{61}$$

Finally, we apply (56) with

$$G_1(t) \equiv 1,\; G_2(\tau) = |(\nabla\theta_f^\delta(x,\tau) - \nabla\theta_\Gamma(x,\tau)) - (\nabla\theta_f^\delta(x,\tau-h) - \nabla\theta_\Gamma(x,\tau-h))|^2$$

to establish

$$\|\nabla\varphi^{\delta,h}\|_{L^2(\Omega_T)} \le 2\,h\|\nabla(\theta_f^\delta - \theta_\Gamma)\|_{L^2(\Omega_T)} \le C\,h, \tag{62}$$

where we have used the uniform a priori estimate (49). From the estimate (62) and the uniform bounds (49) we hence obtain

$$\left|\int_{\Omega_T} \mathbb{K}^{*,\delta}(\lambda_f(S_f^\delta)\nabla\theta_f^\delta + \lambda_{w,f}(S_f^\delta)\nabla \mathsf{P}_f^\delta)\cdot\nabla\varphi^{\delta,h}\,dxdt\right| \le C\,h\,\delta\,\hat{k}_m^2. \tag{63}$$

Collecting the estimates (60), (61), (63), from (57) we get

$$\int_h^T\int_\Omega \Phi^\delta(S_f^\delta(x,t) - S_f^\delta(x,t-h))(\theta_f^\delta(x,t) - \theta_f^\delta(x,t-h))\,dxdt \le C\,\delta\,\sqrt{h}.$$

Now, the desired estimate follows from $\Phi^\delta \ge c\delta$, for some c independent of δ, and the monotonicity of $S \mapsto \beta_f(S)$.

□

5.2 The fully homogenized model

In this subsection we present the fully homogenized model for immiscible incompressible two-phase flow in double porosity media with thin fractures. First we state the convergence results holding as $\delta \to 0$.

Theorem 1. *Let assumptions (A.1)–(A.5) be fulfilled. Let $(\mathsf{P}_f^\delta, \theta_f^\delta)$ be a weak solution of the problem* (43)-(45) *and let $S_f^\delta = \mathcal{B}_f(\theta_f^\delta)$. Then there exist functions $\mathsf{P}_f \in L^2(0,T;V) + P_\Gamma$ and $\theta_f \in L^2(0,T;V) + \theta_\Gamma$ such that, up to a subsequence, it holds*

$$\mathsf{P}_f^\delta(x,t) \rightharpoonup \mathsf{P}_f(x,t) \quad \text{weakly in } \; L^2(0,T;H^1(\Omega)), \tag{64}$$

$$\theta_f^\delta(x,t) \rightharpoonup \theta_f(x,t) \quad \text{weakly in } \; L^2(0,T;H^1(\Omega)) \tag{65}$$

as $\delta \to 0$. Moreover, $0 \le \theta_f(x,t) \le \theta_f^\star$ a.e. in Ω_T. Furthermore,

$$S_f^\delta(x,t) \to S_f(x,t) \quad \text{strongly in } \; L^2(\Omega_T) \text{ and a.e. in } \; \Omega_T, \tag{66}$$

where $S_f = \mathcal{B}_f(\theta_f)$. Here (P_f, θ_f) is a weak solution in Ω_T of problem:

$$
\begin{aligned}
&\text{div } (\lambda_f(S_f) k^\star \nabla P_f) = 0, \\
&\Phi_f \partial_t S_f - \text{div } \big(k^\star \left(\lambda_f(S_f) \nabla \theta_f + \lambda_{w,f}(S_f) \nabla P_f \right) \big) = -\frac{C_m(x)}{d}\, \partial_t \left[\big(\mathcal{P}(S_f) - \mathcal{P}(S_f^0) \big) * \frac{1}{\sqrt{t}} \right], \\
&S_f = \mathcal{B}_f(\theta_f).
\end{aligned}
\tag{67}
$$

The boundary conditions for the system (67) *are given by:*

$$
\begin{aligned}
&P_f = P_\Gamma \quad \text{and} \quad \theta_f = \theta_\Gamma \quad \text{on } \Gamma_{inj} \times (0,T), \\
&k^\star \lambda_f(S_f) \nabla P_f \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{imp} \times (0,T), \\
&k^\star \left(\lambda_f(S_f) \nabla \theta_f + \lambda_{w,f}(S_f) \nabla P_f \right) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{imp} \times (0,T),
\end{aligned}
\tag{68}
$$

and the initial condition is

$$
\theta_f(x,0) = \theta_f^0(x) \text{ in } \Omega. \tag{69}
$$

The effective permeability tensor $k^\star$ and the function C_m are given by (22) *and* (38).

Proof. Weak convergences in (64) and (65) follow from (49). The boundedness of θ_f^δ and $S_f = \mathcal{B}_f(\theta_f)$ follows directly from the strong convergence (66). In order to prove (66) we use Lemma 3 and Lemma 1.9 from [32], which we repeat for reader's convenience:

Lemma 4. *Suppose that the sequence $(u_\delta)_\delta$ converges weakly to u in $L^2(0,T; H^1(\Omega))$. Let F be a continuous, monotone and bounded function in $\mathbb{R}$. Assume that*

$$
\int_h^T \int_\Omega \big(F(u_\delta(x,t)) - F(u_\delta(x,t-h)) \big) \big(u_\delta(x,t) - u_\delta(x,t-h) \big)\, dx\, dt \leq C\, \varpi(h), \tag{70}
$$

for some continuous function ϖ such that $\varpi(0) = 0$, and with a constant C independent of h and δ. Then $F(u_\delta)$ converges to $F(u)$ strongly in $L^2(\Omega_T)$.

Now we apply Lemma 4 to the sequence $(\theta_f^\delta)_\delta$ in the role of $(u_\delta)_\delta$. The conditions on the function $F(z) = \mathcal{B}_f(z)$ in Lemma 4 hold from the definition of $\mathcal{B}_f$.

Due to (50), up to a subsequence we have

$$
\frac{1}{\delta} \partial_t \left(\Phi^\delta S_f^\delta + \big(\mathcal{P}(S_f^\delta) - \mathcal{P}(S_f^0) \big) * \omega^\delta \right) \rightharpoonup \Psi \quad \text{weakly in } \; L^2(0,T; V') \tag{71}
$$

for some $\Psi \in L^2(0,T;V')$. The strong convergence of S_f^δ in (66) allows to identify the limit as

$$
\Psi = \partial_t \left(d \Phi_f S_f + C_m(x) \left[\mathcal{P}(S_f) - \mathcal{P}(S_f^0) \right] * \frac{1}{\sqrt{t}} \right) \in L^2(0,T;V').
$$

We can now pass to the limit as $\delta \to 0$ in the equations (46), (47) and (48), after division by $d\,\delta$, and obtain for any $\zeta, \varphi \in L^2(0,T;V)$:

$$
\int_{\Omega_T} \lambda_f(S_f) k^\star \nabla \mathsf{P}_f \cdot \nabla \zeta \, dx\, dt = 0, \tag{72}
$$

$$\int_0^T \langle \partial_t \Big(\Phi_f S_f + \frac{C_m(x)}{d} [\mathcal{P}(S_f) - \mathcal{P}(S_f^0)] * \frac{1}{\sqrt{t}} \Big), \varphi \rangle dt \\ + \int_{\Omega_T} k^\star \left(\lambda_f(S_f) \nabla \theta_f + \lambda_{w,f}(S_f) \nabla \mathsf{P}_f \right) \cdot \nabla \varphi \, dx \, dt = 0. \tag{73}$$

In the initial condition we take the test function $\varphi \in L^2(0,T;V) \cap W^{1,1}(0,T;L^1(\Omega))$ such that $\varphi(\cdot, T) = 0$ in Ω, and obtain

$$\int_0^T \langle \partial_t \Big(\Phi_f S_f + \frac{C_m(x)}{d} [\mathcal{P}(S_f) - \mathcal{P}(S_f^0)] * \frac{1}{\sqrt{t}} \Big), \varphi \rangle dt \\ + \int_{\Omega_T} \left(\Phi_f (S_f - S_f^0) + \frac{C_m(x)}{d} \big(\mathcal{P}(S_f) - \mathcal{P}(S_f^0) \big) * \frac{1}{\sqrt{t}} \right) \partial_t \varphi \, dx \, dt = 0. \tag{74}$$

We observe that the obtained equations (72), (73) and (74) represent a weak formulation of the problem (67)-(69). This completes the proof of Theorem 1.

□

The system (67)-(69) can be transformed into the phase formulation by reintroducing the phase pressures and thus we finally obtain the global fully homogenized model for immiscible incompressible two-phase flow in double porosity media with thin fractures. Namely,

$$\Phi_f \partial_t S_f - \operatorname{div} \left(\frac{d-1}{d} k_f \, \lambda_{w,f}(S_f) \nabla P_{w,f} \right) = - \frac{C_m(x)}{d} \frac{\partial}{\partial t} \left[\big(\mathcal{P}(S_f) - \mathcal{P}(S_f^0) \big) * \frac{1}{\sqrt{t}} \right], \tag{75}$$

$$-\Phi_f \partial_t S_f - \operatorname{div} \left(\frac{d-1}{d} k_f \, \lambda_{n,f}(S_f) \nabla P_{n,f} \right) = \frac{C_m(x)}{d} \frac{\partial}{\partial t} \left[\big(\mathcal{P}(S_f) - \mathcal{P}(S_f^0) \big) * \frac{1}{\sqrt{t}} \right] \tag{76}$$

with the boundary conditions given by

$$P_{w,f} = P_{w,\Gamma} \quad \text{and} \quad P_{n,f} = P_{n,\Gamma} \quad \text{on } \Gamma_{inj} \times (0,T), \\ k^\star \lambda_{w,f}(S_f) \nabla P_{w,f} \cdot \boldsymbol{\nu} = k^\star \lambda_{n,f}(S_f) \nabla P_{n,f} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{imp} \times (0,T), \tag{77}$$

and the initial condition is

$$S_f(x, 0) = S_f^0(x) \text{ in } \Omega. \tag{78}$$

Here in (75)-(77),

$$k^\star = (d-1) k_f / d \quad \text{and} \quad C_m(x) = 2d \sqrt{\Phi_m k_m \psi_m(x)} / \sqrt{\pi}.$$

6 Concluding remarks

The formal linearization of the imbibition equation presented in Section 4 is borowed from Arbogast [7], and it is not rigourously justified. However, due to the presence of the small parameter δ in the nonlinear problem (31) and in the linear one (24) we expect that the solutions of the two matrix block problems are close to each other as they share the same boundary layer structure. This analysis is subject of our further research. Moreover, we point out that in this paper we focus our attention only on the limiting behaviour of the formally linearized problem.

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